Continuous logic and equivalence relations

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May 7, 2023

A.Hallbäck, M.Malicki, T.Tsankov, "Continuous logic and Borel equivalence relations", J. Symb. Logic,
M.Malicki, "Isomorphism of locally compact Polish metric structures", J. Symb. Logic

Structures

A **structure** is a set M equipped with relations R_i, $i \in I$, functions f_j, $j \in J$, and constants c_k, $k \in K$.

Examples:

- graphs (R, E),
- Boolean algebras $(B, \wedge, \vee, -, 0, 1)$,
- metric spaces $(M, \{d_r\}_{r \in R})$, $R \subseteq \mathbb{R}^+$.

The space of countable structures and the logic action

Let L be a relational signature L, with n_i the arity of relational symbol R_i, $i \in I$. Then $\mathrm{Mod}(L) = \prod_{i \in I} \{0,1\}^{\mathbb{N}^{n_i}}$ is the space of codes of all countable L-structures with universe $\mathbb{N}$.

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The group S_∞, acting on $\mathrm{Mod}(L)$ by permuting the universe, induces the isomorphism equivalence relation $\cong$ on $\mathrm{Mod}(L)$. In particular, Vaught transforms can be used:

For open $U \subseteq S_\infty$, and $A \subseteq \mathrm{Mod}(L)$

$$M \in A^{*U} \Leftrightarrow \forall^* g \in U\, g.M \in A.$$

$\mathcal{L}_{\omega_1\omega}$ and its fragments

We will work in the setting of infinitary logic $\mathcal{L}_{\omega_1\omega}$, i.e., an extension of the finitary logic $\mathcal{L}_{\omega\omega}$ allowing for countably infinite conjunctions $\bigwedge_i \phi_i$, and disjunctions $\bigvee_i \phi_i$.

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A (countable) **fragment** F is a countable set of $\mathcal{L}_{\omega_1\omega}$-formulas containing all $\mathcal{L}_{\omega\omega}$-formulas, and closed under $\wedge$, $\vee$, $\neg$, and $\exists$. We can talk about F-theories, F-types, type spaces $S_n(T)$, spaces $\mathrm{Mod}(T) \subseteq \mathrm{Mod}(L)$ of models of a theory T, isomorphism relations $\cong_T$ on $\mathrm{Mod}(T)$, etc.

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The space $S_n(T)$ of all n-F-types is equipped with the logic topology τ_n with basis consisting of sets $[\phi]$, defined by $\mathrm{tp}(\bar{a}) \in [\phi]$ iff $\phi^M(\bar{a}) = 1$, where $\phi \in F$, $M \in \mathrm{Mod}(T)$, $\bar{a}$ is a tuple in M.

In a similar fashion, we can define a topology t_F on $\mathrm{Mod}(L)$.

$\aleph_0$-categorical and atomic structures

Theorem

Let F be a fragment, let T be an F-theory and let $M \in \mathrm{Mod}(T)$.

1. *M is F-$\aleph_0$-categorical iff $[M]$ is $\mathbf{\Pi}^0_1(t_F)$;*
2. *M is F-atomic iff $[M]$ is $\mathbf{\Pi}^0_2(t_F)$.*

Complexity of equivalence relations

An equivalence relation E on a Polish space X is (**Borel**) **reducible** to an equivalence relation F on a Polish space Y if there is a Borel mapping $f : X \to Y$ such that, for any $x_1, x_2 \in X$,

$$x_1 \; E \; x_2 \leftrightarrow f(x_1) \; F \; f(x_2).$$

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Important types of equivalence relations:

- smooth, i.e., reducible to the identity,
- essentially countable, i.e., reducible to a relation with countable classes,
- classifiable by countable structures, i.e., reducible to isomorphism on a Borel class of countable structures (equivalently: graphs).

$\mathcal{L}_{\omega_1\omega}$ and equivalence relations

Theorem (Hjorth-Kechris)

Let T be a countable theory, and let $\cong_T$ be isomorphism on $\mathrm{Mod}(T)$. TFAE:

1. $\cong_T$ *is smooth;*
2. *There exists a fragment F such that for every $M \in \mathrm{Mod}(T)$, the theory $\mathrm{Th}_F(M)$ is $\aleph_0$-categorical.*

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1' $\cong_T$ is essentially countable,

2' there exists a fragment F such that for every $M \in \mathrm{Mod}(T)$, there is a tuple $\bar{a}$ such that $\mathrm{Th}_F(M, \bar{a})$ is $\aleph_0$-categorical.

