Spectra and Definability

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We will consider various extremal sets of reals, like maximal families of eventually different reals, maximal cofinitary groups, maximal independent families and two specific aspects of their study:

1. possible cardinalities;
2. definability properties.

Maximal Eventually Different Families

Definition

A family $\mathscr{E} \subseteq {}^\omega\omega$ is eventually different(abbreviated e.d.) if for any two distinct $f, g \in \mathscr{E}$ there is $n \in \mathbb{N}$ such that

$$\forall m > n(f(m) \neq g(m)).$$

We write $f \neq^* g$. An e.d. family is maximal if it is not properly contained in any other e.d. family.

We denote such maximal families MED, their minimal cardinality $\mathfrak{a}_e$. For $f, g \in {}^\omega\omega$ if it is not the case that f, g are e.d., we write $f =^\infty g$.

Maximal cofinitary groups

Definition

- A group $\mathscr{G} \leq S_\infty$ is cofinitary if its elements are pairwise eventually different.
- A cofinitary group is maximal if it is not properly contained in any other cofinitary group.
- We denote such groups with MCG and their minimal cardinality $\mathfrak{a}_g$.

It is clear that MED and MCG are close relatives to maximal almost disjoint families and so $\mathfrak{a}_g$, $\mathfrak{a}_e$ are close relatives of $\mathfrak{a}$, the minimal cardinality of an infinite maximal almost disjoint subfamily of $[\omega]^\omega$.

Questions

- Is there a model where there is a MCG of size α with $\omega_1 < \alpha < 2^\omega$, but there is no MED of size α?
- Is there a model where $2^\omega > \omega_1$ and every uncountable cofinitary group $\mathscr{G}_0$ of size $< 2^\omega$ is a subgroup of a MCG of the same cardinality as $\mathscr{G}_0$?
- The analogue to the latter question for MAD families is also open.

To what extent are those distinct?

$\mathrm{non}(\mathscr{M})$ and $\mathfrak{a}$ are independent, while $\mathrm{non}(\mathscr{M}) \leq \mathfrak{a}_g, \mathfrak{a}_e$.

Comparing those combinatorial notions with respect to their projective complexity provides other clear distinctions:

- (A. Mathias) There are no analytic MAD families.
- (H. Horowitz, S. Shelah) There are Borel MED and Borel MCG.

Cardinal Characteristics and their Spectrum

- If every set of cardinality α is meager, then there are no MED, no MCG of cardinality α.
- On the other hand, the corresponding extremal sets of reals are an object of their own interest, which brings us to the study of their spectrum, defined as the set of possible cardinalities.

Examples

- $\mathrm{sp}(\mathfrak{a}) = \{|\mathscr{A}| : \mathscr{A} \text{ is MAD}\}$
- $\mathrm{sp}(\mathfrak{a}_e) = \{|\mathscr{A}| : \mathscr{A} \text{ is MED}\}$
- $\mathrm{sp}(\mathfrak{a}_g) = \{|\mathscr{A}| : \mathscr{A} \text{ is MCG}\}$

What cardinalities are allowed?

The existence of extremal sets of reals of certain cardinality and moreover the existence of nicely definable such families, is highly sensitive to the model where we are!

One real at a time

We can adjoin (via forcing) new desired reals one at a time and so recursively generate a MAD, MED, MCG.

- Almost disjoint coding.
- (Y. Zhang) A new generator for a cof. group.

... the technique does not allow us to obtain families of cardinality $\aleph_\omega$.

Can we do better?

- (S. Hechler) We can adjoin a MAD family of arbitrary size with finite conditions, including families of cardinality $\aleph_\omega$, which eventually produced a model of $\mathfrak{a} = \aleph_\omega$ (J. Brendle, 2003).
- (F., A. Törnquist, 2015) We can also adjoin a MCG of arbitrary cardinality with finite conditions, including such max. groups of cardinality $\aleph_\omega$ and eventually obtain the consistency of $\mathfrak{a}_g = \aleph_\omega$.
- (F., S. Shelah, 2021) We can adjoin a MIF of arbitrary cardinality and in particular of size $\aleph_\omega$! However, the poset is different and the consistency of $\mathfrak{i} = \aleph_\omega$ remains open.

MCG

- (Gao, Zhang) In L there is a MCG with a co-analytic generating set.
- (Kastermans) In L then there is a co-analytic MCG.
- (Horowitz, Shelah) There is a Borel MCG.

Question

What can we say about the existence of such nicely definable combinatorial sets of reals in models of large continuum?

Cohen forcing

Theorem (F., Schrittesser, Törnquist)

Assume $V = L$. Then there is a co-analytic MCG which is indestructible by Cohen forcing.

