A quantifier elimination theorem for Weak König's Lemma with a negated induction axiom

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Preliminaries: second-order arithmetic

This talk is about (weak) fragments of second-order arithmetic.

The language of second-order arithmetic has two sorts of variables:

- first-order: $x, y, z, \dots, i, j, k \dots$ for natural numbers (can be used to code rationals, finite sets and sequences etc.).
- second-order: $X, Y, Z, \dots$ for sets of natural numbers (can be used to code reals, continuous functions etc.).

Non-logical symbols: $+, \cdot, 2^x, \leq, 0, 1; \in$.

Σ^0_n: class of formulas with n first-order quantifier blocks, beginning with $\exists$, then only bounded quantifiers $\exists x \leq t$, $\forall x \leq t$.
Arithmetical formulas have only first-order quantifiers.
Σ^1_n: class of formulas with n second-order quantifier blocks, beginning with $\exists$, followed by an arithmetical formula.
Π^0_n, Π^1_n: dual classes to Σ^0_n, Π^0_n.

Strong fragments of second-order arithmetic

Full second-order arithmetic, Z_2, is axiomatized by:

- the axioms of the nonnegative part of a discrete ordered ring,
- comprehension for all formulas: $\exists X \forall k (k \in X \Leftrightarrow \varphi(x))$,
- induction: $\forall X (0 \in X \wedge \forall k (k \in X \Rightarrow k+1 \in X) \Rightarrow \forall k (k \in X))$.

The intended model is $(\omega, \mathcal{P}(\omega))$.

Strong fragments of Z_2 have a set-theoretic feel to them.
They are, in fact, biinterpretable with various fragments of

$$\mathrm{ZF} \setminus \{\text{Power Set}\} \cup \{\text{every set is countable}\}$$

(with ZF axiomatized using Collection rather than Replacement).

Weaker fragments of second-order arithmetic

ACA_0 is weaker than Z_2 in that it only allows comprehension for arithmetically definable properties.
E.g.: given a tree $T \subseteq \omega^{<\omega}$, $\{v \in T : T$ is infinite below $v\}$ will exist, but $\{v \in T : T$ is well-founded below $v\}$ might not.

RCA_0 is weaker still:

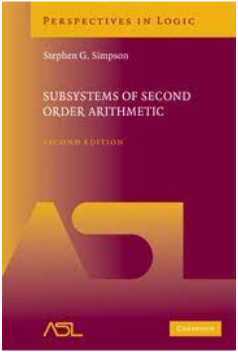
- comprehension only for Δ^0_1-definable properties (i.e. definable by both a Σ^0_1 and a Π^0_1 formula),
- to compensate for that: induction for Σ^0_1-definable properties (not merely the Δ^0_1 ones that correspond to sets).

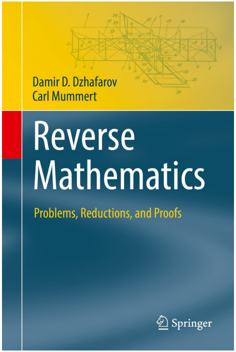
Such fragments have a computability-theoretic feel to them:

- RCA_0 says: "given sets $X_1, \dots, X_n$, any property computable with oracles for $X_1, \dots, X_n$ corresponds to a set".
- ACA_0 can be axiomatized as:
  RCA_0 + "for every set X, the Turing jump of X exists".

Reverse mathematics

RCA_0 typically plays the role of the base theory in a research programme called reverse mathematics.





The idea is to measure the strength of theorems from "everyday mathematics" by proving equivalences and implications between the theorems and some fragments of second-order arithmetic. The equivalences/implications are usually proved over RCA_0.

Reverse mathematics: some examples for ACA_0

Often, the theorems studied are Π^1_2 statements ($\forall X\,\exists Y\ldots$), and their reverse-mathematical strength is connected to how hard it is to compute Y given X.

