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# Filters of countable type

D.H.Fremlin

*University of Essex, Colchester, England*

Following Mauldin Preiss & Weizsäcker 83, I work through the theory of 'filters of countable type' (1A). Results which may be new are that filters of countable type have the Bolzano-Weierstrass property (3C), the density filter is not of countable type (3D), there are $\mathfrak{c}$ isomorphism classes of filters of countable type (5D), and every filter of countable type has coinitiality at most $\mathfrak{d}$ (6A).

## 1 Basics

**1A Definitions (a)** A filter on a set $X$ is **of countable type** if it belongs to the smallest class of filters on $X$ containing the principal ultrafilters and closed under the operations of countable intersection and increasing countable union.

**(b)** If $X$ is a set, $\langle\mathcal{F}_i\rangle_{i\in I}$ is a non-empty family of filters on $X$, and $\mathcal{F}$ is a filter on $I$, I will write $\lim_{i\to\mathcal{F}}\mathcal{F}_i$ for the filter $\{A : A\subseteq X,\ \{i : i\in I,\ A\in\mathcal{F}_i\}\in\mathcal{F}\}$.

Note that if $X=I=\mathbb{N}$ and $\mathcal{F}_n$ is the principal filter generated by $\{n\}$ for each $n$, then $\lim_{n\to\mathcal{F}}\mathcal{F}_n=\mathcal{F}$.

**(c)** Let $\mathcal{F}$ be a filter on a set $X$. For $Y\subseteq X$, set

$$\mathcal{F}\lceil Y=\{A\cap Y : A\in\mathcal{F}\}=\{B : B\subseteq Y,\ B\cup(X\setminus Y)\in\mathcal{F}\};$$

I will call $\mathcal{F}\lceil Y$ the **trace** of $\mathcal{F}$ on $Y$. If $X\setminus Y\in\mathcal{F}$, then $\mathcal{F}\lceil Y=\mathcal{P}Y$; otherwise, $\mathcal{F}\lceil Y$ is a filter on $Y$.

**(d)** I will write $\mathcal{F}_{\mathrm{Fr}}$ for the Fréchet filter on $\mathbb{N}$, the smallest free filter on $\mathbb{N}$.

**1B Proposition** Let $\mathfrak{F}$ be a family of filters on a set $X$. Then the following are equiveridical:

(i) $\bigcap_{n\in\mathbb{N}}\mathcal{F}_n\in\mathfrak{F}$ for every sequence $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{F}$, and $\bigcup_{n\in\mathbb{N}}\mathcal{F}_n\in\mathfrak{F}$ for every non-decreasing sequence $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{F}$;

(ii) $\lim_{n\to\mathcal{F}_{\mathrm{Fr}}}\mathcal{F}_n\in\mathfrak{F}$ whenever $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{F}$;

(iii) $\lim_{n\to\mathcal{F}}\mathcal{F}_n\in\mathfrak{F}$ whenever $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{F}$ and $\mathcal{F}$ is a filter of countable type on $\mathbb{N}$.

**proof (i)⇒(iii)** Assume (i). Let $\mathfrak{G}$ be the set of filters $\mathcal{G}$ on $\mathbb{N}$ such that $\lim_{n\to\mathcal{G}}\mathcal{F}_n\in\mathfrak{F}$.

**($\alpha$)** If $\mathcal{G}$ is the principal filter generated by $\{m\}$, then $\lim_{n\to\mathcal{G}}\mathcal{F}_n=\mathcal{F}_m\in\mathfrak{F}$; so $\mathcal{G}\in\mathfrak{G}$.

**($\beta$)** If $\langle\mathcal{G}_m\rangle_{m\in\mathbb{N}}$ is a sequence in $\mathfrak{G}$ with intersection $\mathcal{G}$, then

$$\begin{aligned}\lim_{n\to\mathcal{G}}\mathcal{F}_n&=\{A : \{n : A\in\mathcal{F}_n\}\in\mathcal{G}\}\\&=\bigcap_{m\in\mathbb{N}}\{A : \{n : A\in\mathcal{F}_n\}\in\mathcal{G}_m\}=\bigcap_{m\in\mathbb{N}}\lim_{n\to\mathcal{G}_m}\mathcal{F}_n\end{aligned}$$

is the intersection of a sequence in $\mathfrak{F}$ so belongs to $\mathfrak{F}$, and $\mathcal{G}\in\mathfrak{G}$.

**($\gamma$)** If $\langle\mathcal{G}_m\rangle_{m\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{G}$ with union $\mathcal{G}$, set $\mathcal{F}'_m=\lim_{n\to\mathcal{G}_m}\mathcal{G}_n\in\mathfrak{F}$ for each $m$. If $k\le m$ then

$$\mathcal{F}'_k=\{A : \{n : A\in\mathcal{F}_n\}\in\mathcal{G}_k\}\subseteq\{A : \{n : A\in\mathcal{F}_n\}\in\mathcal{G}_m\}=\mathcal{F}'_m,$$

so $\langle\mathcal{F}'_m\rangle_{m\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{F}$ and its union belongs to $\mathfrak{F}$. Now

$$\begin{aligned}\lim_{n\to\mathcal{G}}\mathcal{F}_n&=\{A : \{n : A\in\mathcal{F}_n\}\in\mathcal{G}\}\\&=\bigcup_{m\in\mathbb{N}}\{A : \{n : A\in\mathcal{F}_n\}\in\mathcal{G}_m\}=\bigcup_{m\in\mathbb{N}}\mathcal{F}'_m\in\mathfrak{F},\end{aligned}$$

so $\mathcal{G} \in \mathfrak{G}$.

**($\delta$)** Thus every filter on $\mathbb{N}$ of countable type belongs to $\mathfrak{G}$, as required by (iii).

**(iii)⇒(ii)** All we have to observe is that $\mathcal{F}_{\mathrm{Fr}}$ is a filter of countable type. **P** For each $n \in \mathbb{N}$ let $\mathcal{G}_n$ be the principal ultrafilter generated by $\{n\}$; then $\mathcal{F}_{\mathrm{Fr}} = \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} \mathcal{G}_n$ is of countable type. **Q**

**(ii)⇒(i)** Assume (ii), and let $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{F}$.

**($\alpha$)** Let $\langle k_n \rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathbb{N}$ with cofinal repetitions. Then $\mathfrak{F}$ contains

$$\lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_{k_n} = \{A : \{n : A \in \mathcal{F}_{k_n}\} \in \mathcal{F}_{\mathrm{Fr}}\} = \{A : \exists\, m \in \mathbb{N},\ A \in \mathcal{F}_{k_n} \text{ for every } n \ge m\}$$
$$= \{A : A \in \mathcal{F}_n \text{ for every } n \in \mathbb{N}\} = \bigcap_{n\in\mathbb{N}} \mathcal{F}_n.$$

**($\beta$)** If $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ is non-decreasing, then $\mathfrak{F}$ contains

$$\lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_n = \{A : \{n : A \in \mathcal{F}_n\} \in \mathcal{F}_{\mathrm{Fr}}\}$$
$$= \{A : \exists\, m \in \mathbb{N},\ A \in \mathcal{F}_n \text{ for every } n \ge m\} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n.$$

So both clauses of (i) are true.

**1C Countable-type levels (a)** Let $X$ be any non-empty set. Define $\langle \mathfrak{F}_\xi \rangle_{\xi\in\mathrm{On}}$ as follows. $\mathfrak{F}_0$ is the set of principal ultrafilters on $X$. For ordinals $\xi > 0$, $\mathfrak{F}_\xi$ is the set of filters on $X$ expressible in the form $\lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_n$ where $\mathcal{F}_n \in \bigcup_{\eta<\xi} \mathfrak{F}_\eta$ for every $n \in \mathbb{N}$. Since $\mathfrak{F}_0 \subseteq \mathfrak{F}_1$, $\langle \mathfrak{F}_\xi \rangle_{\xi\in\mathrm{On}}$ is non-decreasing.

**(b)** From 1B we see that $\mathfrak{F}_{\omega_1} = \bigcup_{\xi<\omega_1} \mathfrak{F}_\xi$ is the set of filters on $X$ of countable type. We therefore have a rank function $\mathrm{ct} : \mathfrak{F}_{\omega_1} \to \omega_1$ defined by saying that $\mathrm{ct}\,\mathcal{F} = \min\{\xi : \mathcal{F} \in \mathfrak{F}_\xi\}$ for every filter $\mathcal{F}$ of countable type; I will call $\mathrm{ct}\,\mathcal{F}$ the **countable-type level** of $\mathcal{F}$.

**(c)** Let $X$ be a topological space. For $A \subseteq X$, define $\langle A^\sim_\xi \rangle_{\xi\in\mathrm{On}}$ as follows: $A^\sim_0 = A$; for $\xi > 0$, $A^\sim_\xi$ is the set of limits of sequences in $\bigcup_{\eta<\xi} A^\sim_\eta$. Thus $A^\sim_{\omega_1} = \bigcup_{\xi<\omega_1} A^\sim_\xi$ is the sequential closure of $A$.

**1D Lemma** If $\mathcal{F}$, $\mathcal{G}$ are filters on $X$, $Y$ respectively, $\mathcal{F}$ is of countable type and $\mathcal{G} \le_{\mathrm{RK}} \mathcal{F}$, then $\mathcal{G}$ is of countable type and $\mathrm{ct}\,\mathcal{G} \le \mathrm{ct}\,\mathcal{F}$.

**proof** We can induce on $\mathrm{ct}\,\mathcal{F}$, because if $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ is any sequence of filters on $X$, $\mathcal{F}$ is any filter on $\mathbb{N}$, and $g : X \to Y$ is a function, then

$$g[[\lim_{n\to\mathcal{F}} \mathcal{F}_n]] = \{A : \{n : g^{-1}[A] \in \mathcal{F}_n\} \in \mathcal{F}\}$$
$$= \{A : \{n : [A \in g[[\mathcal{F}_n]]\} \in \mathcal{F}\} = \lim_{n\to\mathcal{F}} g[[\mathcal{F}_n]].$$

**1E Proposition** Let $\langle \mathcal{F}_i \rangle_{i\in I}$ be a non-empty family of filters of countable type on a set $X$, and $\mathcal{F}$ a filter of countable type on $I$. Then $\lim_{i\to\mathcal{F}} \mathcal{F}_i$ is a filter of countable type.

**proof** We can induce on $\mathrm{ct}\,\mathcal{F}$, because if $\langle \mathcal{G}_n \rangle_{n\in\mathbb{N}}$ is any sequence of filters on $I$ and $\mathcal{F} = \lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{G}_n$, then

$$\lim_{i\to\mathcal{F}} \mathcal{F}_i = \{A : \{i : A \in \mathcal{F}_i\} \in \mathcal{F}\}$$
$$= \{A : \{n : \{i : A \in \mathcal{F}_i\} \in \mathcal{G}_n\} \in \mathcal{F}_{\mathrm{Fr}}\}$$
$$= \{A : \{n : A \in \lim_{i\to\mathcal{G}_n} \mathcal{F}_i\} \in \mathcal{F}_{\mathrm{Fr}}\} = \lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \lim_{i\to\mathcal{G}_n} \mathcal{F}_i.$$

**1F Proposition** (a) Suppose that $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ is a sequence of filters on a set $X$, and $Y \subseteq X$ is such that $X \setminus Y$ does not belong to $\mathcal{F} = \lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_n$. Let $\langle n_k \rangle_{k\in\mathbb{N}}$ enumerate the infinite set $D = \{n : X \setminus Y \notin \mathcal{F}_n\}$. Then $\mathcal{F}\lceil Y = \lim_{k\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_{n_k}\lceil Y$.

