Measure avoiding and measure supporting filters

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This is a joint (and ongoing) work with **Jonathan Cancino** (Prague) & **Adam Morawski** (Wrocław) and **Artsiom Ranchynski** (Wrocław).

Homogeneity of $\omega^\star$

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Theorem (Frolik, 1967)

$\omega^\star$ is not homogeneous.

Demand (van Douwen)

A *decent* proof of non-homogeneity of $\omega^\star$!

P-points

Definition (Rudin)

An ultrafilter $\mathcal{U}$ on ω is a P-point if for every descending (X_n) of elements of $\mathcal{U}$, there is $X \in \mathcal{U}$ such that $X \subseteq^* X_n$ for each n.

Remark

In the topological setting: $x \in K$ is a P-point if each G_δ containing x has non-empty interior containing x.

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Suppose that $\mathcal{U}$ extends a density filter. Then $\mathcal{U}$ is not a P-point.

Theorem (Shelah, 1977)

Consistently, there are no P-points in $\omega^\star$.

Weak P-points

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Theorem (Kunen, 1980)

$\mathfrak{c}$-OK points exist in $\omega^\star$. $\mathfrak{c}$-OK points are weak P-points.

Measures on ω

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If $\mathcal{U}$ is an ultrafilter on ω, then the following defines a measure

$$\mu_{\mathcal{U}}(A) = \lim_{n\to\mathcal{U}} |A \cap n|/n$$

I. Measure avoiding ultrafilters.

Measure avoiding ultrafilters

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- *$\mathfrak{c}$-OK points avoid measures.*

Remark

There are non-atomic measures on ω and so some ultrafilters do not avoid measures.

Measure avoiding ultrafilters vs weak P-points

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Weak P-points = points avoiding strictly atomic measures.

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In ZFC there are weak P-points which do not avoid non-atomic measures.

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Weak P-points vs. points avoiding non-atomic measures.

Theorem (PBN, Ranchynski, 2023)

In ZFC there are weak P-points which do not avoid non-atomic measures. Under CH there are non weak P-points which avoid non-atomic measures.

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$$|\{T \in \mathcal{T} \colon T \cap X \neq \emptyset, \mathrm{level}(T) = m\}| \leq 2^m/n.$$

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Proposition

If $\mathcal{U}$ avoids trees, then it avoids non-atomic measures.

Theorem (Ranchynski, 2023)

Under CH there is an ultrafilter which is not a P-point but which avoids trees.

P-measures

II. Measures supporting filters.

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Is it consistent that there is a P-measure but there is no P-point?

P-points in the Silver model

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P-points and P-measures in the random model

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If you add ω_1 Cohen reals and then force with the random forcing, you get P-point in the resulting model.

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Theorem (PBN, Sobota, 2022)

Each P-point from the ground model can be extended to a P-measure in the random extension. In particular, there are P-measures in the random model.

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Is there a P-measure in the Silver model?

Remark

Such a measure would necessarily extend a P-measure from the ground model.

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for each $A \subseteq \omega$

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Theorem (PBN, Cancino, Morawski, 2023)

If there is an ultrafilter Rudin-Blass below μ, then μ cannot be extended to a P-measure in the Silver extension.

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Proposition (PBN, Cancino, Morawski, 2023)

If $\mu_{\mathcal{U}}$ is a P-measure, then there is a P-point Rudin-Blass below $\mu_{\mathcal{U}}$.

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Is it consistent that there are no non-meager P-filters?

Thanks.