ACA_0 is equivalent over RCA_0 to, among other things:

- the Bolzano-Weierstrass theorem: every sequence of reals from $[0, 1]$ has a convergent subsequence,
- every countable ring has a maximal ideal,
- Ramsey's theorem for 2-colourings of triples, RT^3_2.

Intuitively, the reason why e.g. RT^3_2 implies ACA_0 is that there is a computable 2-colouring of triples such that every infinite homogeneous set can be used to compute the halting problem.

Weak König's Lemma

WKL says: "every infinite tree $T \subseteq 2^{<\omega}$ has an infinite path".
$\mathsf{WKL}_0 := \mathsf{RCA}_0 + \mathsf{WKL}$.

WKL_0 is strictly in between RCA_0 and ACA_0.

It is equivalent over RCA_0 e.g. to:

- every open cover of $[0, 1]$ contains a finite subcover,
- every countable ring has a prime ideal,
- the completeness theorem for first-order logic,
- Peano's existence theorem for ODEs.

Π^1_1-conservativity of WKL

Theorem (Harrington 1977, independently Ratajczyk 1980's)
WKL_0 is Π^1_1-conservative over RCA_0, i.e. every Π^1_1 sentence provable in WKL_0 is also provable in RCA_0.

The usual proof is by showing that a countable $(M, \mathcal{X}) \models \mathsf{RCA}_0$ can be extended to some $(M, \mathcal{Y}) \models \mathsf{WKL}_0$ (so M stays unchanged).
Adding a path through a single tree $\mathcal{X} \ni T \subseteq 2^{<\omega} \rightsquigarrow$ forcing with infinite subtrees of T. Main difficulty: preserving Σ^0_1 induction.

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It is always the case that $(M, \mathcal{X})$ can be extended to many different (non-elementarily equivalent) $(M, \mathcal{Y})$'s.

Other Π^1_1-conservative statements

WKL is by no means the only Π^1_2 statement that is Π^1_1-conservative over RCA_0. E.g., here is an incomparable one:

COH :=
"for every family $\{R_n : n \in \omega\}$ of subsets of ω, there exists infinite $C \subseteq \omega$ s.t. for each n, either $\forall^\infty k \in C\,(k \in R_n)$ or $\forall^\infty k \in C\,(k \notin R_n)$".

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In fact:

Theorem (Towsner 2015)

The set of Π^1_2 sentences that are Π^1_1-conservative over RCA_0 is a consistent theory that is not c.e. (it is Π^0_2-complete).
In particular, it is not finitely nor even computably axiomatizable.

A weaker base theory

Recall: RCA_0 has comprehension for Δ^0_1-definable properties and induction for Σ^0_1-definable properties.

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RCA^*_0 (Simpson-Smith 1986) is weaker than RCA_0 in that we no longer allow induction for Σ^0_1 properties.

We only have induction for those properties that correspond to sets (i.e. the Δ^0_1-definable ones.)

RCA^*_0 is used e.g. to track essential applications of Σ^0_1 induction, and because it is proof-theoretically more modest than RCA_0.

Π^1_1-conservativity over WKL^*_0

$\mathsf{WKL}^*_0 := \mathsf{RCA}^*_0 + \mathsf{WKL}$.

Theorem (Simpson-Smith 1986)

WKL^*_0 is Π^1_1-conservative over RCA^*_0.

The proof is quite similar to the one over RCA_0.

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Question (essentially Towsner 2015)
Is the set of Π^1_2 sentences that are Π^1_1-conservative over RCA^*_0 also Π^0_2-complete?

Main results

Theorem

Let $(\mathsf{M}, \mathcal{X}) \models \mathsf{RCA}_0^* + \neg\mathsf{RCA}_0$, and let $(\mathsf{M}, \mathcal{Y}), (\mathsf{M}, \mathcal{W}) \models \mathsf{WKL}_0^*$ be countable with $\mathcal{Y}, \mathcal{W} \supseteq \mathcal{X}$. Then $(\mathsf{M}, \mathcal{Y}) \simeq (\mathsf{M}, \mathcal{W})$.