(b) Let $\mathcal{F}$ be a filter on a set $X$ and $Y$ a subset of $X$ such that $X \setminus Y \notin \mathcal{F}$. If $\mathcal{F}$ is of countable type, so is $\mathcal{F}\lceil Y$, and $\mathrm{ct}(\mathcal{F}\lceil Y) \le \mathrm{ct}\,\mathcal{F}$.

**proof (a)** For $B \subseteq Y$,

$$\begin{aligned} B \in \mathcal{F}\lceil Y &\iff B \cup (X \setminus Y) \in \mathcal{F} \\ &\iff \text{there is an } m \in \mathbb{N} \text{ such that } B \cup (X \setminus Y) \in \mathcal{F}_n \text{ for every } n \ge m \\ &\iff \text{there is an } m \in \mathbb{N} \text{ such that } B \cup (X \setminus Y) \in \mathcal{F}_n \\ &\qquad\qquad \text{whenever } n \in D \text{ and } n \ge m \\ &\iff \text{there is an } m \in \mathbb{N} \text{ such that } B \cup (X \setminus Y) \in \mathcal{F}_{n_k} \text{ for every } k \ge m \\ &\iff \text{there is an } m \in \mathbb{N} \text{ such that } B \in \mathcal{F}_{n_k}\lceil Y \text{ for every } k \ge m \\ &\iff B \in \lim_{k \to \mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_{n_k}\lceil Y. \ \mathbf{Q} \end{aligned}$$

**(b)** The result is now an easy induction on $\mathrm{ct}\,\mathcal{F}$.

**1G Proposition** Let $X$ be a countably compact topological space, $I$ a non-empty set and $\mathcal{F}$ a filter on $I$ of countable type. Then $f[[\mathcal{F}]]$ has a cluster point in $X$ for every $f : I \to X$.

**proof (a)** If $f : I \to X$ is a function, $\langle \mathcal{F}_n \rangle_{n \in \mathbb{N}}$ is a sequence of filters on $I$, and $x_n$ is a cluster point of $f[[\mathcal{F}_n]]$ for every $n \in \mathbb{N}$, then any cluster point $x$ of $\langle x_n \rangle_{n \in \mathbb{N}}$ will also be a cluster point of $f[[\lim_{n \to \mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_n]]$. $\mathbf{P}$ If $G$ is an open set containing $x$ and $A \in f[[\lim_{n \to \mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_n]]$, then there is an $m \in \mathbb{N}$ such that $f^{-1}[A] \in \mathcal{F}_n$ for every $n \ge m$. Now there is an $n \ge m$ such that $x_n \in G$, in which case $f[f^{-1}[A]] \in f[[\mathcal{F}_n]]$ and

$$\emptyset \ne G \cap f[f^{-1}[A]] \subseteq G \cap A.$$

As $G$ and $A$ are arbitrary, $x$ is a cluster point of $f[[\lim_{n \to \mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_n]]$.

**(b)** Inducing on $\mathrm{ct}\,\mathcal{F}$ we get the result.

**1H Theorem** Let $X$ be a topological space and $A$ a subset of $X$.

(a) If $\xi < \omega_1$ and $x \in A^\sim_\xi$, then there are an $f : \mathbb{N} \to A$ and a filter $\mathcal{F}$ on $\mathbb{N}$, of countable-type level at most $\xi$, such that $f[[\mathcal{F}]] \to x$.

(b) If $X$ is sequentially compact, $I$ is a non-empty set, $f : I \to A$ is a function, and $\mathcal{F}$ is a filter on $I$ of countable-type level $\xi$, then

(i) there is a filter $\mathcal{G}$ on $I$, of countable-type level at most $\xi$, such that $\mathcal{G} \supseteq \mathcal{F}$ and $f[[\mathcal{G}]]$ has a limit;

(ii) if $f[[\mathcal{F}]] \to x$ and $X$ is Hausdorff then $x \in A^\sim_\xi$.

**proof (a)** Induce on $\xi$. If $\xi = 0$, then $x \in A$ and we can take $f$ to be the constant function with value $x$ and $\mathcal{F}$ the principal ultrafilter generated by $\{0\}$. For the inductive step to $\xi > 0$, let $\langle x_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\bigcup_{\eta < \xi} A^\sim_\eta$ converging to $x$. For each $n \in \mathbb{N}$ let $f_n : \mathbb{N} \to A$ and $\mathcal{F}_n$ be such that $\mathrm{ct}\,\mathcal{F}_n < \xi$ and $f_n[[\mathcal{F}_n]] \to x_n$; set $g_n(i) = 2^n(2i+1) - 1$ for $i \in \mathbb{N}$; set $\mathcal{G}_n = g_n[[\mathcal{F}_n]]$, so that $\mathrm{ct}\,\mathcal{G}_n < \xi$. Set $\mathcal{F} = \lim_{n \to \mathcal{F}_{\mathrm{Fr}}} \mathcal{G}_n$, so that $\mathrm{ct}\,\mathcal{F} \le \xi$; define $f : \mathbb{N} \to A$ by setting $f(g_n(i)) = f_n(i)$ for $n$, $i \in \mathbb{N}$.

For each $n \in \mathbb{N}$, $f[[\mathcal{G}_n]] = f_n[[\mathcal{F}_n]] \to x_n$. So if $H$ is any open set containing $x$,

$$\{n : f^{-1}[H] \in \mathcal{G}_n\} \supseteq \{n : x_n \in H\} \in \mathcal{F}_{\mathrm{Fr}},$$

and $f^{-1}[H] \in \mathcal{F}$. Accordingly $f[[\mathcal{F}]] \to x$ and the induction proceeds.

**(b)** Again induce on $\xi$. If $\xi = 0$ then $\mathcal{F}$ is the principal ultrafilter on $\mathbb{N}$ generated by $\{i\}$ for some $i$; take $\mathcal{G} = \mathcal{F}$. Then $f[[\mathcal{G}]]$ is the principal ultrafilter on $X$ generated by $f(i)$, and converges to $f(i)$. If the topology on $X$ is $\mathrm{T}_1$ and $f[[\mathcal{F}]] \to x$, then $x = f(i) \in A = A^\sim_0$. So the induction starts.

For the inductive step to $\xi > 0$, express $\mathcal{F}$ as $\lim_{n \to \mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_n$ where $\mathrm{ct}\,\mathcal{F}_n < \xi$ for every $n$. For each $n \in \mathbb{N}$ we have a filter $\mathcal{G}_n \supseteq \mathcal{F}_n$ such that $\mathrm{ct}\,\mathcal{G}_n < \xi$ and $f[[\mathcal{G}_n]]$ has a limit $x_n$ say. Now $\langle x_n \rangle_{n \in \mathbb{N}}$ has a convergent subsequence; suppose that $\langle n_k \rangle_{k \in \mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$ such that $\langle x_{n_k} \rangle_{k \in \mathbb{N}}$ has a limit $z$ in $X$. Set $\mathcal{G} = \lim_{k \to \mathcal{F}_{\mathrm{Fr}}} \mathcal{G}_{n_k}$; then $\mathrm{ct}\,\mathcal{G} \le \xi$. If $J \in \mathcal{F}$, then there is an $m \in \mathbb{N}$ such that $J \in \mathcal{F}_n$ for every $n \ge m$, and now

$$\{k : J \in \mathcal{G}_{n_k}\} \supseteq \{k : J \in \mathcal{F}_{n_k}\} \supseteq \{k : n_k \ge m\} \in \mathcal{F}_{\mathrm{Fr}},$$

so $J \in \mathcal{G}$. Thus $\mathcal{F} \subseteq \mathcal{G}$. If $H$ is an open set including $z$, then

$$\{k : f^{-1}[H] \in \mathcal{G}_{n_k}\} \supseteq \{k : x_{n_k} \in H\} \in \mathcal{F}_{\mathrm{Fr}},$$

so $f^{-1}[H] \in \mathcal{G}$; thus $\mathcal{G} \to z$.

This deals with (i). As for (ii), if $f[[\mathcal{F}]] \to x$ then $f[[\mathcal{G}]] \to x$; as $X$ is Hausdorff, $x = z = \lim_{k\to\infty} x_{n_k}$; also the inductive hypothesis now tells us that

$$x_{n_k} = \lim f[[\mathcal{G}_{n_k}]] \in \bigcup_{\eta<\xi} A^{\sim}_{\eta}$$

for every $k$, so $x \in A^{\sim}_{\xi}$ and the induction proceeds.

## 2 Borel filters and the Fatou property

**2A Proposition** If $I$ is a non-empty set and $\mathcal{F}$ is a filter of countable type on $I$, then $\mathcal{F}$ is a Baire subset of $\mathcal{P}I$, therefore a Borel subset of $\mathcal{P}I$.

**2B Proposition** Let $\mathcal{F}$ be a filter on $\mathbb{N}$ which is a Borel subset of $\mathcal{P}\mathbb{N}$, $X$ a set and $\Sigma$ a $\sigma$-algebra of subsets of $X$.

(a) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of $\Sigma$-measurable functions from $X$ to $[-\infty,\infty]$, then $\liminf_{n\to\mathcal{F}} f_n$ is $\Sigma$-measurable.

(b) If $Y$ is a Polish space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of $\Sigma$-measurable functions from $X$ to $Y$, then $\lim_{n\to\mathcal{F}} f_n$ is $\Sigma$-measurable and has domain in $\Sigma$.

**proof (a)** Set $f = \liminf_{n\to\mathcal{F}} f_n$. For $x \in X$, $\alpha \in \mathbb{R}$ set $J(x,\alpha) = \{n : x(n) \ge \alpha\}$. Then, for given $\alpha$, $x \mapsto J(x,\alpha) : X \to \mathcal{P}\mathbb{N}$ is $\Sigma$-measurable, so $F_\alpha = \{x : J(x,\alpha) \in \mathcal{F}\}$ belongs to $\Sigma$. Now

$$\begin{aligned}\{x : f(x) > \alpha\} &= \{x : \text{ there is an } I \in \mathcal{F} \text{ such that } \inf_{n\in I} f_n(x) > \alpha\}\\ &= \{x : \text{ there are an } I \in \mathcal{F} \text{ and a rational } q > \alpha \text{ such that } I \subseteq J(x,q)\}\\ &= \{x : \text{ there is a rational } q > \alpha \text{ such that } J(x,q) \in \mathcal{F}\}\\ &= \bigcup_{q\in\mathbb{Q},q>\alpha} F_q \in \Sigma.\end{aligned}$$

As $\alpha$ is arbitrary, $f$ is measurable.