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Corollary

Isomorphism of finitely generated countable groups is essentially countable.

Metric structures

A **metric structure** is a complete metric space (M, d) with $d \leq 1$, equipped with uniformly continuous functions $R_i : M^{n_i} \to [0, 1]$, $i \in I$ (relations), uniformly continuous functions $f_j : M^{n_j} \to M$, $j \in J$, and constants c_k, $k \in K$.

A metric signature consists of relation (including the metric), function, and constant symbols, as well as arities, and moduli of continuity $\Delta : [0, 1]^n \to [0, 1]$. Each of the relations and functions of a metric structure in a given signature must respect its modulus of continuity.

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Examples:

- Complete metric spaces (M, d) with $d \leq 1$;
- Probability measure algebras $(B, d, \wedge, \vee, 0, 1)$;
- Unit balls of Banach spaces, C^*-algebras, etc.

The space of Polish metric structures

Let L be a countable relational signature L, with n_i the arity of relation R_i, $i \in I$, where $R_0 = d$. Then $\mathrm{Mod}(L) \subseteq \prod_{i \in I} [0,1]^{\mathbb{N}^{n_i}}$ is the space of codes of all Polish metric structures with universe containing $\mathbb{N}$ as a (tail-)dense subset of M.

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Remark: No Vaught transforms. However, for $M \in \mathrm{Mod}(L)$, let $D \subseteq M^{\mathbb{N}}$ be the Polish space of all tail-dense sequences in M, and $\pi : D \to [M]$ a natural projection from D onto the isomorphism class $[M]$ of M. For $A \subseteq \mathrm{Mod}(L)$, $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$, and $u \in \mathbb{Q}^+$, put

$$M \in A^{*\bar{a},u} \Leftrightarrow \forall^* y \in B_{<u}^{D(M)}(\bar{a})(\pi(y) \in A).$$

Continuous $\mathcal{L}_{\omega\omega}$ and $\mathcal{L}_{\omega_1\omega}$

Formulas of continuous finitary logic $\mathcal{L}_{\omega\omega}$ are defined using

- continuous functions $s : [0,1]^n \to [0,1]$ as connectives.
  Alternatively: polynomials or just $\{0, 1, \frac{x}{2}, \dot{+}, \dot{-}\}$,
- inf and sup as quantifiers.

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- inf and sup as quantifiers.

Analogs of infinite conjunctions and disjunctions in the continuous infinitary logic $\mathcal{L}_{\omega_1\omega}$ are defined with $\inf_i \phi_i$, $\sup_i \phi_i$ as infinitary connectives, provided that all ϕ_i respect a single modulus of continuity.

Type spaces

Topologies τ_n and t_n on $S_n(T)$ and $\mathrm{Mod}(L)$ can be defined analogously to the classical setting.

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a (possibly non-separable) complete metric ∂ on $S_n(T)$; for $F = \mathcal{L}_{\omega\omega}$, it can be defined by

$$\partial(p, q) = \inf\{d^M(\bar{a}, \bar{b}) : M \models T,\ \bar{a}, \bar{b} \in M^n,\ \mathrm{tp}(\bar{a}) = p, \mathrm{tp}(\bar{b}) = q\};$$

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in general,

$$\partial(p, q) = \sup_{\phi \in F_1} |p(\phi) - q(\phi)|,$$

where F_1 are 1-Lipschitz formulas.

Atomic models

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Theorem (Cúth, Doležal, Doucha, Kurka)

The isometry classes of the Gurarij space and L^p for $p \geq 1$ are G_δ sets in $(\mathrm{Mod}(T_0), t_{qf})$, where T_0 is the theory of Banach spaces.

Continuous $\mathcal{L}_{\omega_1\omega}$ and equivalence relations

Theorem (Hallbäck, M., Tsankov)

Let T be a countable theory, and let $\cong_T$ be the isomorphism relation on $\mathrm{Mod}(T)$. TFAE:

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Continuous $\mathcal{L}_{\omega_1\omega}$ and equivalence relations

Definition: A type p is $\aleph_0$-**rigid** if whenever $(M, \bar{a})$ and $(N, \bar{b})$ are two realizations of p with M and N separable, then $M \cong N$.