Corollary

The existence of a Π^1_1 MCG of cardinality $\aleph_1$ is consistent with $\mathfrak{c}$ begin arbitrarily large.

Our construction is inspired by the forcing method...

Definition: Coding a real into a group element

Let σ be a partial function from $\mathbb{N}$ to $\mathbb{N}$. Then

1. σ codes a finite string $t \in 2^l$ with parameter $m \in \mathbb{N}$ iff
$$(\forall k < l)\sigma^k(m) = t(k) \mod 2.$$
2. σ exactly codes t with parameter m iff
it codes t and $\sigma^l(m)$ is undefined.
3. σ codes $z \in 2^{\mathbb{N}}$ with parameter m iff
$$(\forall k \in \mathbb{N})\sigma^k(m) = z(k) \mod 2.$$

Outline

The group is recursively defined, in ω_1 steps, adding one permutation at a time, so that each new permutation codes a given real.

Definition: The partial order $\mathbb{Q}^z_{\mathscr{G}}$

Conditions of $\mathbb{Q}$ are triples $p = (s^p, F^p, \bar{m}^p)$ such that:

1. $(s^p, F^p) \in \mathbb{Q}_{\mathscr{G}}$, $\bar{m}^p$ is a partial function from F^p to $\mathbb{N}$
2. For any $w \in \mathrm{dom}(\bar{m}^p)$ there is $l \in \omega$ such that $w[s^p]$ exactly codes $z \restriction l$ with parameter $\bar{m}^p(w)$
3. ...

with extension relation:

1. $(s^q, F^q, \bar{m}^q) \leq (s^p, F^p, \bar{m}^p)$ if and only if $(s^q, F^q) \leq_{\mathbb{Q}} (s^p, F^p)$ and $\bar{m}^q$ extends $\bar{m}^p$ as a function.

The generic group

Theorem

Let $\mathscr{G} \leq S_\infty$, $z \in 2^\mathbb{N}$, let G be $(M, \mathbb{Q}^z_\mathscr{G})$-generic filter and let

$$\sigma_G = \bigcup_{p \in G} s^p \in S_\infty.$$

1. Then $\langle \mathscr{G}, \sigma_G \rangle$ is cofinitary, isomorphic to $\mathscr{G} * \mathbb{F}(X)$.
2. If $\tau \in (S_\infty \setminus \mathscr{G}) \cap M$, then $\langle \mathscr{G}, \sigma_G, \tau \rangle$ is not cofiniatry.
3. Any new permutation in $\langle \mathscr{G}, \sigma_G \rangle$ codes z.

To summarize

1. The existence of a co-analytic MCG of cardinality $\aleph_1$ is consistent with $\mathfrak{a}_g = \mathfrak{b} < \mathfrak{d} = \mathfrak{c}$.
2. The existence of a co-analytic MED of cardinality $\aleph_1$ is consistent with $\mathfrak{a}_e = \mathfrak{b} < \mathfrak{d} = \mathfrak{c}$.

How to obtain a model in which there is a co-analytic MED family of cardinality $\aleph_1$ and $\mathfrak{d} < \mathfrak{c}$?

Theorem (F., Schrittesser)

In the constructible universe L there is a co-analytic MED which remains maximal after countable support iterations or countable support products of Sacks forcing.

To summarize

The existence of a co-analytic MED family of cardinality $\aleph_1$ is consistent with

$$\mathfrak{a}_e = \mathfrak{d} = \aleph_1 < \mathfrak{c}.$$

Definition

A forcing notion $\mathbb{P}$ has the property ned iff for every countable $\mathscr{F}_0 \subseteq {}^\omega\omega$ and every $\mathbb{P}$-name $\dot{f}$ for a function in ${}^\omega\omega$ such that

$$\Vdash_{\mathbb{P}} \dot{f} \text{ is e.d. from } \check{\mathscr{F}}_0,$$

there are $h \in {}^\omega\omega$ which is e.d. from $\mathscr{F}_0$ and $p \in \mathbb{P}$ with

$$p \Vdash_{\mathbb{P}} \check{h} =^\infty \dot{f}.$$

Theorem

Sacks forcing, as well as its countable support products and iterations have property ned.

Theorem

Suppose $\mathscr{E}$ is a Σ^1_2 MED family. Then, there is a Π^1_1 MED family $\mathscr{E}'$ such that for any forcing $\mathbb{P}$, if $\mathscr{E}$ is $\mathbb{P}$-indestructible, then so is $\mathscr{E}'$.