**(b)** Let $\rho$ be a complete metric on $Y$ inducing the topology of $Y$. Let $\langle y_i\rangle_{i\in\mathbb{N}}$ run over a dense subset of $Y$. (I am passing over the trivial case $X = Y = \emptyset$. For $i$, $j \in \mathbb{N}$, $F_{ij} = \{x : \limsup_{n\to\mathcal{F}} \rho(f_n(x), y_i) \le 2^{-j}\}$ belongs to $\Sigma$, by (a), inverted. Setting $f = \lim_{n\to\mathcal{F}} f_n$, $\operatorname{dom} f = \bigcap_{j\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} F_{ij} \in \Sigma$. Moreover, for $i$, $j \in \mathbb{N}$, $\{x : \rho(f(x), y_i) \le 2^{-j}\} = F_{ij} \cap \operatorname{dom} f$ belongs to $\Sigma$, so $f$ is measurable.

**2B Definition** I will say that a filter $\mathcal{F}$ on $\mathbb{N}$ has the **Fatou property** if whenever $(X,\Sigma,\mu)$ is a measure space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-negative functions defined on $X$, then $\overline{\int} \liminf_{n\to\mathcal{F}} f_n d\mu \le \liminf_{n\to\mathcal{F}} \overline{\int} f_n d\mu$.

**2C Lemma** If $\mathcal{F}$ is a filter on $\mathbb{N}$, the following are equiveridical:

(i) $\mathcal{F}$ has the Fatou property;

(ii) whenever $(X,\Sigma,\mu)$ is a probability space, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\Sigma$, and $X = \bigcup_{I\in\mathcal{F}} \bigcap_{n\in I} E_n$, then $\lim_{n\to\mathcal{F}} \mu E_n = 1$.

**proof** (i)$\Rightarrow$(ii) is elementary. In the reverse direction, suppose that $\mathcal{F}$ does not have the Fatou property. Let $(X,\Sigma,\mu)$ be a measure space and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of non-negative functions defined on $X$ such that $\overline{\int} \liminf_{n\to\mathcal{F}} f_n d\mu > \liminf_{n\to\mathcal{F}} \overline{\int} f_n d\mu$. Take $\alpha$ such that $\overline{\int} \liminf_{n\to\mathcal{F}} f_n d\mu > \alpha > \liminf_{n\to\mathcal{F}} \overline{\int} f_n d\mu$; set $J = \{n : \overline{\int} f_n d\mu\} \le \alpha$; then $J$ meets every member of $\mathcal{F}$. For $n \in J$, let $g_n : X \to [0,\infty[$ be a $\Sigma$-measurable function such that $f_n \le_{\text{a.e.}} g_n$ and $\int g_n d\mu \le \alpha$. Then $G = \{x : \sup_{n\in J} g_n(x) > 0\}$ is a countable union of sets of finite measure, and there is a negligible set $H \in \Sigma$ such that $f_n(x) \le g_n(x)$ whenever $x \in X \setminus H$ and $n \in J$. Note that this means that $\liminf_{n\to\mathcal{F}} f_n(x) = 0$ whenever $x \notin G \cup H$, so that $\overline{\int}_{G\cup H} \liminf_{n\to\mathcal{F}} f_n d\mu > \alpha$.

Let $\mathcal{G}$ be the filter $\{I \cap J : I \in \mathcal{F}\}$ on $J$. Then

$$\liminf_{n\to\mathcal{G}} f_n(x) = \sup_{I\in\mathcal{F}} \inf_{n\in I\cap J} f_n(x) \ge \liminf_{n\to\mathcal{F}} f(x)$$

for every $x$, so

$$\overline{\int}_{G\cup H} \liminf_{n\to\mathcal{G}} g_n d\mu \ge \overline{\int}_{G\cup H} \liminf_{n\to\mathcal{G}} f_n d\mu \ge \overline{\int}_{G\cup H} \liminf_{n\to\mathcal{F}} f_n d\mu > \alpha.$$

Let $\lambda$ be the product measure on $(G \cup H) \times \mathbb{R}$, and consider the ordinate sets $W_n = \{(x, \alpha) : x \in G \cup H, 0 \le \alpha < g_n(x)\}$ for $n \in J$. Set

$$W = \bigcup_{I\in\mathcal{G}} \bigcap_{n\in I} W_n = \{(x, \alpha) : x \in G \cup H, \alpha \ge 0, \{n : n \in J, \alpha < g_n(x)\} \in \mathcal{G}\};$$

setting $g = \liminf_{n\to\mathcal{G}} g_n$,

$$\{(x, \alpha) : x \in G \cup H, 0 \le \alpha < g(x)\} \subseteq W \subseteq \{(x, \alpha) : x \in G \cup H, 0 \le \alpha \le g(x)\}.$$

So

$$\lambda^* W = \overline{\int}_{G\cup H} \liminf_{n\to\mathcal{G}} g_n d\mu > \alpha$$

(Fremlin 01, 252Yh).

There is therefore a set $V \subseteq (G \cup H) \times \mathbb{R}$ such that $\lambda V < \infty$ and $\lambda^*(V \cap W) > \alpha$. Let $\nu$ be the subspace measure on $V \cap W$. Set

$$\begin{aligned} V_n &= V \cap W \cap W_n \text{ if } n \in J \\ &= V \cap W \text{ if } n \in \mathbb{N} \setminus J. \end{aligned}$$

Then

$$\begin{aligned} \liminf_{n\to\mathcal{F}} \nu V_n &= \sup_{I\in\mathcal{F}} \inf_{n\in I} \nu V_n \le \sup_{n\in J} \nu V_n \\ &\le \sup_{n\in J} \lambda W_n = \sup_{n\in J} \int g_n d\mu \le \alpha. \end{aligned}$$

On the other hand,

$$\bigcup_{I\in\mathcal{F}} \bigcap_{n\in I} V_n = \bigcup_{I\in\mathcal{F}} \bigcap_{n\in I\cap J} V \cap W \cap W_n = V \cap W \cap \bigcup_{I\in\mathcal{G}} \bigcap_{n\in I} W_n = V \cap W$$

and $\nu(V \cap W) = \lambda^*(V \cap W) > \alpha$. Moving to a normalization of $\nu$, we see that (ii) is false.

**2D Proposition** Let $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ be a sequence of filters on $\mathbb{N}$ with the Fatou property, and $\mathcal{F}$ a filter with the Fatou property. Then $\mathcal{G} = \lim_{n\to\mathcal{F}} \mathcal{F}_n$ has the Fatou property.

**proof (a)** The point is that if $\langle t_n \rangle_{n\in\mathbb{N}}$ is any sequence in $[0, \infty]$ then

$$\liminf_{n\to\mathcal{G}} t_n = \liminf_{n\to\mathcal{F}} \liminf_{i\to\mathcal{F}_n} t_i.$$

**P** For any $\alpha \in [0, \infty[$,

$$\begin{aligned} \liminf_{n\to\mathcal{G}} t_n > \alpha &\iff \exists\, \beta > \alpha,\ \{n : t_n \ge \beta\} \in \mathcal{G} \\ &\iff \exists\, \beta > \alpha,\ A \in \mathcal{F},\ \{i : t_i \ge \beta\} \in \mathcal{F}_n\ \forall\, n \in A \\ &\iff \exists\, \beta > \alpha,\ A \in \mathcal{F},\ \liminf_{i\to\mathcal{F}_n} t_i \ge \beta\ \forall\, n \in A \\ &\iff \liminf_{n\to\mathcal{F}} \liminf_{i\to\mathcal{F}_n} t_i > \alpha. \quad \mathbf{Q} \end{aligned}$$

**(b)** So if we have a measure space $(X, \Sigma, \mu)$ and a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of non-negative functions defined on $X$,

$$\begin{aligned} \overline{\int} \liminf_{n\to\mathcal{G}} f_n &= \overline{\int} \liminf_{n\to\mathcal{F}} \liminf_{i\to\mathcal{F}_n} f_i \le \liminf_{n\to\mathcal{F}} \overline{\int} \liminf_{i\to\mathcal{F}_n} f_i \\ &\le \liminf_{n\to\mathcal{F}} \liminf_{i\to\mathcal{F}_n} \overline{\int} f_i = \liminf_{n\to\mathcal{G}} \overline{\int} f_n. \end{aligned}$$

**2E Corollary** A filter on $\mathbb{N}$ of countable type has the Fatou property.

**proof** Principal ultrafilters on $\mathbb{N}$, and the Fréchet filter, have the Fatou property; use 1B(ii).

**2E Example** There is a filter $\mathcal{F}$ on $\mathbb{N}$, an $\mathrm{F}_\sigma$ subset of $\mathcal{P}\mathbb{N}$, which does not have the Fatou property.

**construction** Let $\mu$ be the usual measure on $X = \{0,1\}^{\mathbb{N}}$. Let $\langle E_n\rangle_{n\in\mathbb{N}}$ enumerate the family of sets of the form $\{x : x \in X,\ x{\restriction}k \in J\}$ where $k \in \mathbb{N}$ and $J \subseteq \{0,1\}^k$ has $k+1$ members. Then $\lim_{n\to\infty} \mu E_n = 0$. For $x \in X$ set $\phi(x) = \{n : x \in E_n\}$; then $\phi : X \to \mathcal{P}\mathbb{N}$ is continuous so $K = \phi[X]$ is compact. Set

$$\mathcal{F} = \bigcup_{k\in\mathbb{N}} \{a \cup (b_0 \cap \ldots \cap b_k \setminus k) : a \subseteq \mathbb{N},\ b_0, \ldots , b_k \in K\};$$

then $\mathcal{F}$ is an $\mathrm{F}_\sigma$ set. If $x_0, \ldots , x_k \in X$, there is an $n \ge k$ such that $x_0, \ldots , x_k$ all belong to $E_n$; accordingly $\emptyset \notin \mathcal{F}$ and $\mathcal{F}$ is a filter on $\mathbb{N}$.

Since $\mathcal{F}$ contains all cofinite subsets of $\mathbb{N}$,

$$\lim_{n\to\mathcal{F}} \mu E_n = \lim_{n\to\infty} \mu E_n = 0.$$

If $x \in X$, then $\phi(x) \in \mathcal{F}$ and

$$x \in \bigcap_{n\in\phi(x)} E_n \subseteq \bigcup_{I\in\mathcal{F}} \bigcap_{n\in I} E_n.$$

So 2C(ii) is false and $\mathcal{F}$ does not have the Fatou property.

## 3 The Bolzano-Weierstrass property

**3A Definition** (Filipów Mrožek Recław & Szuca 07) A filter $\mathcal{F}$ on a set $X$ has the **Bolzano-Weierstrass property** if for every $g : X \to [0, 1]$ there is an $I \subseteq X$, meeting every member of $\mathcal{F}$, such that $(g{\restriction}I)[[\mathcal{F}{\restriction}I]]$ is convergent.

**3B Basic facts (a)** Any filter which is not free has the Bolzano-Weierstrass property. $\mathcal{F}_{\mathrm{Fr}}$ has the Bolzano-Weierstrass property. If $\mathcal{F}$ is a filter on $X$ with the Bolzano-Weierstrass property, $Z$ is a compact metrizable space and $g : X \to Z$ is a function, there is an $I \subseteq X$, meeting every member of $\mathcal{F}$, such that $(g{\restriction}I)[[\mathcal{F}{\restriction}I]]$ is convergent (Filipów Mrožek Recław & Szuca 07, §2.3).