Theorem (Hallbäck, M., Tsankov)

Let T be a theory with locally compact models. TFAE:

1. $\cong_T$ *is essentially countable,*
2. *there exists a fragment F such that for every $M \in \mathrm{Mod}(T)$, there is $k \in \mathbb{N}$ such that the set*

$$\{\bar{a} \in M^k : \mathrm{Th}_F(M, \bar{a}) \text{ is } \aleph_0\text{-rigid}\}$$

has non-empty interior in M^k.

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Corollary (Kechris)

Every orbit equivalence relation induced by a locally compact Polish group is essentially countable.

Coding actions of locally compact Polish metric groups

Assume that

- X is a compact space with compatible metric d bounded by 1;
- $\{a_i\}_{i\in\mathbb{N}}$ is a dense sequence in X;
- $G \leq \mathrm{Homeo}(X)$ is a locally compact group with a proper right-invariant metric d_R.

For $x \in X$, let $A(x)$ be a structure with universe (G, d_R), and unary predicates

$$P_i^x(h) = d(h.x, a_i).$$

Note that because $\{a_i\}$ is dense, P_i^x code x uniquely:

$$x = y \text{ iff } P_i^x(1_G) = P_i^y(1_G) \text{ for all } i.$$

Coding actions of locally compact Polish metric groups

Proposition

The map $x \mapsto A(x)$ is a reduction from the orbit equivalence relation of the evaluation action of G on X to the isomorphism relation.

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Let A be a proper metric structure. Then (A, a) is $\aleph_0$-categorical (in $\mathcal{L}_{\omega\omega}$) for every $a \in A$; in particular, $\mathrm{tp}(a)$ is $\aleph_0$-rigid.

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Theorem (M.)

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Corollary (M.)

Isometry of locally compact Polish metric spaces is reducible to graph isomorphism.

A locally compact Polish metric space (K, d), regarded as an element $\mathcal{K}(\mathbb{U})$ of the hyperspace of the Urysohn space, can be coded in a Borel way as $M_K \in \mathrm{Mod}(L)$ with the trivial signature, and metric bounded by 1: using the Kuratowski–Ryll-Nardzewski theorem, pick a countable tail-dense subset of K, and replace d with $d/(1+d)$.

Countable structures

For a countable M, and $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$, put

$$\mathrm{tp}^0(\bar{a}) = \mathrm{tp}(\bar{a}),\ \mathrm{tp}^\alpha(\bar{a}) = \{\mathrm{tp}^\beta(\bar{b}) : \beta < \alpha,\ \bar{b} \in \mathbb{N}^{<\mathbb{N}},\ \bar{a} \subseteq \bar{b}\},$$

$$\mathrm{Th}^\alpha(M) = \mathrm{tp}^\alpha(\emptyset).$$

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Theorem

Suppose that, for a fragment F, F-theory T, and $M \in \mathrm{Mod}(T)$, we have that $[M] \in \mathbf{\Pi}^0_{1+\alpha}(t_F)$, $\alpha \geq 1$. Then

$$[M] = \{N \in \mathrm{Mod}(T) : \mathrm{Th}^\alpha(N) = \mathrm{Th}^\alpha(M)\}.$$

α-AE families

Let $\beta = 0$ or a limit ordinal.

- An (-1)-AE family $P(\bar{x})$ is a formula $\phi(\bar{x})$ in F.
- a β-AE family $P(\bar{x})$ is a collection of γ-AE families $p_k(\bar{x})$, $k \in \mathbb{N}$, $\gamma < \beta$.
- a $(\beta + 1)$-AE family $P(\bar{x})$ is a collection of γ-AE families $p_{k,l}(\bar{x}_{k,l})$, $\gamma < \beta$, $k, l \in \mathbb{N}$, $\bar{x} \subseteq \bar{x}_{k,l}$,
- a $(\beta + n)$-AE family $P(\bar{x})$, $2 \leq n < \omega$, is a collection of $(\beta + n - 2)$-AE families $p_{k,l}(\bar{x}_{k,l})$, $k, l \in \mathbb{N}$, $\bar{x} \subseteq \bar{x}_{k,l}$.