(We may also require the iso to fix a given finite tuple pointwise.)

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Corollaries:

- $\mathrm{WKL}_0^* + \neg\mathrm{RCA}_0$ proves the collapse of the second-order quantifier hierarchy to Δ_1^1 (even a bit more).
- WKL_0^* is the strongest Π_2^1 sentence that is Π_1^1-conservative over $\mathrm{RCA}_0^* + \neg\mathrm{RCA}_0$.

Note: $\neg\mathrm{RCA}_0$ is a false Σ_1^1 statement,
but the Π_1^1 consequences of $\mathrm{RCA}_0^* + \neg\mathrm{RCA}_0$ are a true theory.
In fact, they are contained in the Π_1^1 consequences of ACA_0.

Plan for rest of talk

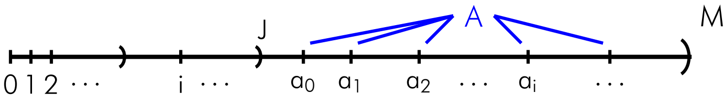
- Comment on how the isomorphism theorem is proved.
- Explain the already mentioned consequences.
- Explain what this has to do with Ramsey's theorem for pairs.

The isomorphism theorem: role of failure of induction

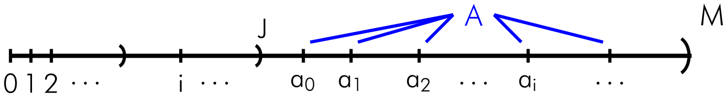
Theorem (Recalled)

Let $(M, \mathcal{X}) \models \mathsf{RCA}_0^* + \neg\mathsf{RCA}_0$, and let $(M, \mathcal{Y}), (M, \mathcal{W}) \models \mathsf{WKL}_0^*$ be countable with $\mathcal{Y}, \mathcal{W} \supseteq \mathcal{X}$. Then $(M, \mathcal{Y}) \simeq (M, \mathcal{W})$.

In the proof, the main reason why $\neg\mathsf{RCA}_0$ matters is as follows. When Σ^0_1-induction fails, ω behaves like a singular cardinal: there is a Σ^0_1-definable proper cut J and an infinite set $A \in \mathcal{X}$ s.t. $A = \{a_i : i \in J\}$ enumerated in increasing order.



The isomorphism theorem: ideas behind proof

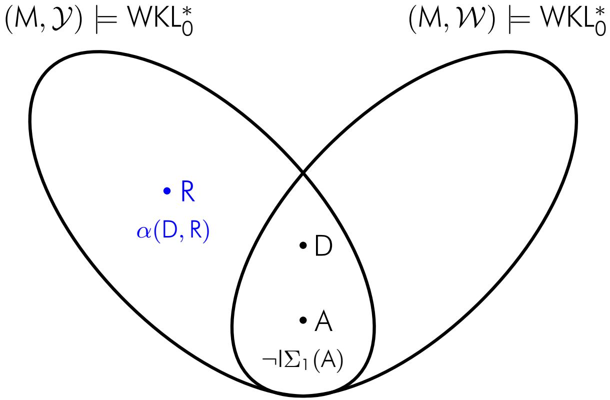


- We use back-and-forth. At each step, we have finite tuples $\bar{r}, \bar{R}$ in the domain, $\bar{s}, \bar{S}$ in the range of the partial iso. The inductive invariant is roughly: for each Δ_0 formula δ, each $i, k \in J$,

$$(M, \mathcal{Y}) \models \delta(a_i, k, \bar{r}, \bar{R}) \text{ iff } (M, \mathcal{W}) \models \delta(a_i, k, \bar{s}, \bar{S}).$$