**(b)** (Fridy 93, or Filipów Mrožek Recław & Szuca 07, §3) Let $\mathcal{F}_d$ be the filter of subsets of $\mathbb{N}$ with asymptotic density 1. Then $\mathcal{F}_d$ does not have the Bolzano-Weierstrass property. **P** Let $g = \langle g(n)\rangle_{n\in\mathbb{N}}$ be a sequence in $[0, 1]$ which is equidistributed for Lebesgue measure $\mu$. Then $d^*(I) \le \mu\overline{g[I]}$ for every $I \subseteq \mathbb{N}$. If $I \subseteq \mathbb{N}$ meets every member of $\mathcal{F}_d$, then $d^*(I) > 0$. Let $m \ge 1$ be such that $\frac{2}{m} < d^*(I)$, and for $k \le m$ set

$$J_k = \{n : n \in I,\ \tfrac{k-1}{m} \le g(n) \le \tfrac{k+1}{m}\}.$$

Then

$$d^*(I \cap J_k) \le \mu\overline{g[I \cap J_k]} \le \frac{2}{m} < d^*(I)$$

so $I \cap J_k \notin \mathcal{F}_d{\restriction}I$ for every $k$. But this means that $(g{\restriction}I)[[\mathcal{F}_d{\restriction}I]]$ cannot contain any interval of the form $[0, 1] \cap [\frac{k-1}{m}, \frac{k+1}{m}]$ and cannot be convergent. **Q**

**(c)** For $I \subseteq \mathbb{N}$ write $d^*_s(I)$ for its **Banach upper density**, that is,

$$d^*_s(I) = \inf_{m\ge1} \sup_{k\in\mathbb{N}} \frac{1}{m}\#(I \cap [k, k + m[) = \lim_{m\to\infty} \sup_{k\in\mathbb{N}} \frac{1}{m}\#(I \cap [k, k + m[)$$

(Fremlin n05). Let $\mathcal{F}_s$ be the **Banach density filter**, that is, the filter of sets $I \subseteq \mathbb{N}$ such that $d^*_s(\mathbb{N}\setminus N) = 0$. Then $\mathcal{F}_s$ does not have the Bolzano-Weierstrass property. **P** Argue as in (b), but with a well-distributed sequence. **Q**

**3C Theorem** Let $\mathcal{G}$ be a filter on a set $X$, and suppose that there is a filter $\mathcal{F}$ on $X$, of countable type, such that $\mathcal{G} \subseteq \mathcal{F}$. Then $\mathcal{G}$ has the Bolzano-Weierstrass property.

**proof** Induce on the countable-type level $\operatorname{ct}\mathcal{F}$ of $\mathcal{F}$.

**(a)** If $\mathcal{F}$ is the principal filter generated by $\{x\}$, then $\mathcal{G}$ is not free, so has the Bolzano-Weierstrass property.

**(b)** If $\operatorname{ct}\mathcal{F} = \xi > 0$, there is a sequence $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ of filters on $X$, all of countable-type level less than $\xi$, such that $\mathcal{F} = \lim_{n\to\mathcal{F}_{\mathrm{Fr}}} \mathcal{F}_n$. Let $g : X \to [0,1]$ be any function.

**(i)** Suppose that there are a set $I \subseteq X$ and an $n \in \mathbb{N}$ such that $I$ meets every member of $\mathcal{F}_n$ and $\mathcal{G}\lceil I \subseteq \mathcal{F}_n\lceil I$. In this case, $\mathcal{G}\lceil I \ne \mathcal{P}I$, so $\mathcal{G}\lceil I$ is a filter, while $\operatorname{ct}(\mathcal{F}_n\lceil I) \le \operatorname{ct}\mathcal{F}_n < \xi$ (1Fb). So $\mathcal{G}\lceil I$ has the Bolzano-Weierstrass property, by the inductive hypothesis. Applying this to the function $g\restriction I$, we see that there is a $J \subseteq I$, meeting every member of $\mathcal{G}\lceil I$, such that $(g\restriction J)[[(\mathcal{G}\lceil I)\lceil J]]$ converges; but in this case $J$ meets every member of $\mathcal{G}$, $\mathcal{G}\lceil J = (\mathcal{G}\lceil I)\lceil J$ and $(g\restriction J)[[\mathcal{G}\lceil J]]$ converges.

**(ii)** Otherwise, choose $\langle I_k\rangle_{k\in\mathbb{N}}$, $\langle J_k\rangle_{k\in\mathbb{N}}$ and $\langle n_k\rangle_{k\in\mathbb{N}}$ inductively, as follows. $I_0 = \mathbb{N}$. Given that $I_k$ meets every set in $\mathcal{F}$, there must be an $n_k$, greater than $n_i$ for any $i < k$, such that $I_k$ meets every member of $\mathcal{F}_{n_k}$. In this case, since (i) is false, there is a $J_k \in (\mathcal{G}\lceil I_k) \setminus (\mathcal{F}_{n_k}\lceil I_k)$. As $J_k \cup (X \setminus I_k) \in \mathcal{G} \subseteq \mathcal{F}$, $J_k = I_k \cap (J_k \cup (X \setminus I_k))$ meets every member of $\mathcal{F}$; let $I_{k+1} \subseteq J_k$ be a set meeting every member of $\mathcal{F}$ such that $\operatorname{diam} g[I_{k+1}] \le 2^{-k-1}$. Continue.

At the end of the induction, set $I = \bigcup_{k\in\mathbb{N}} I_k \setminus J_k$. As $I_{k+1} \subseteq J_k \subseteq I_k$ for every $k$, $I \cap I_{k+1} = (I \cap I_k)\setminus(I_k \setminus J_k)$ for every $k$. Now, for each $k$, $J_k \notin \mathcal{F}_{n_k}\lceil I_k$, so $I_k \setminus J_k$ meets every member of $\mathcal{F}_{n_k}\lceil I_k$ and $I$ meets every member of $\mathcal{F}_{n_k}$. Consequently $I$ meets every member of $\mathcal{F} \supseteq \mathcal{G}$. On the other hand, $\mathbb{N} \setminus (I_k \setminus J_k) \in \mathcal{G}$ for every $k$, so (inducing on $k$) $I \cap I_k \in \mathcal{G}\lceil I$ for every $k$. Since $\operatorname{diam} g[I_k] \le 2^{-k}$ for every $k$, $(g\lceil I)[[\mathcal{G}\lceil I]]$ is Cauchy, therefore convergent.

**(iii)** Thus in either case the condition of Definition 3A is satisfied. As $g$ is arbitrary, $\mathcal{G}$ has the Bolzano-Weierstrass property, and the induction continues.

**3D Corollary** Neither the density filter nor the Banach density filter is of countable type.

**3E Proposition** $\mathcal{F}_d$ has the Fatou property.

**proof (a)** Let $(X, \Sigma, \mu)$ be a probability space and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence in $\Sigma$ such that $\bigcup_{I\in\mathcal{F}_d} \bigcap_{n\in I} E_n = X$; set $\alpha = \liminf_{n\to\mathcal{F}_d} \mu E_n$. Take any $\epsilon > 0$. Set $I = \{n : \mu E_n \le \alpha + \epsilon\}$; then $d^*(I) > 0$. Take any $\eta \in \left]0, \frac{1}{2}d^*(I)\right]$. Set $M = \{m : m > 0, \#(I \cap m) \ge m(d^*(I) - \eta)\}$; then $M \subseteq \mathbb{N}$ is infinite. For $k \in \mathbb{N}$ set

$$H_k = \bigcap_{m\in M\setminus k} \{x : x \in X, \#(\{n : n \in I \cap m, x \in E_n\}) \ge m(d^*(I) - 2\eta)\}.$$

Then $\langle H_k\rangle_{k\in\mathbb{N}}$ is a non-decreasing sequence in $\Sigma$ and $\bigcup_{k\in\mathbb{N}} H_k = X$. **P** If $x \in X$, there is a $J \in \mathcal{F}_d$ such that $x \in E_n$ for every $n \in J$. Now $d^*(\mathbb{N} \setminus J) = 0$ so there is a $k \in \mathbb{N}$ such that $\#(m \setminus J) \le \eta m$ for every $m \ge k$. If $m \in M$ and $m \ge k$, then

$$\#(\{n : n \in I \cap m, x \in E_n\}) \ge \#(m \cap I \cap J) \ge \#(m \cap I) - \eta m \ge m(d^*(I) - 2\eta)\},$$

so $x \in H_k$. **Q**

**(b)** There is therefore an $m \in M$ such that $\mu H_m \ge 1 - \epsilon$ and $\#(I \cap m) \le m(d^*(I) + \eta)$. In this case

$$m(1-\epsilon)(d^*(I) - 2\eta) \le m(d^*(I) - 2\eta)\mu H_m \le \int \#(\{n : n \in I \cap m, x \in E_n\})\mu(dx)$$
$$= \sum_{n\in I\cap m} \mu E_n \le (\alpha + \epsilon)\#(I \cap m) \le m(\alpha + \epsilon)(d^*(I) + \eta).$$

So $(1-\epsilon)(d^*(I) - 2\eta) \le (\alpha + \epsilon)(d^*(I) + \eta)$. As $\eta$ and $\epsilon$ are arbitrary, and $d^*(I) > 0$, $\alpha = 1$. By 2C, $\mathcal{F}$ has the Fatou property.

**3F Proposition** $\mathcal{F}_s$ has the Fatou property.

**proof (a)** Let $(X, \Sigma, \mu)$ be a probability space and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence in $\Sigma$ such that $\bigcup_{I\in\mathcal{F}_s} \bigcap_{n\in I} E_n = X$; set $\alpha = \liminf_{n\to\mathcal{F}_s} \mu E_n$. Take any $\epsilon > 0$. Set $I = \{n : \mu E_n \le \alpha + \epsilon\}$; then $d_s^*(I) > 0$. Take any $\eta \in \left]0, \frac{1}{2}d_s^*(I)\right]$. Then there is a sequence $\langle k_m\rangle_{m\in\mathbb{N}}$ such that $k_m + m \le k_{m+1}$ and $\#(I \cap [k_m, k_m + m[) \ge m(d_s^*(I) - \eta)$ for every $m \in \mathbb{N}$; set $K_m = [k_m, k_m + m[$ for each $m$. For $l \in \mathbb{N}$ set

$$H_l = \bigcap_{m\ge l}\{x : x\in X,\ \#(\{n : n\in I\cap K_m,\ x\in E_n\}) \ge m(d_s^*(I)-2\eta)\}.$$

Then $\langle H_l\rangle_{l\in\mathbb{N}}$ is a non-decreasing sequence in $\Sigma$ and $\bigcup_{l\in\mathbb{N}} H_l = X$. **P** If $x\in X$, there is a $J\in\mathcal{F}_s$ such that $x\in E_n$ for every $n\in J$. Now $d_s^*(\mathbb{N}\setminus J)=0$ so there is an $l\in\mathbb{N}$ such that $\#(K_m\setminus J)\le\eta m$ for every $m\ge l$. If $m\ge l$, then

$$\#(\{n : n\in I\cap K_m,\ x\in E_n\}) \ge \#(I\cap J\cap K_m) \ge \#(I\cap K_m)-\eta m \ge m(d_s^*(I)-2\eta)\},$$

so $x\in H_l$. **Q**

**(b)** There is therefore an $m\ge 1$ such that $\mu H_m\ge 1-\epsilon$ and $\#(I\cap K_m)\le m(d_s^*(I)+\eta)$. In this case

$$\begin{aligned} m(1-\epsilon)(d_s^*(I)-2\eta) \le m(d^*(I)-2\eta)\mu H_m &\le \int \#(\{n : n\in I\cap K_m,\ x\in E_n\})\mu(dx) \\ &= \sum_{n\in I\cap K_m}\mu E_n \le (\alpha+\epsilon)\#(I\cap K_m) \le m(\alpha+\epsilon)(d_s^*(I)+\eta). \end{aligned}$$

So $(1-\epsilon)(d^*(I)-2\eta)\le(\alpha+\epsilon)(d_s^*(I)+\eta)$. As $\eta$ and $\epsilon$ are arbitrary, and $d_s^*(I)>0$, $\alpha=1$. By 2C, $\mathcal{F}_s$ has the Fatou property.