An α-AE family $P(\bar{x}) = \{p_{k,l}(\bar{x}_{k,l})\}$, $\alpha \geq 1$, comes equipped with a fixed $u_P \geq 0$ such that $u_P \geq u_{p_{k,l}}$, $k, l \in \mathbb{N}$.

α-AE families

For $\beta = 0$ or a limit ordinal, a tuple $\bar{a}$ in $M \in \mathrm{Mod}(L)$ realizes a

- (-1)-AE family $P(\bar{x}) = \phi(\bar{a})$ if $\phi^M(\bar{a}) = 0$,
- β-AE family $P(\bar{x})$ if it realizes every $p(\bar{x}) \in P(\bar{x})$,
- $(\beta + n)$-AE family $P(\bar{x}) = \{p_{k,l}(\bar{x}_{k,l})\}$, $1 \leq n < \omega$, if it holds in M that

$$\forall \bar{b} \in B_{u_P}^{M^{<\omega}}(\bar{a}) \forall v > 0 \forall k \exists \bar{c} \in B_v^{M^{<\omega}}(\bar{b}) \exists l \, (\bar{c} \text{ realizes } p_{k,l}(\bar{x}_{k,l}) \text{ in } M).$$

If $\emptyset$ in M realizes $P(\emptyset)$, we say that M **models** P.

Remark: A countable M models P if

$$\forall \bar{b} \forall k \exists \bar{c} \supseteq \bar{b} \exists l (\bar{c} \text{ realizes } p_{k,l}(\bar{x}_{k,l}) \text{ in } M).$$

AE families and Borel complexity

Let F be fragment in signature L, and let $1 \leq \alpha < \omega_1$.

Theorem (M.)

Suppose that $A \in \mathbf{\Pi}^0_{1+\alpha}(t_F)$ for some $A \subseteq \mathrm{Mod}(L)$. For every $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$, and $u \in \mathbb{Q}^+$, there exists an α-AE family $P(\bar{x})$ such that

$$A^{*\bar{a},u} = \{N \in \mathrm{Mod}(L) : \bar{a} \text{ realizes } P(\bar{x}) \text{ in } N\}.$$

Corollary

Suppose that $[M] \in \mathbf{\Pi}^0_{1+\alpha}(t_F)$ for some $M \in \mathrm{Mod}(L)$, $\alpha \geq 1$. There exists an α-AE family P_M such that

$$[M] = \{N \in \mathrm{Mod}(L) : N \text{ models } P_M\}.$$

Locally compact structures

Fix a countable basis $\mathcal{U}_n$ for each τ_n. For $\bar{a} \in \mathbb{N}^n$, $U \in \mathcal{U}_n$, and $\epsilon \in \mathbb{Q}^+$, define

$$T^0_{U,\epsilon}(\bar{a}) = \overline{B_{<\epsilon}(\mathrm{tp}(\bar{a})) \cap U}^{\tau} \subseteq S_n(T),$$

if (U, ϵ) is $\bar{a}$-good, $T^0_{U,\epsilon}(\bar{a}) = \emptyset$, otherwise, and

$$T^\alpha_{U,\epsilon}(\bar{a}) = \{T^\beta_{U',\epsilon'}(\bar{a}') : \beta < \alpha, |\bar{a}'| \geq |\bar{a}|, U' \in \mathcal{U}_{|\bar{a}'|}, U' \restriction |\bar{a}| \subseteq U, \epsilon' \leq \epsilon\}$$

for $\alpha > 0$. Also, for $u > 0$, put

$$T^\alpha_u(\bar{a}) = \{T^\beta_{U,v}(\bar{b}) : \beta < \alpha, \bar{b} \in B^{M^{<\omega}}_u(\bar{a}), |\bar{b}| \geq |\bar{a}|, U \in \mathcal{U}_{|\bar{b}|}, v > 0\},$$

$$T^\alpha(M) = T^\alpha_1(\emptyset).$$

Locally compact structures

For a theory T, locally compact $M \in \mathrm{Mod}(T)$, $n \in \mathbb{N}$, and n-tuple $\bar{a}$ in M, let

$$\rho(\bar{a}) = \sup\{r \in \mathbb{R} : \overline{B^{M^n}_{<r}(\bar{a})} \text{ is compact}\},$$

$$\Theta_n(M) = \{\mathrm{tp}(\bar{b}) : \bar{b} \in M^n\}.$$

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Put $\mathcal{U} = \bigcup_n \mathcal{U}_n$. For $U \in \mathcal{U}_n$, and $\epsilon > 0$, (U, ϵ) is $\bar{a}$-**good** in M if

