The definability of the nonstationary ideal

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The non-stationary ideal on ω_1

A set $S \subset \omega_1$ is non-stationary iff there is some club $C \subset \omega_1$ with $S \cap C = \emptyset$. Hence NS_{ω_1} is Σ_1-definable in the parameter ω_1. A paper by Hoffelner-Larson-Sch-Wu shows that in the presence of BPFA (the Bounded Proper Forcing Axiom) NS_{ω_1} may also be Π_1-definable in the parameter ω_1.

Theorem

(Hoffelner-Larson-Sch-Wu) *In the presence of* PFA *(the Proper Forcing Axiom),* NS_{ω_1} *may be* Π_1*-definable in some parameter* $A \subset \omega_1$.

Theorem

(Larson-Sch-Sun) *Martin's Maximum or (*) both imply that* NS_{ω_1} *cannot be* Π_1*-definable in any parameter from* H_{ω_2}.

Proof of the first theorem

Let us say that

$$\vec{A} = (A_i \colon i < \omega_1)$$

splits ω_1 *into stationary sets* iff

(i) each A_i is a stationary subset of ω_1,

(ii) $\omega_1 = \bigcup_{i<\omega_1} A_i$, and

(iii) $A_i \cap A_j = \emptyset$ for $i \neq j$.

Lemma

(Folklore) *Let* $g \colon \omega_1 \to \omega_1$ *be* $\mathrm{Col}(\omega_1, \omega_1)$*-generic over* V*. For* $i < \omega_1$ *write* $A_i = \{\alpha < \omega_1 \colon g(\alpha) = i\}$*. Then in* $V[g]$*,* $(A_i \colon i < \omega_1)$ *splits* ω_1 *into stationary sets.*

Proof of the first theorem, cont'd

Let $\vec{A} = (A_i \colon i < \omega_1)$ split ω_1 into stationary sets. Let $S \subset \omega_1$, $S \neq \emptyset$, and let $\kappa \geq \omega_1$, $\kappa < \omega_2$, be an ordinal. We say that S is *coded at* κ (*modulo* $\vec{A}$) iff there is some

$$(X_i \colon i < \omega_1)$$

such that

(a) $X_i \in [\kappa]^\omega$ for all i,

(b) $X_i \subsetneq X_j$ for $i < j$,

(c) $X_\lambda = \bigcup_{i<\lambda} X_i$ for $\lambda < \omega_1$ a limit ordinal, and

(d) $\kappa = \bigcup_{i<\omega_1} X_i$.

Proof of the first theorem, cont'd

Let $f_\kappa\colon \omega_1 \to \omega_1$ denote "the" canonical function associated with κ, i.e., $f_\kappa(i) = \mathrm{otp} f''i$, where $f\colon \omega_1 \to \kappa$ is bijective. We say that S is *honestly coded at* κ (*modulo* $\vec{A}$) iff for all $\alpha < \omega_1$,

$$\alpha \in S \iff \{i < \omega_1 \colon f_\kappa(i) \in A_\alpha\} \text{ is stationary.}$$

Proof of the first theorem, cont'd

Let $g\colon \omega_1 \to \omega_1$, and let $\vec{A} = (A_i\colon i < \omega_1)$ be *induced by* g, i.e., $A_i = \{\alpha < \omega_1\colon g(\alpha) = i\}$ for $i < \omega_1$. Let us assume $\vec{A} = (A_i\colon i < \omega_1)$ to split ω_1 into stationary sets. Let $S \subset \omega_1$, $S \neq \emptyset$, and let $\kappa \geq \omega_1$ (possibly, $\kappa \geq \omega_2$, in fact we will mostly assume κ to be a measurable cardinal). Let $\mathbb{P}_{S,\kappa}(g)$ denote the following forcing. $p \in \mathbb{P}_{S,\kappa}(g)$ iff there is some countable ordinal θ such that $p = (X_i\colon i \leq \theta)$, where

(α) $X_i \in [\kappa]^\omega$ for all i,

(β) $X_i \subsetneq X_j$ for $i < j$,

(γ) $X_\lambda = \bigcup_{i<\lambda} X_i$ for $\lambda \leq \theta$ a limit ordinal, and

(δ) $\mathrm{otp}(X_i) \in \bigcup_{\alpha \in S} A_\alpha$ for all i.