  To express this properly, one uses a truth definition for Δ_0 formulas.
- In the inductive step, we add say new set R^* to domain and need to find corresponding S^* to add to range. Inductive assumption gives a tree of finite approximations to S^*. WKL gives a path, which can be used as S^*. □

Elimination of second-order quantifiers



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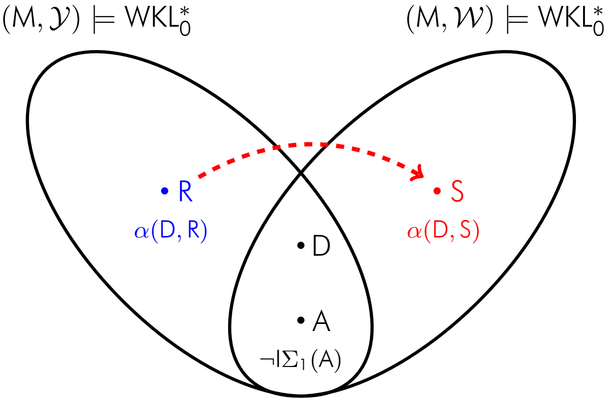

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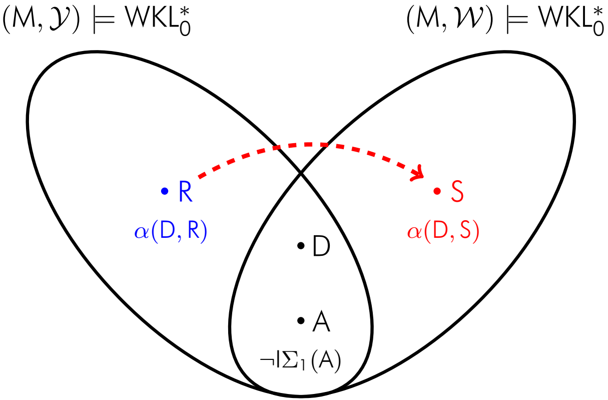

This is (almost) the usual model-theoretic criterion for q.e., if we treat arithmetical formulas as quantifier-free!

Elimination of second-order quantifiers (cont'd)

Corollary

For every formula $\psi(\bar{X}, \bar{x})$ there exists an arithmetical formula $\alpha(\bar{X}, \bar{x}, Y)$ such that

$$\mathsf{WKL}_0^* \vdash \forall A \, [\neg \mathsf{I}\Sigma_1(A) \to (\forall \bar{x} \, \forall \bar{X} \, (\psi(\bar{X}, \bar{x}) \leftrightarrow \alpha(\bar{X}, \bar{x}, A)))].$$

(Using formalized forcing, this can be given a more informative proof than the compactness argument hidden inside the model-theoretic criterion for q.e.)

Since $\mathsf{WKL}_0^* + \neg\mathsf{RCA}_0$ is Π^1_1-conservative over $\mathsf{RCA}_0^* + \neg\mathsf{RCA}_0$ and can eliminate (second-order) quantifiers, it is the model companion of $\mathsf{RCA}_0^* + \neg\mathsf{RCA}_0$, in a setting where we ignore first-order quantification.

Towsner's problem

By general properties of model companions:

$\rightsquigarrow$ Any model of $\mathsf{WKL}_0^* + \neg\mathsf{RCA}_0$ is Σ^1_1-closed: extending the second-order universe will not make new Σ^1_1 formulas true.

$\rightsquigarrow$ $\mathsf{WKL}_0^* + \neg\mathsf{RCA}_0$ is the strongest Π^1_2-axiomatized theory that is Π^1_1-conservative over $\mathsf{RCA}_0^* + \neg\mathsf{RCA}_0$.

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So, we get the following in answer to Towsner's problem:

Corollary

- The set of Π^1_2 sentences that are Π^1_1-conservative over RCA_0^* is Π^0_2-complete. [By a rather boring proof.]
- But, the set of Π^1_2 sentences that are Π^1_1-conservative over $\mathsf{RCA}_0^* + \neg\mathsf{RCA}_0$ is axiomatized by WKL. So, it is c.e.