- $\mathrm{tp}(\bar{a}) \in U$,
- $2\epsilon < \rho(\bar{a})$,
- there is $\delta > 0$ such that $U \cap B_{<2\epsilon}(\mathrm{tp}(\bar{a})) \subseteq B_{<\epsilon-\delta}(\mathrm{tp}(\bar{a}))$.

Locally compact structures

- For every $\delta > 0$ there exist $U \in \mathcal{U}$ and $0 < \epsilon < \delta$ such that (U, ϵ) is $\bar{a}$-good,
- if (U, ϵ) is $\bar{a}$-good, then

$$\overline{B_{<\epsilon}(\mathrm{tp}(\bar{a})) \cap U}^{\tau} \subseteq \Theta_{|\bar{a}|}(M),$$

- if (U, ϵ) is $\bar{a}$-good, there is $\delta > 0$ such that $d(\bar{a}, \bar{a}') < \delta$ implies that (U, ϵ) is $\bar{a}'$-good, and

$$U \cap B_{<\epsilon}(\mathrm{tp}(\bar{a})) = U \cap B_{<\epsilon}(\mathrm{tp}(\bar{a}')).$$

Locally compact structures

Theorem (M.)

Let F be a fragment, and let T be an F-theory. Suppose that $M, N \in Mod(T)$ are locally compact, and $T^{\alpha}_{u}(\bar{a}) = T^{\alpha}_{u'}(\bar{a}')$ for some tuples $\bar{a}$, $\bar{a}'$ in M, N, respectively. Then every α-AE family $P(\bar{x})$ with $u_P \leq u$ realized by $\bar{a}'$, is also realized by $\bar{a}$.

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Theorem (M.)

Let F be a fragment, and let T be an F-theory with locally compact models. Suppose that $[M] \in \mathbf{\Pi}^0_{1+\alpha}(t_F)$, $\alpha \geq 1$, for some $M \in \mathrm{Mod}(T)$. Then

$$[M] = \{N \in \mathrm{Mod}(T) : T^{\alpha}(N) = T^{\alpha}(M)\}.$$

Coding $T^\alpha(M)$ with countable models

For $M \in \mathrm{Mod}(T)$, C_M consists of elements

$$x = (\overline{B_\epsilon(\mathrm{tp}(\bar{a})) \cap U}^\tau, |\bar{a}|, U, \epsilon),$$

where $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$, $U \in \mathcal{U}_{|\bar{a}|}$, $\epsilon \in \mathbb{Q}^+$, and (U, ϵ) is $\bar{a}$-good, and relations O_l, $R_{k,l,\delta}$, $k, l \in \mathbb{N}$, $\delta \in \mathbb{Q}^+$, and E, defined as follows:

- $O_l(x)$ iff $U_{l,|\bar{a}|} \cap \overline{B_\epsilon(\mathrm{tp}(\bar{a})) \cap U}^\tau = \emptyset$,
- $R_{k,l,\delta}(x)$ iff $k = |\bar{a}|$, $U = U_{l,n}$, $\delta = \epsilon$,
- xEx' iff $|\bar{a}'| \geq |\bar{a}|$, $U' \restriction |\bar{a}| \subseteq U$, $\epsilon' \leq \epsilon$.

Proposition

$C_M \cong C_N$ iff $T^\alpha(M)$=$T^\alpha(N)$ for every $\alpha < \omega_1$.

Borel isomorphism relations

A relation E on a standard Borel space X is **potentially $\mathbf{\Pi}^0_\alpha$** if there is a Polish topology t inducing the Borel structure of X, and such that $E \in \mathbf{\Pi}^0_\alpha(t \times t)$.