We write $p \leq q$ iff p end-extends q.

Proof of the first theorem, cont'd

Lemma

Let $g\colon \omega_1 \to \omega_1$, *let* $\vec{A} = (A_i \colon i < \omega_1)$ *be induced by* g. *Let us assume that* $\vec{A} = (A_i \colon i < \omega_1)$ *splits* ω_1 *into stationary sets. Let* $S \subset \omega_1$, $S \neq \emptyset$, *and let* κ *be a measurable cardinal. Write* $\mathbb{P} = \mathbb{P}_{S,\kappa}(g)$. *Let* g *be* $\mathbb{P}$*-generic over* V, *and write* $(X_i \colon i < \omega_1) = \bigcup g$. *Then*
(1) $\mathbb{P}$ *is semi-proper,*
(2) in $V[g]$, S *is coded at* κ, *and*
(3) in $V[g]$, S *is honestly coded at* κ *as being witnessed by* $f_\kappa(i) = \mathrm{otp}(X_i)$.

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Definition

Let $(\mathbb{P}_\eta, \dot{\mathbb{Q}}_\xi : \eta \leq \beta, \xi < \beta)$ be a countable support iteration of forcings. We call this iteration *appropriate* iff the following hold true.

(i) $\mathbb{P}_0 = \mathrm{Col}(\omega_1, \omega_1)$

and if $\xi > 0$, then either

(ii) $\Vdash_{\mathbb{P}_\xi} \dot{\mathbb{Q}}_\xi$ is proper

or else there are $\dot{S}$ and κ such that

(iii) $\Vdash_{\mathbb{P}_\xi} \dot{S} \neq \emptyset$ is a stationary subset of ω_1, κ is a measurable cardinal, and

$$\dot{\mathbb{Q}}_\xi = \mathbb{P}_{\dot{S},\kappa}(\dot{g}_0).$$

Proof of the first theorem, cont'd

Let us now fix a supercompact cardinal δ. By a *Laver function* for δ we mean some $F\colon \delta \to V_\delta$ such that for all $X \in V$ there are $\bar\delta < \bar\theta < \delta$ and $\theta > \delta$ together with an elementary embedding

$$j\colon H_{\bar\theta} \to H_\theta$$

such that

(i) $\mathrm{crit}(j) = \bar\delta$,

(ii) $j(\bar\delta) = \delta$,

(iii) $X \in H_\theta$, and

(iv) $j(f(\bar\delta) = X$.

Proof of the first theorem, cont'd

As every supercompact cardinal has a Laver function, let us fix a Laver function f for δ. Let us then define an appropriate iteration

$$(\mathbb{P}_\eta, \dot{\mathbb{Q}}_\xi : \eta \leq \delta, \xi < \delta)$$

of length $\delta + 1$ as follows.

(i) $\mathbb{P}_0 = \mathsf{Col}(\omega_1, \omega_1)$,

if $\xi > 0$ is a limit ordinal, $\xi < \delta$, then

(ii) $\dot{\mathbb{Q}}_\xi = f(\xi)$, provided that $\Vdash_{\mathbb{P}_\xi} f(\xi)$ is a proper forcing; and $\dot{\mathbb{Q}}_\xi$ is trivial otherwise,

and if ξ is a successor, $\xi < \delta$, and κ is the least measurable cardinal strictly above $\mathsf{Card}(\mathbb{P}_\xi)$, then

(iii) $\dot{\mathbb{Q}}_\xi = \mathbb{P}_{f(\xi-1),\kappa}(\dot{g}_0)$, provided that $f(\xi) = \dot{S}$ for some $\dot{S} \in V^{\mathbb{P}_{\xi-1}}$ such that for some γ, $\Vdash_{\mathbb{P}_{\xi-1}} \dot{S}$ is a stationary subset of ω_1 with $\gamma \in \dot{S}$.