(Fine print: what Towsner really asked about was conservativity over $\mathsf{I}\Delta^0_n$ for $n \geq 2$. The answers are similar except that we seem to lose finite axiomatizability.)

Reverse mathematics of combinatorial statements

A major problem in reverse mathematics:
describe the Π^1_1 consequences of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$.
(Here RT^2_2 is Ramsey's Thm for 2-colourings of pairs.
There are some other combinatorial statements of apparently similar Π^1_1 strength.)

$\mathsf{RCA}_0 + \mathsf{RT}^2_2$ proves $\mathsf{I}\Delta^0_2$, and it is plausible
that its Π^1_1 consequences coincide with $\mathsf{I}\Delta^0_2$.

It is also known (Cholak-Jockusch-Slaman 2001)
that it suffices to understand RT^2_2 over $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2 + \neg\mathsf{I}\Sigma^0_2$.

But if $(M, \mathcal{X}) \models \mathsf{RCA}_0 + \mathsf{I}\Delta^0_2 + \neg\mathsf{I}\Sigma^0_2$,
then $(M, \Delta^0_2\text{-Def}(M, \mathcal{X})) \models \mathsf{RCA}^*_0 + \neg\mathsf{RCA}_0$!
Is there a simple way to tell when this is also a model of WKL?

The cohesive set principle COH

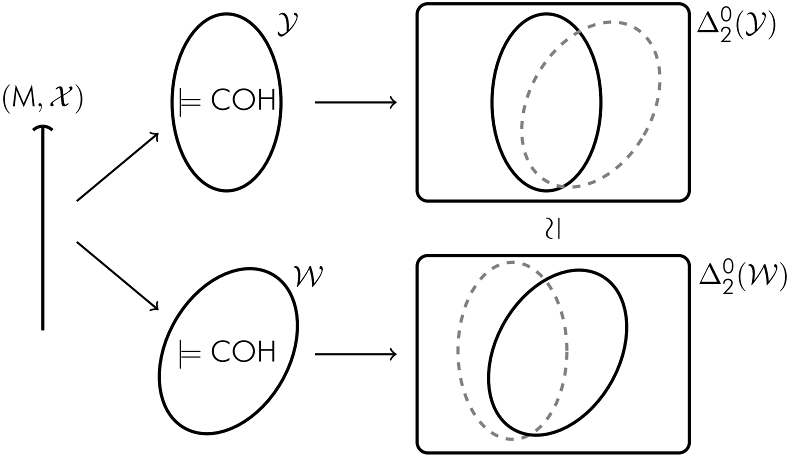
Recall COH := "for every family $\{R_n : n \in \omega\}$ of subsets of ω, there exists infinite $C \subseteq \omega$ s.t. for each n, either $\forall^\infty k \in C\,(k \in R_n)$ or $\forall^\infty k \in C\,(k \notin R_n)$".

- Every countable $(M, \mathcal{X}) \models \mathrm{RCA}_0 + \mathrm{I}\Delta^0_2$ can be extended to $(M, \mathcal{Y}) \models \mathrm{RCA}_0 + \mathrm{I}\Delta^0_2 + \mathrm{COH}$ (Chong-Slaman-Yang 2012).
- For $(M, \mathcal{X}) \models \mathrm{RCA}_0 + \mathrm{I}\Delta^0_2$, we have $(M, \Delta^0_2\text{-Def}(M, \mathcal{X})) \models \mathrm{WKL}$ iff $(M, \mathcal{X}) \models \mathrm{COH}$ (Belanger).

