Around Mycielski and Eggleston theorems

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Mycielski theorem

- Let $X \subseteq [0,1] \times [0,1]$ be comeager.
  Then there exists a perfect set P such that

$$P \times P \subseteq X \cup \Delta.$$

- Assume that $X \subseteq [0,1] \times [0,1]$ has measure 1.
  Then there exists a perfect set P such that

$$P \times P \subseteq X \cup \Delta.$$

J. Mycielski, Algebraic independence and measure, Fundamenta Mathematicae 61 (1967), 165-169.

Solecki, Spinas

For every coanalytic set $X \subseteq (\omega^\omega)^2$ with complement in the product ideal $\{\emptyset\} \otimes \mathcal{K}_\sigma$ there exists a superperfect set $P \subseteq \omega^\omega$ such that $P \times P \subseteq X \cup \Delta$.

Spinas

Assume that $X_0 \cup X_1 \cup ... \cup X_n = (\omega^\omega)^2$ and X_k is symmetric and Borel for each $k \in \{0, 1, ..., n\}$. Then there exists a superperfect set $P \subseteq \omega^\omega$ such that $P \times P \subseteq X_k \cup \Delta$ for some $k \in \{0, 1, ..., n\}$.

- S. Solecki, O. Spinas, Dominating and unbounded free sets, Journal of Symbolic Logic 64 (1999), 75-80.
- O. Spinas, Ramsey and freeness properties of Polish planes, Proceedings of the London Mathematical Society 82 (2001), 31-63.

Banakh, Zdomskyy

Let $X \subseteq (\omega^\omega)^2$ be comeager. Then there exists nowhere meager set $Q \subseteq \omega^\omega$ such that $Q \times Q \subseteq X \cup \Delta$.

T. Banakh, L. Zdomskyy, Non-meager free sets for meager relations on Polish spaces, Proceedings of the American Mathematical Society 143 (2015), 2719-2724.

A tree $T \subseteq A^{<\omega}$ ($A \in \{2, \omega\}$) is called

- a Miller or superperfect tree, if $(\forall \sigma \in T)(\exists \tau \in \omega\text{-split}(T))(\sigma \subseteq \tau)$;
- a Laver tree, if $(\exists \sigma)(\forall \tau \in T)(\tau \subseteq \sigma \vee (\sigma \subseteq \tau \wedge \tau \in \omega\text{-split}(T)))$.

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- a Laver tree, if $(\exists \sigma)(\forall \tau \in T)(\tau \subseteq \sigma \vee (\sigma \subseteq \tau \wedge \tau \in \omega\text{-split}(T)))$.
- uniformly perfect, if for every $n \in \omega$ either $A^n \cap T \subseteq \text{split}(T)$ or $A^n \cap \text{split}(T) = \emptyset$;
- a Silver tree, if $(\forall \sigma, \tau \in T)(|\sigma| = |\tau| \Rightarrow (\forall a \in A)(\sigma^\frown a \in T \Leftrightarrow \tau^\frown a \in T))$.

Mycielski, category case

Laver tree

There exists a dense G_δ set $G \subseteq \omega^\omega$ such that $[T] \not\subseteq G$ for every Laver tree T.

Proof.

$G = \{x \in \omega^\omega : (\exists^\infty n \in \omega)(x(n) = 0)\}$. □

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Miller tree

(Solecki, Spinas) There exists an open dense set $U \subseteq \omega^\omega \times \omega^\omega$ such that $[T] \times [T] \not\subseteq U \cup \Delta$ for every Miller tree T.

S. Solecki, O. Spinas, Dominating and unbounded free sets, Journal of Symbolic Logic 64 (1999), 75-80.

Miller tree..

There exists a dense G_δ set $G \subseteq \omega^\omega \times \omega^\omega$ such that $[T_1] \times [T_2] \not\subseteq G \cup \Delta$ for any Miller trees T_1, T_2.

Proof.

Let $Q = \{q^n : n \in \omega\}$ and $K(q) = \max\{q_1(n), q_2(n) : n \in \omega\}$

$$G = \bigcap_{n \in \omega} \bigcup_{k > n} [q^k \restriction (\mathrm{supp}(q^k) + K(q^k))],$$

$\square$

Silver tree

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Proof.