1. (Törnquist) The existence of a Σ^1_2 definable MAD implies the existence of a Π^1_1 MAD.
2. (Brendle, F., Khomskii) The existence of a Σ^1_2 definable MIF implies the existence of a Π^1_1 MIF.
3. (F., Schilhan) The existence of a Σ^1_2 definable tower implies the existence of a Π^1_1 tower.

However the question if the existence of a Σ^1_2 definable MCG implies the existence of a Π^1_1 one is still open.

Tightness

Observations

- If X is a set of functions, then $\bigcup X \subseteq \omega^2$.
- Similarly if $T \subseteq \omega^{<\omega}$ is a tree then $\bigcup T \subseteq \omega^2$.

Definition

Let $X \subseteq {}^{\omega}\omega$, $T \subseteq {}^{<\omega}\omega$ be a tree. We say that X almost covers T if

$$\bigcup T \subseteq^* \bigcup X.$$

The tree ideal generated by $\mathscr{E}$

Definition (F., C. Switzer)

1. The tree ideal generated by $\mathscr{E}$, denotes $\mathscr{I}_{tr}(\mathscr{E})$, is the set of all trees $T \subseteq \omega^{<\omega}$ so that there are

$$t \in T \text{ and a finite } X \subseteq \mathscr{E}$$

   so that

$$\bigcup T_t \subseteq^* \bigcup X.$$

2. A tree $T \subseteq \omega^{<\omega}$ is said to be in $\mathscr{I}_{tr}(\mathscr{E})^+$ if for each $t \in T$ it is not the case that $\bigcup T_t$ can be almost covered by a finite $X \subseteq \mathscr{E}$.

Tight eventually different families

Definition

Let $T \subseteq \omega^{<\omega}$ be a tree, $g \in {}^{\omega}\omega$. We say that g densely diagonalizes T, if for every $t \in T$ there is a branch h through t in T such that $h =^{\infty} g$.

Definition

An eventually different family $\mathscr{E}$ is tight if for any $\{T_n\}_{n\in\omega} \subseteq \mathscr{I}_{tr}(\mathscr{E})^+$ there is a single $g \in \mathscr{E}$ which densely diagonalizes all the T_n's.

Observations

- If $\mathscr{E}$ is a tight eventually different family, then it is maximal.
- MA(σ-linked) implies that every e.d. family $\mathscr{E}_0$, $|\mathscr{E}_0| < \mathfrak{c}$ is contained in a tight e.d. family.
- CH implies that tight eventually different families exist.

Moreover...

tight eventually different families are never analytic, which is a strong distinction with the Borel MED family of Horowitz-Shelah.

... and moreover:

1. tight eventually different families are Cohen indestructible;
2. in L there is a co-analytic tight e.d. family;
3. thus (once again!) $\mathfrak{a}_e$ has a co-analytic witness in a model of $\mathfrak{a}_e = \mathfrak{b} = \aleph_1 < \mathfrak{d} = \mathfrak{c}$.

Strong Preservation of Tightness

Definition: Strong preservation

Let $\mathbb{P}$ be a proper forcing notion and $\mathscr{E}$ a tight e.d. family. We say that $\mathbb{P}$ strongly preserves the tightness of $\mathscr{E}$ if for every sufficiently large θ and $M \prec H_\theta$ such that $p, \mathbb{P}, \mathscr{E}$ are elements of M,

$$\text{if } g \text{ densely diagonalizes every elements of } M \cap \mathscr{I}_T(\mathscr{E})^+,$$

then there is an $(M, \mathbb{P})$-generic $q \leq p$ such that q forces that

$$g \text{ densely diagonalizes every element of } M[\dot{G}] \cap \mathscr{I}_T(\mathscr{E})^+.$$

Such a q is called an $(M, \mathbb{P}, \mathscr{E}, g)$-generic condition.

Theorem

Suppose $\mathscr{E}$ is a tight e.d. family. If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ is a countable support iteration of proper forcing notions such that for all α,

$$\Vdash_\alpha \dot{\mathbb{Q}}_\alpha \text{ strongly preserves the tightness of } \check{\mathscr{E}},$$

then $\mathbb{P}_\gamma$ strongly preserves the tightness of $\mathscr{E}$.

Lemma

- Suppose $\mathbb{P}$ strongly preserves the tightness of $\mathscr{E}$ and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a poset, which strongly preserves the tightness of $\mathscr{E}$. Then $\mathbb{P} * \dot{\mathbb{Q}}$ strongly preserves the tightness of $\mathscr{E}$.
- Moreover, if p is $(M, \mathbb{P}, \mathscr{E}, g)$-generic and forces $\dot{q}$ to be $(M[\dot{G}], \mathbb{P}, \mathscr{E}, g)$-generic then $(p, \dot{q})$ is $(M, \mathbb{P}, \mathscr{E}, g)$-generic.