Proof of the first theorem, cont'd

Theorem

If $(\mathbb{P}_\eta, \dot{\mathbb{Q}}_\xi : \eta \leq \delta, \xi < \delta)$ *is the iteration as being defined above, and if* g *is* $\mathbb{P}_\delta$*-generic over* V*, then the following hold true.*

(1) Every $\mathbb{P}_\eta$*,* $\eta \leq \delta$*, is proper.*

(2) In $V[g]$*, if* $S \subset \omega_1$*, then there is some* $\kappa < \omega_2$ *such that* S *is coded at* κ *if and only if* S *is stationary.*

(3) In $V[g]$*,* PFA *holds true and* NS_{ω_1} *is* Π_1*-definable in some parameter* $A \subset \omega_1$*.*

Proof of the second theorem

We give the proof under MM.

Definition

Let $F\colon \mathbb{R} \to \mathbb{R}$ be universally Baire. Let Ω be an uncountable cardinal, and let G be $\mathrm{Col}(\omega, \Omega)$–generic over V. Let $\mathfrak{A} \in V[G]$ be a transitive model of ZFC^- which is countable in $V[G]$. We say that $\mathfrak{A}$ is *closed under* F (or, F*–closed*) iff for all posets $\mathbb{P} \in \mathfrak{A}$ and for all $g \in V[G]$ which are $\mathbb{P}$–generic over $\mathfrak{A}$, $\mathfrak{A}[g]$ is closed under F^G, i.e., $F^G(x) \in \mathfrak{A}[g]$ for all $x \in \mathbb{R} \cap \mathfrak{A}[g]$ in the domain of F^G.

Proof of the second theorem, cont'd

Definition

Let X be a set, and let $\varphi(x)$ be a Σ_1 formula in the language of set theory equipped with a predicate for NS_{ω_1}. We say that $\varphi(X)$ is *honestly consistent* iff for every $F\colon \mathbb{R} \to \mathbb{R}$ which is universally Baire there is an F–closed transitive model $\mathfrak{A}$ such that

(a) $\mathfrak{A} \in V^{\mathrm{Col}(\omega,\Omega)}$ for some large enough Ω,

(b) $(H_{\omega_2})^V \cup \{X\} \subset \mathfrak{A}$,

(c) if $T \subset \omega_1^V$, $T \in V$, $V \models T$ is stationary, then $\mathfrak{A} \models T$ is stationary, and

(d) $\mathfrak{A} \models \mathrm{ZFC}^- + \varphi(X)$.

Proof of the second theorem, cont'd

Now suppose that $\psi(-)$ is Σ_1 in a parameter from H_{ω_2} such that for all $S \subset \omega_1$,

$$S \text{ is stationary} \iff \psi(S).$$

Proof of the second theorem, cont'd

Lemma

The conjunction of the following statements is honestly consistent.
(1) There is some S with $\psi(S)$ such that $T \setminus S$ is stationary for all $T \in V$ such that $V \models T$ is a stationary subset of ω_1.
(2) $\mathrm{cf}(\omega_2^V) = \omega$.

Proof of the second theorem, cont'd

Given this lemma, we may use a variant of my "$(*)$-forcing" with D. Asperó to force the existence of some S with $\psi(S)$ such that $T \setminus S$ is stationary for all $T \in V$ such that $V \models T$ is a stationary subset of ω_1 by a stationary set preserving forcing.

In a second step we may then shoot a club through the complement of S in the usual fashion.

Proof of the second theorem, cont'd

Now look at the statement "There is some non-stationary S with $\psi(S)$." It is true in a stationary set preserving forcing extension of V, hence by MM it is true in V.

Open questions

- Under PFA, can NS_{ω_1} be Π_1 in the parameter ω_1?
- Can NS_{ω_1} be Π_1 in a parameter from H_{ω_2} under PFA plus "NS_{ω_1} is saturated"?
- Under MM, is NS_{ω_1} a complete Σ_1 set? (And in which sense of the word "complete"?)

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