For $\alpha < \omega_1$, $\mathcal{P}^0(\mathbb{N}) = \mathbb{N}$, $\mathcal{P}^\alpha(\mathbb{N}) =$ all countable subsets of $\mathcal{P}^{<\alpha}(\mathbb{N}) \cup \mathbb{N}$, where $\mathcal{P}^{<\alpha}(\mathbb{N}) = \bigcup_{\beta<\alpha} \mathcal{P}^\beta(\mathbb{N})$, and $=_\alpha$ is the equality on $\mathcal{P}^\alpha(\mathbb{N})$.

Theorem (Hjort, Kechris, Louveau)

Let F be a fragment in the classical $\mathcal{L}_{\omega_1\omega}$, and let T be an F-theory. If $\cong_T$ is potentially $\mathbf{\Pi}^0_{\alpha+2}$, where $\alpha \geq 1$, then $\cong_T$ is Borel reducible to $=_{\alpha+1}$.

Borel isomorphism relations

A relation E on a standard Borel space X is **potentially** $\mathbf{\Pi}^0_\alpha$ if there is a Polish topology t inducing the Borel structure of X, and such that $E \in \mathbf{\Pi}^0_\alpha(t \times t)$.

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Theorem (M.)

Let F be a fragment in the continuous $\mathcal{L}_{\omega_1\omega}$, and let T be an F-theory with locally compact models. If $\cong_T$ is potentially $\mathbf{\Pi}^0_{\alpha+2}$, where $\alpha \geq 1$, then $\cong_T$ is Borel reducible to $=_{\alpha+1}$.

Borel isomorphism relations

Theorem

Let L be a signature, let t be a Polish topology on $\mathrm{Mod}(L)$ consisting of Borel subsets of the standard topology, and let $\alpha < \omega_1$. There exists a fragment F such that $A^{*\bar{a},1/k} \in \mathbf{\Pi}^0_\alpha(t_F)$ for every $A \in \mathbf{\Pi}^0_\alpha(t)$, $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$, and $k > 0$.

Corollary

Let L be a signature, and let T be a theory such that $\cong_T$ is potentially $\mathbf{\Pi}^0_\alpha$. There exists a fragment F such that $[M] \in \mathbf{\Pi}^0_\alpha(t_F)$ for every $M \in \mathrm{Mod}(T)$.

Borel isomorphism relations

For a fragments F, F', and a formula ϕ,

- $\mathrm{rk}_F(\phi) = 0$ if $\phi \in F$,
- $\mathrm{rk}_F(\phi) = \sup\{\mathrm{rk}_F(\phi_i) + 1\}$ if $\phi = \bigvee_i \phi_i$ or $\phi = \bigwedge_i \phi_i$,
- $\mathrm{rk}_F(\phi) = \mathrm{rk}_F(\psi)$ if ϕ is in the fragment gen. by F and ψ,
- $\mathrm{rk}_F(F') = \sup\{\mathrm{rk}_F(\phi) : \phi \in F'\}$.

Remark: ϕ can be coded as an element of $\mathcal{P}^\alpha(\mathbb{N})$ if $\mathrm{rk}_F(\phi) \leq \alpha$.

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Remark: ϕ can be coded as an element of $\mathcal{P}^\alpha(\mathbb{N})$ if $\mathrm{rk}_F(\phi) \leq \alpha$.

Theorem (M.)

Let F be a fragment, and let T be an F-theory with locally compact models. Suppose that $[M] \in \mathbf{\Pi}^0_{\alpha+2}(t_F)$ for some $M \in \mathrm{Mod}(T)$, $\alpha \geq 1$. There is a fragment $F_M \supseteq F$ such that $[M] \in \mathbf{\Pi}^0_2(t_{F_M})$, and $\mathrm{rk}_F(F_M) = \alpha$.

Borel isomorphism relations

Theorem (Hallbäck, M., Tsankov)

Let F be a fragment and let T be an F-theory. For any $M \in Mod(T)$, $[M]$ is G_δ in the topology t_F iff M is an atomic model of $\mathrm{Th}_F(M)$.

Lemma (Tsankov)

Let L be a signature. For every fragment F, there exists a fragment $F' \supseteq F$ such that if $M \in \mathrm{Mod}(L)$ is F-atomic, then $\mathrm{Th}_{F'}(M)$ is $\aleph_0$-categorical.

Thank You!