Lemma

Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ be a countable support iteration of proper forcing notions such that for all α,

$$\Vdash_\alpha \dot{\mathbb{Q}}_\alpha \text{ densely preserves the tightness of } \check{\mathscr{E}},$$

θ sufficiently large and $M \prec H_\theta$ containing $\mathbb{P}_\gamma, \gamma, \mathscr{E}$. For each $\alpha \in M \cap \gamma$ and every $(M, \mathbb{P}_\alpha, \mathscr{E}, g)$-generic condition $p \in \mathbb{P}_\alpha$ the following holds:

If $\dot{q}$ is a $\mathbb{P}_\alpha$-name, $p \Vdash_\alpha \dot{q} \in \mathbb{P}_\gamma \cap M$ and $p \Vdash_\alpha \dot{q} \restriction \alpha \in \dot{G}_\alpha$, then there is an $(M, \mathbb{P}_\gamma, \mathscr{E}, g)$-generic condition $\bar{p} \in \mathbb{P}_\gamma$ so that

$$\bar{p} \restriction \alpha = p \text{ and } \bar{p} \Vdash_\gamma \dot{q} \in \dot{G}.$$

The notion of a tight eventually different family gives a uniform framework which applies to a long list of partial orders, including:

- Sacks,
- Miller rational perfect set forcing,
- Miller partition forcing,
- Infinitely often equal forcing,
- Shelah's poset for diagonalizing a maximal ideal

and gives rise to a MED family indestructible by the above posets.

Theorem (F., Switzer)

The following inequalities are all consistent and in each case there is a tight eventually different family and a tight eventually different set of permutations of cardinality $\aleph_1$, respectively.

1. $\mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p < \mathfrak{d} = \mathfrak{a}_T = 2^{\aleph_0}$
2. $\mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{d} < \mathfrak{a}_T = 2^{\aleph_0}$
3. $\mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{d} = \mathfrak{u} < non(\mathcal{N}) = cof(\mathcal{N}) = 2^{\aleph_0}$.
4. $\mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{i} = cof(\mathcal{N}) < \mathfrak{u}$.

Moreover, if we work over the constructible universe, we can provide co-analytic witnesses of cardinality $\aleph_1$ to each of

$$\mathfrak{a}, \mathfrak{a}_e, \mathfrak{a}_p, \mathfrak{i}, \mathfrak{u}$$

in the above inequalities.

Definition

We refer to a MCG $\mathscr{G}$ of cardinality μ as witnesses to

$$\mu \in \text{sp}(\mathfrak{a}_g) = \{|\mathscr{G}| : \mathscr{G} \text{ is mcg}\}$$

and to values $\mu \in \text{sp}(\mathfrak{a}_g)$ such that

$$\aleph_1 < \mu < \mathfrak{c}$$

as intermediate cardinalities (or values).

Definition: Good projective witnesses

A good projective witness to

$$\mu \in \mathrm{sp}(\mathfrak{a}_g)$$

is a MCG $\mathscr{G}$ of cardinality μ which is also of

lowest projective complexity,

i.e. there are no witnesses to μ whose definitional complexity lies strictly below that of $\mathscr{G}$ in terms of the projective hierarchy.

Question

What can we say about the definability properties of maximal cofinitary groups $\mathscr{G}$ such that

$$\aleph_1 < |\mathscr{G}| < \mathfrak{c}?$$

Observation

Note that a Σ^1_2 MCG must be either of size $\aleph_1$ or continuum (being the union of $\aleph_1$ many Borel sets). Therefore the lowest possible projective complexity of a witness to intermediate values in $\mathrm{sp}(\mathfrak{a}_g)$ is Π^1_2.

Theorem (F., Friedman, Schrittesser, Törnquist)

It is relatively consistent with ZFC that:

- $\mathfrak{c} \geq \aleph_3$ and
- there is a Π^1_2 MCG of size $\aleph_2$.

Thus, it is consistent that there is a Π^1_2 good projective witness to an intermediate value in $\mathrm{sp}(\mathfrak{a}_g)$.

Remark

The same holds for the spectrum of MED and MAD.

Theorem (F., Friedman, Schrittesser, Törnquist)

Let $2 \leq M < N < \aleph_0$ be given. There is a cardinal preserving generic extension of the constructible universe L in which

$$\mathfrak{a}_g = \mathfrak{b} = \mathfrak{d} = \aleph_M < \mathfrak{c} = \aleph_N$$

and there is a Π^1_2 definable maximal cofinitary group fo size $\aleph_M$.