## 4 Medial limits

**4A** Recall that a **medial limit** is a non-negative additive functional $\theta$ on $\mathbb{P}\mathbb{N}$ such that $\int w\,d\theta = \lim_{n\to\infty} w(n)$ whenever $w\in\mathbb{R}^{\mathbb{N}}$ is a convergent sequence and $\iint f_n(x)\theta(dn)\mu(dx)$ is defined and equal to $\iint f_n(x)\mu(dx)\theta(dn)$ whenever $(X,\Sigma,\mu)$ is a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a uniformly bounded sequence of integrable real-valued functions on $X$; here $\int\ldots d\theta$ is defined as in Fremlin 02, 363L. See Fremlin n02 or Hoffman-Jørgensen 78 for the basic theory of medial limits, in particular, for the proof that if $\mathfrak{p}=\mathfrak{c}$ then there is a medial limit.

If $\mathcal{F}$ is a filter on $\mathbb{N}$ a medial limit $\theta$ **refines** $\mathcal{F}$ if $\liminf_{n\to\mathcal{F}} w(n)\le\int w\,d\theta$ for every $w\in\ell^\infty$.

**4B Proposition** Let $\mathfrak{F}$ be the set of filters $\mathcal{F}$ on $\mathbb{N}$ such that there is an additive functional $\theta:\mathcal{P}\mathbb{N}\to[0,1]$ such that

(†) $\iint f_n(x)\theta(dn)\mu(dx)$ is defined and equal to $\iint f_n(x)\mu(dx)\theta(dn)$ whenever $(X,\Sigma,\mu)$ is a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a uniformly bounded sequence of integrable real-valued functions on $X$,

(‡) $\liminf_{n\to\mathcal{F}} w(n)\le\int w\,d\theta$ for every $w\in\ell^\infty$.

Then $\lim_{n\to\mathcal{F}}\mathcal{F}_n\in\mathfrak{F}$ whenever $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{F}$ and $\mathcal{F}\in\mathfrak{F}$.

**proof (a)** Write $\mathcal{G}$ for $\lim_{n\to\mathcal{F}}\mathcal{F}_n$. For each $n\in\mathbb{N}$ let $\theta_n$ witness that $\mathcal{F}_n\in\mathfrak{F}$, and let $\theta$ witness that $\mathcal{F}\in\mathfrak{F}$. Set $\theta^*(a)=\int\theta_n(a)\theta(dn)$ for $a\subseteq\mathbb{N}$; then $\theta^*:\mathcal{P}\mathbb{N}\to[0,1]$ is an additive functional.

**(b)** $\theta^*$ has the property (†). **P** If $(X,\Sigma,\mu)$ is a probability space and $\langle f_i\rangle_{i\in\mathbb{N}}$ is a uniformly bounded sequence of integrable real-valued functions on $X$,

$$\begin{aligned} \iint f_i(x)\theta^*(di)\mu(dx) &= \iiint f_i(x)\theta_n(di)\theta(dn)\mu(dx) = \iiint f_i(x)\theta_n(di)\mu(dx)\theta(dn) \\ &= \iiint f_i(x)\mu(dx)\theta_n(di)\theta(dn) = \iiint f_i(x)\mu(dx)\theta^*(di). \ \mathbf{Q} \end{aligned}$$

**(c)** For any $w\in\ell^\infty$,

$$\liminf_{i\to\mathcal{G}} w(i)\le\liminf_{n\to\mathcal{F}}\liminf_{i\to\mathcal{F}_n} w(i).$$

**P** Suppose that $\gamma>\liminf_{n\to\mathcal{F}}\liminf_{i\to\mathcal{F}_n} w(i)$, and that $A\in\mathcal{G}$. Then $B=\{n : A\in\mathcal{F}_n\}$ belongs to $\mathcal{F}$, so there is an $n\in B$ such that $\liminf_{i\to\mathcal{F}_n} w(i)\le\gamma$. Now $A\in\mathcal{F}_n$, so

$$\inf_{i\in A} w(i)\le\liminf_{i\to\mathcal{F}_n} w(i)\le\gamma.$$

As $A$ is arbitrary, $\liminf_{i\to\mathcal{G}} w(i)\le\gamma$. **Q**

Accordingly,

$$\liminf_{i\to\mathcal{G}} w(i) \le \liminf_{n\to\mathcal{F}} \liminf_{i\to\mathcal{F}_n} w(i) \le \int \liminf_{i\to\mathcal{F}_n} w(i)\theta(dn)$$
$$\le \iint w\, d\theta_n\theta(dn) = \int w\, d\theta^*.$$

So $\theta^*$ has property (‡), and witnesses that $\mathcal{G} \in \mathfrak{F}$.

**4C Corollary** If there is any medial limit, and $\mathcal{F}$ is a free filter on $\mathbb{N}$ of countable type, then there is a medial limit refining $\mathcal{F}$.

**proof** In 4B, it is easy to check that $\mathfrak{F}$ contains all principal ultrafilters on $\mathbb{N}$; if $\mathcal{F}$ is generated by $\{k\}$, set $\theta a = \chi a(k)$ for $a \subseteq \mathbb{N}$; then $\theta$ witnesses that $\mathcal{F} \in \mathfrak{F}$. By definition, a medial limit is a witness that $\mathcal{F}_{\mathrm{Fr}} \in \mathfrak{F}$. So an induction on the countable-type level of $\mathcal{F}$ shows that every filter on $\mathbb{N}$ of countable type belongs to $\mathfrak{F}$. If $\mathcal{F}$ is a free filter and belongs to $\mathfrak{F}$, then a witness that $\mathcal{F} \in \mathfrak{F}$ is a medial limit refining $\mathcal{F}$.

**4D Proposition** Suppose that $\mathcal{F}$ is a filter on $\mathbb{N}$ such that whenever $I \subseteq \mathbb{N}$ meets every member of $\mathcal{F}$ there is a medial limit refining

$$\mathcal{F} \vee \{I\} = \{A : A \subseteq \mathbb{N},\, A \cup (\mathbb{N} \setminus I) \in \mathcal{F}\},$$

the filter generated by $\mathcal{F} \cup \{I\}$. Then $\mathcal{F}$ has the Fatou property.

**proof** Let $J$ be any set meeting every member of $\mathcal{F}$, and $\theta$ a medial limit refining $\mathcal{F}\vee\{J\}$. Let $(X, \Sigma, \mu)$ be a probability space and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence in $\Sigma$ such that $X = \bigcup_{I\in\mathcal{F}} \bigcap_{n\in I} E_n$, that is, $\liminf_{n\to\mathcal{F}} \chi E_n(x) = 1$ for every $x \in X$. Then

$$1 = \int \liminf_{n\to\mathcal{F}} \chi E_n(x)\mu(dx) \le \iint \chi E_n(x)\theta(dn)\mu(dx)$$

(since $\theta$ also refines $\mathcal{F}$)

$$= \iint \chi E_n(x)\mu(dx)\theta(dn) = \int \mu E_n\theta(dn) \le \limsup_{n\to\mathcal{F}\vee\{J\}} \mu E_n \le \sup_{n\in J} \mu E_n.$$

As $J$ is arbitrary, $\liminf_{n\to\mathcal{F}} \mu E_n = 1$. As $(X, \Sigma, \mu)$ and $\langle E_n\rangle_{n\in\mathbb{N}}$ are arbitrary, $\mathcal{F}$ has the Fatou property.

**4E Remark** The filter $\mathcal{F}$ of Example 2E is a Borel subset of $\mathcal{P}\mathbb{N}$ but is not refined by any medial limit. **P?** Otherwise, follow the argument of 4D with $J = \mathbb{N}$; we should get $1 \le \limsup_{n\to\mathcal{F}} \mu E_n = 0$. **XQ**

**4F Proposition** If there is a medial limit, there is a medial limit refining the density filter.

**proof** Let $\theta$ be a medial limit, and define $\theta^*$ by setting

$$\theta^*(a) = \int \tfrac{1}{n+1}\#(a \cap (n+1))\theta(dn)$$

for $a \subseteq \mathbb{N}$. Then

$$\int w\, d\theta^* = \int \tfrac{1}{n+1}\textstyle\sum_{i=0}^n w(i)\theta(dn)$$

for $w \in \ell^\infty$. So if $(X, \Sigma, \mu)$ is a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a uniformly bounded sequence of real-valued functions on $X$,

$$\iint f_n(x)\theta^*(dn)\mu(dx) = \iint \frac{1}{n+1}\sum_{i=0}^n f_i(x)\theta(dn)\mu(dx) = \iint \frac{1}{n+1}\sum_{i=0}^n f_i(x)\mu(dx)\theta(dn)$$
$$= \int \frac{1}{n+1}\sum_{i=0}^n \int f_i(x)\mu(dx)\theta(dn) = \iint f_n(x)\mu(dx)\theta(dn).$$

Thus $\theta^*$ is a medial limit. Now $\theta^*(a) \le d^*(a)$ for every $a \subseteq \mathbb{N}$; in particular, $\theta^*(a) = 0$ whenever $d^*(a) = 0$ and $\theta^*(a) = 1$ for every $a \in \mathcal{F}_d$; it follows that $\liminf_{n\to\mathcal{F}_d} w(n) \le \int w\,d\theta^*$ for every $w \in \ell^\infty$, and $\theta^*$ refines $\mathcal{F}_d$.

## 5 Isomorphism classes of filters of countable type

**5A Homogeneous and critical filters (a)** I will say that a filter $\mathcal{F}$ on a set $X$ is **homogeneous** if $\mathcal{F}$ is isomorphic to its trace $\mathcal{F}\lceil Y$ whenever $Y \subseteq X$ and $X \setminus Y \notin \mathcal{F}$. Observe that in this case $\mathcal{F}$ must contain every set $A \subseteq X$ such that $\#(X \setminus A) < \#(X)$; in particular, $\mathcal{F}$ must be free, unless $X$ is a singleton. An ultrafilter on $X$ is homogeneous iff it is uniform.