Set

$$U = \bigcup_{n \in \omega} \left[\left(q_1^n \upharpoonright \left(\mathrm{supp}(q^n) \right) \right)^\frown (0,0) \right] \times \left[\left(q_2^n \upharpoonright \left(\mathrm{supp}(q^n) \right) \right)^\frown (1,1) \right].$$

□

Mycielski, category case, positive result!

For every comeager set G of $\omega^\omega \times \omega^\omega$ there exists a Miller tree $T_M \subseteq \omega^{<\omega}$ and a uniformly perfect tree $T_P \subseteq T_M$ such that

$$[T_P] \times [T_M] \subseteq G \cup \Delta.$$

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For every comeager set G of $\omega^\omega \times \omega^\omega$ there exists a Miller tree $T_M \subseteq \omega^{<\omega}$ and a uniformly perfect tree $T_P \subseteq T_M$ such that

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but..

There exists a dense G_δ set G such that $[T] \not\subseteq G$ for every uniformly perfect Miller tree T.

Measure case

Miller tree

Let μ be a strictly positive probabilistic measure on ω^ω. Then there exists an F_σ set F of measure 1 such that $[T] \not\subseteq F$ for every Miller tree T.

Small set

$A \subseteq 2^\omega$ is a small set if there is a partition $\mathcal{A}$ of ω into finite sets and a collection $(J_a)_{a \in \mathcal{A}}$ such that $J_a \subseteq 2^a$, $\sum_{a \in \mathcal{A}} \frac{|J_a|}{2^{|a|}} < \infty$ and

$$A = \{x \in 2^\omega : \ (\exists^\infty a \in \mathcal{A})(x \restriction a \in J_a)\}.$$

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Silver tree

There exist a small set $A \subseteq 2^\omega \times 2^\omega$ such that $(A \cap [T] \times [T]) \setminus \Delta \neq \emptyset$ for any Silver tree $T \subseteq 2^{<\omega}$.

Proof.

Let $\{I_n\}_{n \in \omega}$ be a partition of ω, $|I_n| \geq n$.
Clearly, $\{I_n \times I_m\}_{n,m \in \omega}$ forms a partition of $\omega \times \omega$.

$$J_{n,m} = \begin{cases} \emptyset & \text{if } \ n \neq m \\ \{(x,x) : \ x \in 2^{I_n}\} & \text{if } \ n = m \end{cases}$$

$$A = \{(x,y) \in 2^\omega \times 2^\omega : \ (\exists^\infty n \in \omega)(x \restriction I_n = y \restriction I_n)\}$$

□

Silver tree

There exist a small set $A \subseteq 2^\omega \times 2^\omega$ such that $(A \cap [T] \times [T]) \setminus \Delta \neq \emptyset$ for any Silver tree $T \subseteq 2^{<\omega}$.

Silver tree..

Every closed subset of 2^ω of positive Lebesgue measure contains a Silver tree.

Measure case, positive result

Let F be a subset of $2^\omega \times 2^\omega$ of full measure. Then there exists a uniformly perfect tree $T \subseteq 2^{<\omega}$ satisfying $[T] \times [T] \subseteq F \cup \Delta$.

Sketch of the proof.

$F = \bigcup_{n\in\omega} F_n$, where $(F_n : n \in \omega)$ is an ascending sequence of closed sets. Fix $\varepsilon_n = \frac{1}{2^{2n+3}}$, $n \in \omega$. Construct:

- a collection of clopen sets $\{[\tau_\sigma] : \sigma \in 2^{<\omega}\}$;
- two sequences $(k_n : n \in \omega)$ and $(N_n : n \in \omega\setminus\{0\})$;
- a sequence of pairs $((x_n, y_n) : n \in \omega\setminus\{0\})$ from $2^\omega \times 2^\omega$;
- a collection of points $\{t_\sigma : \sigma \in 2^{<\omega}\}$ from 2^ω;
- a sequence $(B_n : n \in \omega)$ of subsets of $2^\omega \times 2^\omega$;

satisfying the following conditions for all $\sigma, \eta \in 2^{<\omega}$ and $n \in \omega$:

1. $\tau_\sigma \subseteq \tau_\eta \Leftrightarrow \sigma \subseteq \eta$;
2. $|\sigma| = |\eta| \to [\tau_\sigma] \times [\tau_\eta] \cap F_{k_{|\sigma\cap\eta|}} \neq \emptyset$;
3. $|\sigma| = |\eta| \Rightarrow |\tau_\sigma| = |\tau_\eta|$;
4. $B_n = \bigcap_{\sigma,\eta\in\{0,1\}^n} \Big(((([\tau_\sigma] \times [\tau_\eta]) \cap F_{k_{|\sigma\cap\eta|}}) - (t_\sigma, t_\eta) \Big)^s$ has a positive measure.

Eggleston theorem

Assume that $A \subseteq [0,1] \times [0,1]$ has measure 1.
Then there exists a perfect set P and a perfect set Q, $\lambda(Q) > 0$ such that

$$P \times Q \subseteq A.$$

H. G. Eggleston, Two measure properties of Cartesian product sets, The Quarterly Journal of Mathematics 5 (1954), 108–115.

Eggleston like

ideal $\mathcal{N}$

- If $G \in \mathrm{Bor}([0,1]^2)$ and $G^c \in \mathcal{N}$ then there are $P \in \mathit{Perf}([0,1])$ and $B \in \mathit{Bor}([0,1])$ such that $B^c \in \mathcal{N}$ and $P \times B \subseteq G$.
- If $G \in \mathrm{Bor}([0,1]^2)$ and $G \notin \mathcal{N}$ then there are $P \in \mathit{Perf}([0,1])$ and $B \in \mathit{Bor}([0,1])$ such that $B \notin \mathcal{N}$ and $P \times B \subseteq G$.

Eggleston like

ideal $\mathcal{N}$

- If $G \in \text{Bor}([0,1]^2)$ and $G^c \in \mathcal{N}$ then there are $P \in \mathit{Perf}([0,1])$ and $B \in \mathit{Bor}([0,1])$ such that $B^c \in \mathcal{N}$ and $P \times B \subseteq G$.
- If $G \in \text{Bor}([0,1]^2)$ and $G \notin \mathcal{N}$ then there are $P \in \mathit{Perf}([0,1])$ and $B \in \mathit{Bor}([0,1])$ such that $B \notin \mathcal{N}$ and $P \times B \subseteq G$.

ideal $\mathcal{M}$

- If $G \in \text{Bor}(\mathbb{R}^2)$ and $G^c \in \mathcal{M}$ then there are $P \in \mathit{Perf}(\mathbb{R})$ and $B \in \mathit{Bor}(\mathbb{R})$ such that $B^c \in \mathcal{M}$ and $P \times B \subseteq G$.
- If $G \in \text{Bor}(\mathbb{R}^2)$ and $G \notin \mathcal{M}$ then there are $P \in \mathit{Perf}(\mathbb{R})$ and $B \in \mathit{Bor}(\mathbb{R})$ such that $B \notin \mathcal{M}$ and $P \times B \subseteq G$.

Sz. Żeberski, Nonstandard proofs of Eggleston like theorems, Proceedings of the Ninth Topological Symposium (2001), 353–357.

Fubini product

$A \in \mathcal{I} \otimes \mathcal{J} \Leftrightarrow \{x : A_x \notin \mathcal{J}\} \in \mathcal{I}$

ideal $\mathcal{N} \cap \mathcal{M}$

For every set $G \in Bor(\mathbb{R}^2) \setminus (ctbl \otimes (\mathcal{N} \cap \mathcal{M}))$ there are $P \in Perf(\mathbb{R})$ and $B \in Bor(\mathbb{R}) \setminus (\mathcal{N} \cap \mathcal{M})$ such that $P \times B \subseteq G$.