Remark

The analogous result holds for maximal families of eventually different reals, maximal families of eventually different permutations, maximal families of almost disjoint sets.

	$\aleph_1$	μ	$\mathfrak{c}$
	Π^1_1	Π^1_2	Borel
MED	✓	?	✓
MED	?	✓	✓
MCG	✓	?	✓
MCG	?	✓	✓

Independent Families

A family $\mathscr{A} \subseteq [\omega]^\omega$ is said to be independent for any two non-empty finite disjoint subfamilies $\mathscr{A}_0$ and $\mathscr{A}_1$ the set

$$\bigcap \mathscr{A}_0 \setminus \bigcup \mathscr{A}_1$$

is infinite. It is a maximal independent family if it is maximal under inclusion and

$$\mathfrak{i} = \min\{|\mathscr{A}| : \mathscr{A} \text{is a m.i.f.}\}$$

Boolean combinations

For finite $h : \mathscr{A} \to \{0,1\}$, we refer to $\mathscr{A}^h = \bigcap h^{-1}(0) \setminus \bigcup h^{-1}(1)$ as a boolean combination. If $h' \supseteq h$, we say that $\mathscr{A}^{h'}$ strengthen $\mathscr{A}^h$.

... and once again Maximality

Let $\mathscr{A}$ be an independent family.

- Note that, if $\mathscr{A}$ is maximal, then $\forall X \in [\omega]^\omega \backslash \mathscr{A} \exists h \in \mathrm{FF}(\mathscr{A})$ such that X does not split $\mathscr{A}^h$.
- If for each $X \in [\omega]^\omega \backslash \mathscr{A}$ and every $h \in \mathrm{FF}(\mathscr{A})$ there is a strengthening of $\mathscr{A}^h$ such that X does not split $\mathscr{A}^h$, we say that $\mathscr{A}$ is densely maximal.

Remark

The notion of dense maximality appears for the first time in the work of M. Goldstern and S. Shelah on the consistency of $\mathfrak{r} < \mathfrak{u}$.

Density filter

Let $\mathscr{A}$ be an independent family. The family of all $Y \subseteq \omega$ with the property that every $\mathscr{A}^h$ has a strengthening contained in Y is a filter, referred to as the the density filter and denoted $\mathrm{fil}(\mathscr{A})$.

Definition: Selective independence

A densely maximal independent family $\mathscr{A}$ is said to be selective if $\mathrm{fil}(\mathscr{A})$ is Ramsey.

Theorem (Shelah)

- Selective independent families exists under *CH*.
- They are indestructible by a countable support iterations and countable support products of Sacks forcing.

Remark

It is consistent that $\mathfrak{i} < \mathfrak{c}$. In fact the construction can be extracted from Shelah's proof of $\mathfrak{i} < \mathfrak{u}$.

Theorem (A. Miller)

There are no analytic maximal independent families.

Theorem (Brendle, F., Khomskii)

It is relatively consistent that $\mathfrak{i} = \aleph_1 < \mathfrak{c}$ with a co-analytic witness to $\mathfrak{i}$.

Recall that existence of a Σ^1_2 MIF implies the existence of a Π^1_1 MIF.

Optimal spectra?

	$\aleph_1$	μ	$\mathfrak{c}$	
MIF	✓	–	?	$V^{\mathbb{S}_\lambda} \vDash \mathrm{sp}(\mathfrak{i}) = \{\aleph_1, \mathfrak{c}\}$
MIF	–	–	✓	$V^{\mathbb{P}} \vDash \mathfrak{r} = \mathfrak{i} = \mathfrak{c}$

It is still open how to guarantee the existence of

- a good projective witnesses for two distinct cardinals in $\mathrm{sp}(\mathfrak{i})$, or
- a good projective witness for intermediate values.

Theorem (F., Shelah)

The $\mathrm{sp}(\mathfrak{i})$ is closed with respect to singular limits of countable cofinality, but subject to this restriction it can be arbitrarily large. In particular

- For any finite set of uncountable cardinals $\{\kappa_i\}_{i\in n}$, consistently

$$\mathrm{sp}(\mathfrak{i}) = \{\kappa_i\}_{i\in n}.$$

- For any infinite $C \subseteq \{\aleph_n\}_{n\in\omega}$ and $\lambda > \aleph_\omega$ of uncountable cofinality, consistently

$$\mathrm{sp}(\mathfrak{i}) = C \cup \{\aleph_\omega, \lambda\},\ \mathfrak{c} = \lambda$$

Independent family of arbitrary cardinality:

- Let κ be uncountable and S a κ-splitting tree of height ω_1. For each $\alpha < \kappa$ let S_α be the α-th level of S.
- The poset which adjoins a MIF of cardinality κ is of the form $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \omega_1, \beta < \omega_1 \rangle$ where for each $\alpha < \omega_1$, in $V^{\mathbb{P}_\alpha}$

$$\mathbb{Q}_\alpha = \prod_{\eta \in S_\alpha} \mathbb{Q}_{\alpha,\eta}.$$

Even though we know how to adjoin via forcing a MIF of cardinality $\aleph_\omega$, it remains an open question if consistently $\mathfrak{i} = \aleph_\omega$.