**(b)** I will say that a filter $\mathcal{F}$ of countable type on a set $I$ is **critical** if there are a sequentially compact Hausdorff space $X$ and a function $f : I \to X$ such that $f[[\mathcal{F}]]$ converges to a point of $X \setminus \bigcup_{\eta<\mathrm{ct}\,\mathcal{F}} f[I]^\sim_\eta$.

**5B Lemma** (a) Let $X$ be a discrete space. Then there is a locally countable sequentially compact Hausdorff space $Y$ with an open set homeomorphic to $X$.

(b) Let $(X, \mathfrak{T})$ be a Hausdorff space, and suppose that whenever $\langle A_i\rangle_{i\in I}$ is a countable family of countably infinite subsets of $X$ and $x \in X$ then

either there are an $i \in I$ and a sequence of distinct points in $A_i$ convergent to a point of $X$

or there are distinct $i$, $j \in I$ such that $A_i \cap A_j$ is infinite

or there is a disjoint family $\langle G_i\rangle_{i\in I}$ of open sets such that $A_i \setminus G_i$ is finite for every $i$ and $x \notin \overline{\bigcup_{i\in I} G_i}$.

Suppose that we are also given a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in $X$ with no convergent subsequence in $X$. Then $X$ can be embedded as an open set in a sequentially compact Hausdorff space $Z$ such that $\langle x_n\rangle_{n\in\mathbb{N}}$ is convergent in $Z$ to a point $z^*$ such that no sequence in $Z \setminus (\{z^*\} \cup \{x_n : n \in \mathbb{N}\})$ converges to $z^*$.

**proof (a)(i)** We can suppose that $X$ is a cardinal $\kappa$; let $\lambda > \kappa$ be a cardinal of uncountable cofinality such that the cardinal power $\lambda^\omega$ is equal to $\lambda$, and let $\langle A_\xi\rangle_{\kappa\le\xi<\lambda}$ enumerate $[\lambda]^\omega$. Define $\langle B_\xi\rangle_{\xi<\lambda}$ inductively by setting

$$\begin{aligned} B_\xi &= A_\xi \text{ if } \kappa \le \xi < \lambda \text{ and } A_\xi \cap B_\eta \text{ is finite for every } \eta < \xi \\ &= \emptyset \text{ otherwise.} \end{aligned}$$

Observe that $B_\xi \cap B_\eta$ is finite for all distinct $\xi$, $\eta < \lambda$, and that if $C \in [\lambda]^\omega$ there is a $\xi < \lambda$ such that $C \cap B_\xi$ is infinite.

On $Y = \lambda$, let $\mathfrak{T}$ be the topology

$$\{G : G \subseteq Y,\ B_\xi \setminus G \text{ is finite for every } \xi \in G\}.$$

**(ii)** Since $B_\xi = \emptyset$ for $\xi < \kappa$, every subset of $X$ is an open subset of $Y$, and $X$, with its discrete topology, is a subspace of $Y$.

**(iii)** If $\langle \xi_n\rangle_{n\in\mathbb{N}}$ is a sequence in $Y$, then either it has a constant subsequence, which is surely convergent, or $C = \{\xi_n : n \in \mathbb{N}\}$ is infinite. In this case, let $\xi < \lambda$ be such that $B_\xi \cap C$ is infinite; then $\langle \xi_n\rangle_{n\in\mathbb{N}}$ has a subsequence converging to $\xi$.

Thus $Y$ is sequentially compact.

**(iv)** If $\alpha$, $\beta$ are distinct points of $Y$, enumerate $\mathbb{N} \times \mathbb{N}$ as $\langle (i_n, j_n)\rangle_{n\in\mathbb{N}}$ in such a way that $i_n \le n$ for every $n$, and define $\langle (U_n, V_n)\rangle_{n\in\mathbb{N}}$ as follows. $U_0 = \{\alpha\}$, $V_0 = \{\beta\}$. Given that $U_n$ and $V_n$ are disjoint countable sets such that $B_\xi \cap U_n$ is finite for every $\xi \in \lambda \setminus U_n$, while $B_\xi \cap V_n$ is finite for every $\xi \in \lambda \setminus V_n$, let $\langle \xi_{nj}\rangle_{j\in\mathbb{N}}$ run over $U_n$ and $\langle \eta_{nj}\rangle_{j\in\mathbb{N}}$ run over $V_n$. Set

$$U_{n+1} = U_n \cup (B_{\xi_{i_n,j_n}} \setminus V_n), \quad V_{n+1} = V_n \cup (B_{\eta_{i_n,j_n}} \setminus U_{n+1})$$

and continue. At the end of the induction, set $G = \bigcup_{n\in\mathbb{N}} U_n$ and $H = \bigcup_{n\in\mathbb{N}} V_n$; then $G = \{\xi_{ij} : i, j \in \mathbb{N}\}$ and $H = \{\eta_{ij} : i, j \in \mathbb{N}\}$ are disjoint and open, $\alpha \in G$ and $\beta \in H$.

Thus $Y$ is Hausdorff.

**(v)** Note that in the construction of (iv), both $G$ and $H$ are necessarily countable; it follows at once that $Y$ is locally countable.

**(b)(i)** Let $\mathcal{C}$ be the family of all infinite subsets $C$ of $X$ such that there is no convergent sequence of distinct points of $C$. Note that $A^* = \{x_n : n \in \mathbb{N}\}$ belongs to $\mathcal{C}$. Let $\mathcal{A} \subseteq \mathcal{C}$ be a maximal set, containing $A^*$, such that $A \cap B$ is finite for all distinct $A$, $B$ in $\mathcal{A}$. Let $\langle z_A \rangle_{A \in \mathcal{A}}$ be a family of distinct points not in $X$, and set $Z_0 = \{z_A : A \in \mathcal{A}\}$.

**(ii)** By (a), we can construct a locally countable sequentially compact Hausdorff space $(Z_1, \mathfrak{T}_1)$, disjoint from $X$, such that $Z_0 \subseteq Z_1$ and the subspace topology on $Z_0$ is discrete. Set $Z = Z_1 \cup X$, and give $Z$ the topology

$$\begin{aligned}\mathfrak{S} = \{G : G \subseteq Z,\, G \cap Z_1 \in \mathfrak{T}_1,\, G \cap X \in \mathfrak{T},\\ A \setminus G \text{ is finite whenever } A \in \mathcal{A} \text{ and } z_A \in G\}.\end{aligned}$$

**(iii)** $X$ is an open subset of $Z$ and $\mathfrak{T}$ is the subspace topology induced by $\mathfrak{S}$ on $X$.

**(iv)** If $\langle t_n \rangle_{n \in \mathbb{N}}$ is a sequence in $Z$, then

either it has a constant subsequence, which is surely convergent;

or it has a subsequence in $Z_1$, which must in turn have a subsequence converging in $Z_1$ and in $Z$;

or it has a convergent subsequence in $X$, which will converge in $Z$;

or $C = \{t_n : n \in \mathbb{N}\} \cap X$ belongs to $\mathcal{C}$, and there is an $A \in \mathcal{A}$ such that $C \cap A$ is infinite, in which case $\langle t_n \rangle_{n \in \mathbb{N}}$ has a subsequence convergent to $z_A$.

So $Z$ is sequentially compact.

**(v)** $Z$ is Hausdorff. **P** Suppose that $w$, $z$ are distinct points of $Z$.

**($\alpha$)** If $w$, $z$ both belong to $X$, they are separated by disjoint open sets in $X$, which are still open in $Z$.

**($\beta$)** If $w \in X$ and $z \in Z_1$, let $V$ be a countable neighbourhood of $z$ in $Z_1$, and consider the countable set $\mathcal{D} = \{A : A \in \mathcal{A},\, z_A \in V\}$. By the hypothesis on $X$, there is a family $\langle H_A \rangle_{A \in \mathcal{D}}$ of open sets in $X$ such that $A \setminus H_A$ is finite for every $A \in \mathcal{D}$ and $w \notin \overline{\bigcup_{A \in \mathcal{D}} H_A}$. Set $G = X \setminus \overline{\bigcup_{A \in \mathcal{D}} H_A}$, $H = V \cup \bigcup_{A \in \mathcal{D}} H_A$; then $G$ and $H$ are open subsets of $Z$ separating $w$ from $z$.

**($\gamma$)** Similarly, $w$ and $z$ can be separated if $w \in Z_1$ and $z \in X$.

**($\delta$)** Suppose that $w$, $z$ both belong to $Z_1$. Let $U$, $V \subseteq Z_1$ be disjoint countable sets, both open in $Z_1$, containing $w$, $z$ respectively. Set $\mathcal{D} = \{A : A \in \mathcal{A},\, z_A \in U \cup V\}$; let $\langle G_A \rangle_{A \in \mathcal{D}}$ be a disjoint family of open subsets in $X$ such that $A \setminus G_A$ is finite for every $A \in \mathcal{D}$; set $G = U \cup \{G_A : A \in \mathcal{D},\, z_A \in U\}$, $H = V \cup \{G_A : A \in \mathcal{D},\, z_A \in V\}$. Then $G$ and $H$ are open sets in $Z$ separating $w$ from $z$. **Q**

**(vi)** Setting $z^* = z_{A^*}$, we see that $A^* \setminus G$ is finite for every neighbourhood $G$ of $z^*$, so that $z^* = \lim_{n\to\infty} x_n$. The construction ensured that $\{z^*\}$ is an open set in $Z_1$, so that $X \cup \{z^*\}$ is an open set in $Z$, as required. If $\langle z_n \rangle_{n \in \mathbb{N}}$ is a sequence in $Z \setminus (\{z^*\} \cup \{x_n : n \in \mathbb{N}\})$, then

either it has a subsequence in $Z_1$, which cannot converge to $z^*$ because $z^*$ is isolated in $Z_1$,

or it has a subsequence which is a convergent sequence in $X$, and does not converge to $z^*$,

or $C = \{z_n : n \in \mathbb{N}\} \cap X$ belongs to $\mathcal{C}$, so meets some $A \in \mathcal{A}$ in an infinite set; as $A \neq A^*$, $\langle z_n \rangle_{n \in \mathbb{N}}$ has a subsequence converging to some $z_A \neq z^*$.

In any case, we see that $\langle z_n \rangle_{n \in \mathbb{N}}$ does not converge to $z^*$, which is what we needed to know.

**5C Lemma** Suppose that $\langle \mathcal{F}_n \rangle_{n \in \mathbb{N}}$ is a sequence of critical filters of countable type on $\mathbb{N}$ such that $\langle \operatorname{ct} \mathcal{F}_n \rangle_{n \in \mathbb{N}}$ is non-decreasing. Let $\langle I_n \rangle_{n \in \mathbb{N}}$ be a partition of $\mathbb{N}$ into infinite sets, and $g_n : \mathbb{N} \to I_n$ a bijection for each $n$; let $\mathcal{F}$ be $\lim_{n \to \mathcal{F}_{\mathrm{Fr}}} g_n[[\mathcal{F}_n]]$. Then $\mathcal{F}$ is a critical filter, and $\operatorname{ct} \mathcal{F} = \sup_{n \in \mathbb{N}} (\operatorname{ct} \mathcal{F}_n + 1)$.