Corollary (of the isomorphism theorem for $\mathrm{WKL}^*_0 + \neg\mathrm{RCA}_0$)

Let $(M, \mathcal{X})$ be a countable model of $\mathrm{RCA}_0 + \mathrm{I}\Delta^0_2 + \neg\mathrm{I}\Sigma^0_2$. If $\mathcal{Y}, \mathcal{W} \supseteq \mathcal{X}$ countable s.t. $(M, \mathcal{Y}), (M, \mathcal{W}) \models \mathrm{RCA}_0 + \mathrm{I}\Delta^0_2 + \mathrm{COH}$, then $(M, \Delta^0_2\text{-Def}(M, \mathcal{Y})) \simeq (M, \Delta^0_2\text{-Def}(M, \mathcal{W}))$.

The isomorphism theorem for COH, pictured



In general, $(M, \mathcal{Y}) \not\equiv (M, \mathcal{W})$.

Π^1_1-conservation over $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2 + \neg\mathsf{I}\Sigma^0_2$

Given a model of $\mathsf{RCA}_0^* + \neg\mathsf{RCA}_0$, we could witness new Σ^1_1 formulas by extending to a model of WKL, and nothing more could be done.

Given a model of $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2 + \neg\mathsf{I}\Sigma^0_2$, we can extend to a model of COH. And then we can do a bit more, by turning Δ^0_2-sets that are "kind-of-low" – their jumps are themselves Δ^0_2 – into sets.

Corollary

A Π^1_2 statement $\psi := \forall X\, \exists Y\, \alpha(X, Y)$ is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2 + \neg\mathsf{I}\Sigma^0_2$ iff $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2$ proves the following statement:

$$\forall A\, [\neg\mathsf{I}\Sigma_2(A) \to \forall X\, \forall W\, (W \text{ solution to appropriate instance of COH} \to \text{there is low-in-}(W \oplus A)\ \Delta^0_2\text{-set } \Upsilon \text{ s.t. } \alpha(X, \Upsilon))].$$

[The statement above roughly says that in any extension of the ground model to a model of COH, we can witness the $\exists Y$ by turning properties "very close to sets" into sets. It is a consequence of $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2 + \psi$.]

The case of RT^2_2

Corollary

$\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2$ iff it is $\forall\Pi^0_5$-conservative over $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2$.

Note:

- $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is $\forall\Pi^0_3$-conservative over $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2$.
  [Patey-Yokoyama 2018]
- We have a slightly better "upper bound" for the $\forall\Pi^0_4$- than for the $\forall\Pi^0_5$-consequences of RT^2_2.
  (By analyzing [Chong-Slaman-Yang 2017].)
- $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ is $\forall\Pi^0_3$- but not $\forall\Pi^0_4$-conservative over RCA^*_0.
  [K-Kowalik-Yokoyama 202X]

Questions for further work

- Are there analogues in other settings?
  Say, elimination of class quantifiers over some weak set theory?
- What are the Π^1_1 consequences of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$?
- Are the Π^1_1 consequences of $\mathsf{RCA}_0 + \mathsf{I}\Delta^0_2 + \neg\mathsf{I}\Sigma^0_2$ finitely axiomatizable?

References

Talk was based on the paper:

Fiori Carones, K, Wong, Yokoyama, An isomorphism theorem for models of Weak König's Lemma without primitive recursion. Accepted to J. Eur. Math. Soc.; arXiv:2112.10876.

Other 21st-century papers referenced:

Belanger, Conservation theorems for the cohesiveness principle, preprint, arXiv: 2212.13011 (earlier version 2015).

Chong, Slaman, Yang, The inductive strength of Ramsey's Theorem for pairs, Adv. Math. 308 (2017), 121–141.

K, Kowalik, Yokoyama, How strong is Ramsey's theorem if infinity can be weak?, J. Symb. Log., to appear. DOI: 10.1017/jsl.2022.46.

Patey, Yokoyama, The proof-theoretic strength of Ramsey's theorem for pairs and two colors, Adv. Math. 330 (2018), 1034–1070.

Towsner, On maximum conservative extensions, Computability 4(1) (2015), 57–68.