Indestructibility

Let $\mathscr{A}$ be a selective independent family. Then $\mathscr{A}$ remains selective after forcing with the countable support iteration of any of:

- (Shelah, 1989) Shelah's poset for diagonalizing a maximal ideal,
- (Cruz-Chapital, F., Guzman, Supina, 2020) Miller partition forcing,
- (J. Bergfalk, F., C. Switzer, 2021) Coding with perfect trees,
- (Switzer, 2022) *h*-perfect trees,
- (F., Switzer, 2023) Miller lite forcing,

leading in particular to the consistency of each of the following

$$\mathfrak{i} < \mathfrak{u}, \mathfrak{u} = \mathfrak{a} = \mathfrak{i} < \mathfrak{a}_T, \mathfrak{i} = \mathfrak{u} < \mathrm{cof}(\mathcal{N}) = \mathrm{non}(\mathcal{N}), \mathfrak{i} = \mathfrak{hm} < \mathfrak{l}_{n,\omega}.$$

Definition

A poset $\mathbb{P}$ is Cohen preserving if every every new dense open subset of $2^{<\omega}$ (or, equivalently $\omega^{<\omega}$) contains an old dense subset.

Remark

More formally, $\mathbb{P}$ is Cohen preserving if for all $p \in \mathbb{P}$ and all $\mathbb{P}$-names $\dot{D}$ so that

$$p \Vdash \text{“}\dot{D} \subseteq 2^{<\omega} \text{ is dense open”}$$

there is a dense $E \subseteq 2^{<\omega}$ in the ground model, $q \leq_{\mathbb{P}} p$ so that

$$q \Vdash \check{E} \subseteq \dot{D}.$$

Theorem (Shelah)

If δ is an ordinal and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \delta, \beta < \delta \rangle$ is a countable support iteration such that for each $\alpha < \delta$

$$\Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha \text{ is proper and Cohen preserving”}$$

then $\mathbb{P}_\delta$ is proper and Cohen preserving.

Lemma

If $\mathbb{P}$ is Cohen preserving and proper, then $\mathbb{P}$ is ${}^\omega\omega$-bounding.

Theorem

Let δ be an ordinal. Let $\mathscr{A}$ be a selective independent family and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle$ be a countable support iteration of proper forcing notions so that for every $\alpha < \delta$,

$$\Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha \text{ is Cohen preserving”}.$$

If for every $\alpha < \delta$,

$$\Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha \text{ preserves the dense maximality of } \mathscr{A}\text{”}$$

then $\mathbb{P}_\delta$ preserves the selectivity of $\mathscr{A}$.

Genericity

Theorem (F., Montoya 2019; F., Switzer 2023)

Let $\mathscr{A}$ be an independent family. Then $\mathscr{A}$ is densely maximal iff $\text{fil}(\mathscr{A})$ is the unique diagonalization filter.

Proof

If $\mathscr{A}$ is densely maximal then $\text{fil}(\mathscr{A})$ is the unique diagonalization filter. If $\text{fil}(\mathscr{A})$ is the unique diagonalization filter, then $\mathscr{A}$ is densely maximal. $\square$

Genericity

Theorem (F., Switzer, 2023)

The generic maximal independent family added by an iteration of Mathias forcing relativized to diagonalization filters is selective.

Ideal Independence

- A family $\mathscr{A} \subseteq [\omega]^\omega$ such that for all finite $\{X_i\}_{i\in n} \subseteq \mathscr{A}$ and $A \in \mathscr{A} \setminus \{X_i\}_{i\in n}$, the set
$$A \setminus \bigcup_{i<n} X_i$$
is infinite, is said to be ideal independent.
- An ideal independent family which is maximal under inclusion is said to be a maximal ideal independent family.
- The least cardinality of an infinite ideal independent family, maximal under inclusion, is denoted $\mathfrak{s}_{mm}$.
- Almost disjoint families and independent families are both examples of ideal independent families.