**proof (a)** Setting $\xi = \sup_{n \in \mathbb{N}} (\operatorname{ct} \mathcal{F}_n + 1)$, then of course

$$\operatorname{ct} \mathcal{F} \le \sup_{n \in \mathbb{N}} (\operatorname{ct} g_n[[\mathcal{F}_n]] + 1) \le \xi$$

by 1D.

**(b)** For each $n \in \mathbb{N}$, let $X_n$ be a sequentially compact Hausdorff space and $f_n : \mathbb{N} \to X_n$ a function such that $\lim f_n[[\mathcal{F}_n]]$ is defined and does not belong to $\bigcup_{\eta<\mathrm{ct}\,\mathcal{F}_n} f[\mathbb{N}]^{\sim}_{\eta}$. Set $Z_n = X_n \times \{n\}$ for each $n$, $X = \bigcup_{n\in\mathbb{N}} X_n$ with the disjoint union topology. Then $X$ satisfies the conditions of 5Bb. **P** Let $\langle A_i\rangle_{i\in I}$ be a countable family of countably infinite subsets of $X$ such that $A_i \cap A_j$ is finite for all $i \neq j$ and there is no non-trivial convergent sequence made up of points from any single $A_i$. As every $X_n$ is sequentially compact, this means that $A_i \cap Z_n$ is finite for every $i$. Take any $(x, k) \in X$.

If $I$ is finite, there is an $n > k$ such that $A_i \cap A_j$ does not meet $Z_m$ for any $m \geq n$ and any $i \neq j$. In this case, for $m \geq n$, $\bigcup_{i\in I} A_i \cap Z_m$ is finite, so we can find disjoint open sets $G_{im} \subseteq X_m$ such that $A_i \cap Z_m \subseteq G_{im} \times \{m\}$ for each $i$; setting $G_i = \bigcup_{m\geq n} G_{im} \times \{m\}$, we have a disjoint family $\langle G_i\rangle_{i\in I}$ of open sets in $X$ such that $A_i \setminus G_i$ is finite for every $i$, and $(x, k) \notin \overline{\bigcup_{i\in I} G_i}$.

If $I$ is infinite, we may take it that $I = \mathbb{N}$. In this case, let $\langle n_i\rangle_{i\in\mathbb{N}}$ be a strictly increasing sequence, starting with $n_0 > k$, such that $A_i \cap A_j \cap Z_m$ is empty whenever $i < j$ and $m \geq n_j$; set $A'_i = A_i \setminus \bigcup_{m<n_i} Z_m$. Then $A_i \setminus A'_i$ is finite and $A'_i \cap A'_j$ is empty whenever $i < j$ in $\mathbb{N}$. This time, $\bigcup_{i\in\mathbb{N}} A_i \cap Z_m$ is finite for every $m \in \mathbb{N}$, so we can use the same method as before to find $G_{im}$, for $i$, $m \in \mathbb{N}$, and $G_m$, for $m \in \mathbb{N}$, such that $A'_i \subseteq G_i$ for every $i$ and $\langle G_i\rangle_{i\in\mathbb{N}}$ is a disjoint family of open sets. If we take the elementary precaution of setting $G_{ik} = \emptyset$ for every $i$, then $(x, k) \notin \overline{\bigcup_{i\in\mathbb{N}} G_i}$. **Q**

**(c)** By Lemma 5Bb, we can embed $X$ in a sequentially compact Hausdorff space $Z$ such that $\langle (x_n, n)\rangle_{n\in\mathbb{N}}$ converges to a point $z^*$ of $Z$ and no sequence in $Z \setminus (\{z^*\} \cup \{(x_n, n) : n \in \mathbb{N}\})$ converges to $z^*$. Let $f : \mathbb{N} \to Z$ be such that $f(g_n(i)) = (f_n(i), n)$ for all $i$, $n \in \mathbb{N}$. Then $f[[\mathcal{F}]] \to z^*$. **P** If $G$ is any neighbourhood of $z^*$, then

$$\begin{aligned}\{n : \{j : f(j) \in G\} \in g_n[[\mathcal{F}_n]]\} &= \{n : \{i : f(g_n(i)) \in G\} \in \mathcal{F}_n\}\\ &= \{n : \{i : (f_n(i), n) \in G\} \in \mathcal{F}_n\}\\ &\supseteq \{n : (x_n, i) \in G\} \in \mathcal{F}_{\mathrm{Fr}},\end{aligned}$$

so $f^{-1}[G] \in \mathcal{F}$. **Q**

If $n \in \mathbb{N}$ and $\eta < \mathrm{ct}\,\mathcal{F}_n$, then $(x_n, n) \notin f[\mathbb{N}]^{\sim}_{\eta}$. **P** Because $Z_n$ is open in $X$ and $Z$,

$$\begin{aligned}Z_n \cap f[\mathbb{N}]^{\sim}_{\eta} &= (Z_n \cap f[\mathbb{N}])^{\sim}_{\eta} = f[I_n]^{\sim}_{\eta}\\ &= (f_n[\mathbb{N}] \times \{n\})^{\sim}_{\eta} = f_n[\mathbb{N}]^{\sim}_{\eta} \times \{n\}\end{aligned}$$

(calculating $f_n[\mathbb{N}]^{\sim}_{\eta}$ in $X_n$, of course); as $x_n \notin f_n[\mathbb{N}]^{\sim}_{\eta}$, $(x_n, n) \notin f[\mathbb{N}]^{\sim}_{\eta}$. **Q**

**?** If $\zeta < \xi$ and $z^* \in f[\mathbb{N}]^{\sim}_{\zeta}$, then (as $z^*$ surely does not belong to $f[\mathbb{N}]$) there is a sequence $\langle z_i\rangle_{i\in\mathbb{N}}$ in $\bigcup_{\eta<\zeta} f[\mathbb{N}]^{\sim}_{\eta}$ converging to $z^*$, with no $z_i$ equal to $z^*$. Now $\langle z_i\rangle_{i\in\mathbb{N}}$ must have a subsequence in common with $\langle (x_n, n)\rangle_{n\in\mathbb{N}}$, so there must be infinitely many $n$ such that $(x_n, n) \in \bigcup_{\eta<\zeta} f[\mathbb{N}]^{\sim}_{\eta}$. However, there is an $m \in \mathbb{N}$ such that $\zeta \leq \mathrm{ct}\,\mathcal{F}_n$ for every $n \geq m$, in which case $(x_n, n) \notin \bigcup_{\eta<\zeta} f_n[\mathbb{N}]^{\sim}_{\eta}$; which is impossible. **X**

Thus we know that $\mathrm{ct}\,\mathcal{F} \leq \xi$, that $f[[\mathcal{F}]] \to z^*$, and that $z^* \notin \bigcup_{\zeta<\xi} f[\mathbb{N}]^{\sim}_{\zeta}$; we conclude that $\mathrm{ct}\,\mathcal{F} = \xi$ and that $f : \mathbb{N} \to Z$ witnesses that $\mathcal{F}$ is critical.

**5D Theorem** For every $\xi < \omega_1$ there is a critical filter on $\mathbb{N}$ with countable-type level $\xi$.

**proof** Induce on $\xi$. For $\xi = 0$, take a principal ultrafilter. For the step to $\xi > 0$, let $\langle \xi_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence such that $\xi = \sup_{n\in\mathbb{N}}(\xi_n + 1)$. For each $n \in \mathbb{N}$, there is a critical filter $\mathcal{F}_n$ on $\mathbb{N}$ such that $\mathrm{ct}\,\mathcal{F}_n = \xi_n$; now the construction of Lemma 5C gives a critical filter with countable-type level $\xi$.

**5E Lemma** For every $n \in \mathbb{N}$ there is a homogeneous critical filter $\mathcal{H}_n$ on $\mathbb{N}$ such that $\mathrm{ct}\,\mathcal{H}_n = n + 1$.

**proof** Induce on $n$.

**(a)** Start with $\mathcal{H}_0 = \mathcal{F}_{\mathrm{Fr}}$. To see that $\mathcal{F}_{\mathrm{Fr}}$ is critical, take $X = \mathbb{N} \cup \{\infty\}$ to be the one-point compactification of $\mathbb{N}$, and $f : \mathbb{N} \to X$ the identity function.

**(b)** For the inductive step to $n + 1$, let $\langle I_k\rangle_{k\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into infinite sets, and let $g_k$ be the increasing enumeration of $I_k$ for each $k$; set $\mathcal{H}_{n+1} = \lim_{k\to\mathcal{F}_{\mathrm{Fr}}} g_k[[\mathcal{H}_n]]$. By 5C, $\mathcal{H}_{n+1}$ is a critical filter and $\mathrm{ct}\,\mathcal{H}_{n+1} = n + 2$.

**(c)(i)** For each $k \in \mathbb{N}$, set $\mathcal{G}_k = g_k[[\mathcal{H}_n]]\lceil I_k$, so that $\mathcal{G}_k$ is isomorphic to $\mathcal{H}_n$ and is homogeneous; observe that

$$\mathcal{H}_{n+1} = \{A : A \subseteq \mathbb{N},\ A \cap I_k \in \mathcal{G}_k\} \text{ for all but finitely many } k\}.$$

**(ii)** Suppose that $A \subseteq \mathbb{N}$ is infinite, and that $Y \subseteq \bigcup_{k\in A} I_k$ is such that $I_k \setminus Y \notin \mathcal{G}_k$ for every $k \in A$. Of course $\mathbb{N} \setminus Y \notin \mathcal{H}_{n+1}$. For $k \in A$, $\mathcal{G}_k$ is isomorphic to its trace $\mathcal{G}'_k = \mathcal{G}_k\lceil(Y \cap I_k)$; let $h_k : I_k \to I_k \cap Y$ be a corresponding isomorphism. Enumerate $A$ in ascending order as $\langle k_i\rangle_{i\in\mathbb{N}}$, and let $h : \mathbb{N} \to Y$ be the bijection defined by setting

$$h(j) = h_{k_i} g_{k_i} g_i^{-1}(j) \text{ if } i \in \mathbb{N} \text{ and } j \in I_i.$$

For $B \subseteq \mathbb{N}$,

$$\begin{aligned}
B \in \mathcal{H}_{n+1} &\iff B \cap I_i \in \mathcal{G}_i \text{ for all but finitely many } i \\
&\iff g_i^{-1}[B \cap I_i] \in \mathcal{H}_n \text{ for all but finitely many } i \\
&\iff g_{k_i}[g_i^{-1}[B \cap I_i]] \in \mathcal{G}_{k_i} \text{ for all but finitely many } i \\
&\iff h_{k_i}[g_{k_i}[g_i^{-1}[B \cap I_i]]] \in \mathcal{G}'_{k_i} \text{ for all but finitely many } i \\
&\iff h[B] \cap I_{k_i} \in \mathcal{G}'_{k_i} \text{ for all but finitely many } i \\
&\iff h[B] \cap I_k \in \mathcal{G}'_k \text{ for all but finitely many } k \in A \\
&\iff \exists\, C \subseteq \mathbb{N},\ h[B] = Y \cap C,\ C \cap I_k \in \mathcal{G}_k \text{ for all but finitely many } k \in A \\
&\iff \exists\, C \subseteq \mathbb{N},\ h[B] = Y \cap C,\ C \cap I_k \in \mathcal{G}_k \text{ for all but finitely many } k \in \mathbb{N} \\
&\iff \exists\, C \in \mathcal{H}_{n+1},\ h[B] = Y \cap C \\
&\iff B \in \mathcal{H}_{n+1}\lceil Y.
\end{aligned}$$

So $h$ is an isomorphism between $\mathcal{H}_{n+1}$ and its trace on $Y$.