Fubini product

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For every set $G \in Bor(\mathbb{R}^2) \setminus (ctbl \otimes (\mathcal{N} \cap \mathcal{M}))$ there are $P \in Perf(\mathbb{R})$ and $B \in Bor(\mathbb{R}) \setminus (\mathcal{N} \cap \mathcal{M})$ such that $P \times B \subseteq G$.

ideal $\mathcal{E}$

If $G \subseteq \mathbb{R}^2$ is a Borel subset such that $G^c \in \mathcal{E} \otimes \mathcal{E}$ then there are $P \in Perf(\mathbb{R})$ and $B \in Bor(\mathbb{R})$ such that $B^c \in \mathcal{E}$ and $P \times B \subseteq G$.

ideal $\mathcal{E}$

If $G \subseteq \mathbb{R}^2$ is a Borel subset such that $G^c \in \mathcal{E} \otimes \mathcal{E}$ then there are $P \in Perf(\mathbb{R})$ and $B \in Bor(\mathbb{R})$ such that $B^c \in \mathcal{E}$ and $P \times B \subseteq G$.

Proof.

We start in V. Extend it to $V' \models \mathrm{add}(\mathcal{E}) \geq \omega_3$
Denote $A = \{x : \mathbb{R} \setminus G_x \in \mathcal{E}\}$
$X \subseteq A$, $|X| = \omega_2$. Let $B \subseteq \bigcap_{x \in X} G_x$.
$X \times B \in G$, $\{x : B \subseteq G_x\}$ is Π^1_1 and has cardinality $\geq \omega_2$
$V' \models \exists P \, \exists B \; P \times B \subseteq G$
By Shoenfield absoluteness theorem it is also true in V.

□

Silver tree

For every dense G_δ-set $G \subseteq (2^\omega \times 2^\omega)$ there are a body of a Silver tree $P \subseteq 2^\omega$ and dense G_δ-set $B \subseteq 2^\omega$ such that $P \times B \subseteq G$.

Silver tree

For every dense G_δ-set $G \subseteq (2^\omega \times 2^\omega)$ there are a body of a Silver tree $P \subseteq 2^\omega$ and dense G_δ-set $B \subseteq 2^\omega$ such that $P \times B \subseteq G$.

Spinas tree

For every dense G_δ-set $G \subseteq (2^\omega \times 2^\omega)$ there are a body of a Spinas tree $P \subseteq 2^\omega$ and dense G_δ-set $B \subseteq 2^\omega$ such that $P \times B \subseteq G$.

$T \subseteq 2^{<\omega}$ is a Spinas tree iff

$$(\forall \sigma \in T)(\exists N)(\forall n \geq N)(\forall i)(\exists \tau \in T)(\sigma \subseteq \tau \wedge \tau(n) = i)$$

Mycielski + Eggleston

Uniformly perfect, category case

Let $G \subseteq 2^\omega \times 2^\omega$ be comeager. Then there exist a uniformly perfect set $P \subseteq 2^\omega$ and a dense G_δ set $D \subseteq 2^\omega$ such that $P \subseteq D$ and $P \times D \subseteq G \cup \Delta$.

Sketch of the proof.

$G = \bigcap_{n\in\omega} U_n$, where $(U_n)_{n\in\omega}$ - descending sequence of open dense sets

(i) $\sigma_\tau{}^\frown i \subseteq \sigma_{\tau^\frown i}$ for $i \in \{0,1\}$;

(ii) $|\tau| = |\tau'| \Rightarrow |\sigma_\tau| = |\sigma_{\tau'}|$;

(iii) $|\tau| = |\tau'| = n \ \wedge \ \tau \neq \tau' \Rightarrow [\sigma_\tau] \times [\sigma_{\tau'}] \subseteq U_n$;

(iv) $V_n \subseteq B_n$;

(v) $|\tau| = n \Rightarrow [\sigma_\tau] \times V_n \subseteq U_n$.

$$P = \bigcap_{n\in\omega} \bigcup_{\tau\in 2^n} [\sigma_\tau]$$

$$D = \bigcap_{n\in\omega} \bigcup_{k\geq n} (V_k \cup \bigcup_{\tau\in 2^k} [\sigma_\tau])$$

□

Thank you for your attention!

M. Michalski, R. Rałowski and Sz. Żeberski, Mycielski among trees, Mathematical Logic Quarterly,67 (2021), 271–281

M. Michalski, R. Rałowski and Sz. Żeberski, Around the Eggleston theorem, still in preparation :)