Earlier investigations (Cancino, Guzman, Miller) of $\mathfrak{s}_{mm}$ shows that

$$\max\{\mathfrak{d}, \mathfrak{r}\} \leq \mathfrak{s}_{mm}$$

and that each of the following inequalities

$$\mathfrak{u} < \mathfrak{s}_{mm}, \quad \mathfrak{s}_{mm} < \mathfrak{i}, \quad \mathfrak{s}_{mm} < \mathfrak{c}$$

is consistent.

Definition

Let $\mathscr{A}$ be an ideal independent family. For any $A \in \mathscr{A}$, the filter $\mathscr{F}(\mathscr{A}, A)$ generated by the family

$$\{A \setminus \bigcup \mathscr{F} : \mathscr{F} \in [\mathscr{A}]^{<\omega} \wedge A \notin \mathscr{F}\}.$$

is referred to as the *complemented filter of* $\mathscr{A}$

Observation

An ideal independent family $\mathscr{A}$ is maximal if and only if every $X \in [\omega]^\omega$ is either in the ideal generated by $\mathscr{A}$ or belongs to at least one of the filters $\mathscr{F}(\mathscr{A}, A)$ (these two possibilities are not mutually exclusive). That is

$$\mathscr{P}(\omega) = \mathscr{I}(\mathscr{A}) \cup \left(\bigcup \{\mathscr{F}(\mathscr{A}, A) : A \in \mathscr{A}\} \right).$$

Theorem (Bardyla, Cancino, F., Switzer)

$\mathfrak{u} \leq \mathfrak{s}_{mm}$.

As a consequence we obtain the independence of $\mathfrak{s}_{mm}$ and $\mathfrak{i}$, as

- the consistency of $\mathfrak{s}_{mm} < \mathfrak{i}$ is shown by Cancino, Guzman, Miller,
- while the consistency of $\mathfrak{i} < \mathfrak{u}$ is follows from the above result and the consistency of $\mathfrak{i} < \mathfrak{u}$.

Therefore $\mathfrak{s}_{mm}$ and $\mathfrak{i}$ are independent.

Observation

- Whenever $\mathscr{A}$ is a maximal ideal independent family such that $|\mathscr{A}| < \mathfrak{c}$, then there are at most finitely many $A \in \mathscr{A}$ for which the corresponding complemented filter is not an ultrafilter.
- This is not necessarily the case for ideal independent families of size $\mathfrak{c}$: Any completely separable maximal almost disjoint family $\mathscr{A}$ is maximal ideal independent and for any $A \in \mathscr{A}$, the corresponding complemented filter $\mathscr{F}(\mathscr{A}, A)$ is the collection of cofinite subsets in A.
- It remains of interest to characterise those ideal independent families $\mathscr{A}$ for which there is $A \in \mathscr{A}$ such that $\mathscr{F}(\mathscr{A}, A)$ is an ultrafilter.

Proposition

If $\mathfrak{p} = \mathfrak{c}$ then there is a maximal ideal independent family $\mathscr{A}$ with the property that for every $A \in \mathscr{A}$ the complemented filter $\mathscr{F}(\mathscr{A}, A)$ is a $p_{\mathfrak{c}}$-point.

Theorem (Bardyla, F., Switzer)

Let $\mathscr{B} = \{B_\alpha : \alpha \in \kappa\}$ be a maximal ideal independent family and for any $\alpha \in \kappa$, $\mathscr{G}_\alpha \subseteq [\omega]^\omega$ be a filter such that

$$\mathscr{F}(\mathscr{B}, B_\alpha) \subseteq \mathscr{G}_\alpha$$

and $\chi(\mathscr{G}_\alpha) \leq \kappa$. Then there exists a maximal ideal independent family $\mathscr{B}' = \{B'_\alpha : \alpha \in \kappa\}$ such that $\mathscr{F}(\mathscr{B}, B'_\alpha) = \mathscr{G}_\alpha$ for any $\alpha \in \kappa$.

Definition

Let $\mathscr{U}$ be an ultrafilter. A maximal ideal independent family $\mathscr{A}$ is called *$\mathscr{U}$-encompassing* if the following conditions hold:

1. $\mathscr{U} \cap \mathscr{A} = \emptyset$, i.e. $\mathscr{A}$ is contained in the dual ideal of $\mathscr{U}$.
2. For every $X \in \mathscr{U}$ the set of $A \in \mathscr{A}$ so that $X \in \mathscr{F}(\mathscr{A}, A)$ is co-countable.

Theorem

Assume CH. For any p-point $\mathscr{U}$ there is a $\mathscr{U}$-encompassing maximal ideal independent family $\mathscr{A}$ such that for all $A \in \mathscr{A}$, the filter $\mathscr{F}(\mathscr{A}, A)$ is a p-point.

Theorem (Bardyla, Cancino, F., Switzer)

Let $\mathscr{U}$ be a p-point and let $\mathbb{P}$ be a proper, ω^ω-bounding forcing notion which preserves p-points. Then $\mathbb{P}$ preserves the maximality of any $\mathscr{U}$-encompassing maximal ideal independent family $\mathscr{A}$ such that for all $A \in \mathscr{A}$, the corresponding complemented filter $\mathscr{F}(\mathscr{A}, A)$ is a p-point.

Note that this theorem implies that under CH, in the generic extension by any proper, ω^ω-bounding, p-point preserving forcing notion $\mathfrak{s}_{mm} = \aleph_1$.

Corollary

1. $\mathfrak{s}_{mm} = \aleph_1$ in the Sacks model.
2. $\mathfrak{s}_{mm} = \aleph_1$ in the Miller partition model and hence $\mathfrak{s}_{mm} < \mathfrak{a}_T$ is consistent.
3. $\mathfrak{s}_{mm} = \aleph_1$ in the *h*-perfect tree forcing model and hence $\mathfrak{s}_{mm} < \text{non}(\mathcal{N})$ is consistent.

An alternation of Miller partition and *h*-perfect tree forcings leads to the consistency of

$$\mathfrak{i} = \mathfrak{s}_{mm} < \mathrm{non}(\mathcal{N}) = \mathfrak{a}_T = \aleph_2.$$

Corollary

$\mathfrak{s}_{mm}$ is independent of $\mathfrak{a}_T$

Proof.

- In the Miller partition model, $\mathfrak{s}_{mm} < \mathfrak{a}_T$.
- On the other hand, $\mathfrak{a}_T < \mathfrak{u}$ holds in the Random model and hence $\mathfrak{a}_T < \mathfrak{s}_{mm}$ holds in that model as well.

□

Question

Is it consistent that for every maximal ideal independent family $\mathscr{I}$ there is an $A \in \mathscr{I}$ so that $\mathscr{F}(\mathscr{I}, A)$ is an ultrafilter?

Remark

A positive answer to this question would imply that there are no completely separable maximal almost disjoint families. It is known that such families exist under either $\mathfrak{s} \leq \mathfrak{a}$ (Mildenberger, Raghavan, Steprans) or $2^{\aleph_0} < \aleph_\omega$ (Hrusak).

The results of the current paper together with those of Cancino, Guzman and Miller, give either a *ZFC* relation, or establish the independence between $\mathfrak{s}_{mm}$ and any other well studied cardinal characteristic, with the exception of $\mathfrak{a}$.

Question

Is it consistent that $\mathfrak{s}_{mm} < \mathfrak{a}$?

The corresponding question for $\mathfrak{i}$, Vaughan's problem, is also still open.

Question

If $\mathfrak{d} = \mathfrak{u} = \aleph_1$ does $\mathfrak{s}_{mm} = \aleph_1$?

Lemma: Eliminating intruders

Let $\mathscr{A}$ be an ideal independent family. There is a *ccc* forcing $\mathbb{P}(\mathscr{A})$ which adds a set z such that in $V^{\mathbb{P}(\mathscr{A})}$:

1. $\mathscr{A} \cup \{z\}$ is an ideal independent family, and
2. for each $y \in V \cap ([\omega]^\omega \setminus \mathscr{A})$ the family $\mathscr{A} \cup \{z, y\}$ is not ideal independent.

Theorem

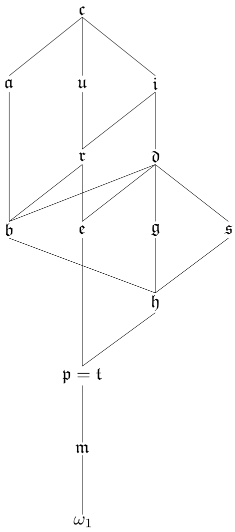
Assume *GCH*. Let C be a set of uncountable cardinals. Then there is a *ccc* generic extension in which

$$C \subseteq \operatorname{spec}(\mathfrak{s}_{mm}) = \{|\mathscr{A}| : \mathscr{A} \text{ is a maximal ideal independent family}\}.$$

Under some restrictions on C, one can obtain a cardinal preserving generic extension in which C is realized as $\operatorname{spec}(\mathfrak{s}_{mm})$.

Questions

- Is it consistent that $\mathfrak{s}_{mm} = \aleph_\omega$?
- What *ZFC* restrictions are there on the set $\mathrm{spec}(\mathfrak{s}_{mm})$? Can it be equal to any set of regular cardinals which includes the continuum?


$\mathfrak{c}$
$\mathfrak{a}$
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ω_1

Thank you for your attention!