**(iii)** Now let $Y$ be any subset of $\mathbb{N}$ such that $\mathbb{N}\setminus Y \notin \mathcal{H}_{n+1}$; set $A = \{k : Y \cap I_k \notin \mathcal{G}_k\}$, $Y' = Y \cap \bigcup_{k\in A} I_k$ and $Y'' = Y \cap \bigcup_{k\in A}(I_k \setminus \{\min(I_k \cap Y)\})$. By (ii), $\mathcal{H}_{n+1}$ is isomorphic to its trace on $Y'$. But observe that $J = Y \setminus Y''$ is an infinite set such that $\mathbb{N} \setminus J$ belongs to $\mathcal{H}_{n+1}$, and that the same is true of $J' = Y' \setminus Y''$. So the traces of $\mathcal{H}_{n+1}$ on $Y$ and $Y'$ are isomorphic, since one is mapped to the other by any bijection between $J$ and $J'$; and the trace of $\mathcal{H}_{n+1}$ on $Y$ is therefore isomorphic to $\mathcal{H}_{n+1}$.

Thus $\mathcal{H}_{n+1}$ is homogeneous.

**5F Lemma** Suppose that $K \subseteq \mathbb{N}$. Then there is a filter $\mathcal{F}$ on $\mathbb{N}$ such that $\operatorname{ct}\mathcal{F} = \omega$ and

$$\begin{aligned}
K = \{n : n \in \mathbb{N} &\text{ and there is an } A \subseteq \mathbb{N} \text{ such that} \\
&\mathcal{F}\lceil A \text{ is a homogeneous filter and } \operatorname{ct}(\mathcal{F}\lceil A) = n + 2\}.
\end{aligned}$$

**proof (a)** Let $\langle n(k)\rangle_{k\in\mathbb{N}}$ be an unbounded sequence in $\mathbb{N}$ such that

$$K = \{n : n \in \mathbb{N},\ \{k : k \in \mathbb{N},\ n(k) = n\} \text{ is infinite}\}.$$

Let $\langle I_k\rangle_{k\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into infinite sets, and $g_k : \mathbb{N} \to I_k$ a bijection for each $k$; let $\langle \mathcal{H}_n\rangle_{n\in\mathbb{N}}$ be the sequence of filters constructed in Lemma 5E, and set $\mathcal{F} = \lim_{k\to\mathcal{F}_{\mathrm{Fr}}} g_k[[\mathcal{H}_{n(k)}]]$. Since

$$\operatorname{ct} g_k[[\mathcal{H}_{n(k)}]] \le \operatorname{ct} \mathcal{H}_{n(k)} < \omega$$

for every $k$, $\operatorname{ct}\mathcal{F} \le \omega$.

**(b)** Suppose that $\langle k_i\rangle_{i\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$ such that $\langle n(k_i)\rangle_{i\in\mathbb{N}}$ is non-decreasing. Set $V = \bigcup_{i\in\mathbb{N}} I_{k_i}$. Then

$$\mathcal{F}\lceil V = \lim_{i\to\mathcal{F}_{\mathrm{Fr}}} g_{k_i}[[\mathcal{H}_{n(k_i)}]]\lceil V$$

(1Fa), so $\operatorname{ct}(\mathcal{F}\lceil V) = \sup_{i\in\mathbb{N}} n(k_i) + 1$ (Lemma 5C, transferred to the partition $\langle I_{k_i}\rangle_{i\in\mathbb{N}}$ of $A$). In particular, since $\langle n(k)\rangle_{k\in\mathbb{N}}$ is unbounded, we can find a strictly increasing sequence $\langle k_i\rangle_{i\in\mathbb{N}}$ such that $\langle n(k_i)\rangle_{i\in\mathbb{N}}$ is also strictly increasing, in which case $\operatorname{ct}\mathcal{F} \ge \operatorname{ct}(\mathcal{F}\lceil V) \ge \omega$, and $\operatorname{ct}\mathcal{F}$ must be exactly $\omega$.

**(c)** If $n \in K$, there is a strictly increasing sequence $\langle k_i \rangle_{i\in\mathbb{N}}$ such that $n(k_i) = n$ for every $i \in \mathbb{N}$. In this case, setting $V = \bigcup_{i\in\mathbb{N}} I_{k_i}$,

$$\mathcal{F}\lceil V = \lim_{i\to\mathcal{F}_{\mathrm{Fr}}} g_{k_i}[[\mathcal{H}_n]]\lceil V \cong \mathcal{H}_{n+1}$$

by the construction in the proof of 5E. By 5E, $\mathcal{F}\lceil V$ is homogeneous with countable-type level $n+2$.

**(d)** Now suppose that $A \subseteq \mathbb{N}$ is such that $\mathbb{N} \setminus A \notin \mathcal{F}$ and $\mathcal{F}\lceil A$ is homogeneous with finite countable-type level. Set

$$J = \{k : k \in \mathbb{N},\ \mathbb{N} \setminus A \notin g_k[[\mathcal{H}_{n(k)}]]\},$$

so that $J$ is infinite; let $\langle k_i \rangle_{i\in\mathbb{N}}$ be a strictly increasing sequence in $J$ such that $\langle n(k_i)\rangle_{i\in\mathbb{N}}$ is either strictly increasing or constant. For each $i \in \mathbb{N}$, set $\mathcal{G}_i = g_{k_i}[[\mathcal{H}_{n(k_i)}]]\lceil I_{k_i}$, so that $\mathcal{G}_i \cong \mathcal{H}_{n(k_i)}$ is homogeneous and $I_{k_i} \setminus A \notin \mathcal{G}_i$; let $h_i : I_{k_i} \to I_{k_i} \cap A$ be an isomorphism between $(I_{k_i}, \mathcal{G}_i)$ and $(I_{k_i} \cap A, \mathcal{G}_i\lceil I_{k_i} \cap A)$. Set $V = \bigcup_{i\in\mathbb{N}} I_{k_i}$, and define $h : V \to V \cap A$ by setting $h(j) = h_i(j)$ if $i \in \mathbb{N}$ and $j \in I_{k_i}$. Just as in part (c-ii) of the proof of 5E, $h$ is an isomorphism between $\mathcal{F}\lceil V$ and $\mathcal{F}\lceil(V \cap A)$. But $\mathcal{F}\lceil(V \cap A)$ must be isomorphic to $\mathcal{F}\lceil A$, so $\mathcal{F}\lceil V$ also is.

**(e)** Looking back at (b), we see that because $\mathcal{F}\lceil V$ has finite countable-type level, $\langle n(k_i)\rangle_{i\in\mathbb{N}}$ must be bounded, and therefore constant, with value belonging to $K$. In the latter case, (c) tells us that

$$\mathrm{ct}(\mathcal{F}\lceil A) = \mathrm{ct}(\mathcal{F}\lceil V) = n + 2.$$

**(f)** So

$$K \subseteq \{n : n \in \mathbb{N} \text{ and there is an } A \subseteq \mathbb{N} \text{ such that } \mathcal{F}\lceil A \text{ is a homogeneous filter and } \mathrm{ct}(\mathcal{F}\lceil A) = n+2\}$$

(by (c))

$$\subseteq K$$

by (e), and we're done.

**5G Theorem** There are $\mathfrak{c}$ non-isomorphic filters of countable-type level $\omega$.

**proof** For every $K \subseteq \mathbb{N}$ we have a corresponding filter $\mathcal{F}_K$ as constructed in 5F. But the formula

$$K = \{n : n \in \mathbb{N} \text{ and there is an } A \subseteq \mathbb{N} \text{ such that } \mathcal{F}_K\lceil A \text{ is a homogeneous filter and } \mathrm{ct}(\mathcal{F}_K\lceil A) = n+2\}$$

tells us that the $\mathcal{F}_K$ are all non-isomorphic, so there are at least $\mathfrak{c}$ isomorphism classes of these filters. In the other direction, of course, $\mathcal{P}\mathbb{N}$ has only $\mathfrak{c}$ Borel sets, so there can be at most $\mathfrak{c}$ filters on $\mathbb{N}$ of countable type.

## 6 The coinitiality of filters of countable type

**6A Theorem** If $\mathcal{F}$ is a filter on $\mathbb{N}$ of countable type, $\mathrm{ci}\,\mathcal{F} \le \mathrm{cf}\,\mathbb{N}^{\mathbb{N}}$.

**proof (a)** Let $\mathfrak{F}$ be the set of filters $\mathcal{F}$ on $\mathbb{N}$ such that there are a function $f : \mathcal{F} \to \mathbb{N}^{\mathbb{N}}$ and a family $\langle A_{\alpha\beta}\rangle_{\alpha,\beta\in\mathbb{N}^{\mathbb{N}}}$ in $\mathcal{F}$ such that

$A_{\alpha\gamma} \subseteq A_{\alpha\beta}$ whenever $\alpha$, $\beta$, $\gamma \in \mathbb{N}^{\mathbb{N}}$ and $\beta \le \gamma$,
whenever $A \in \mathcal{F}$ and $f(A) \le \alpha$ in $\mathbb{N}^{\mathbb{N}}$, there is a $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $A_{\alpha\beta} \subseteq A$.

**(b)** Principal ultrafilters belong to $\mathfrak{F}$. **P** If $\mathcal{F}$ is generated by $\{n\}$, set $f(A) = \mathbf{0}$ for every $A \in \mathcal{F}$ and $A_{\alpha\beta} = \{n\}$ for $\alpha$, $\beta \in \mathbb{N}^{\mathbb{N}}$. **Q**

**(c)** If $\langle \mathcal{F}_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{F}$, then $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ belongs to $\mathfrak{F}$. **P** For each $n \in \mathbb{N}$, let $f_n$, $\langle A^{(n)}_{\alpha\beta}\rangle_{\alpha,\beta\in\mathbb{N}^{\mathbb{N}}}$ witness that $\mathcal{F}_n \in \mathfrak{F}$. If $A \in \mathcal{F}$, set $f(A) = f_n(A)$ where $n = \min\{i : A \in \mathcal{F}_i\}$. If $\alpha$, $\beta \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$, define $<n>^\frown\beta \in \mathbb{N}^{\mathbb{N}}$ by setting

$$(<n>^\frown\beta)(i) = n \text{ if } i = 0$$
$$= \beta(i-1) \text{ if } i > 0,$$

and set

$$A_{\alpha,<n>^\frown\beta} = \bigcap_{k\le n} A^{(k)}_{\alpha\beta} \in \mathfrak{F}.$$

If $m$, $n \in \mathbb{N}$ and $\beta$, $\gamma \in \mathbb{N}^{\mathbb{N}}$ are such that $<m>^\frown\beta \le <n>^\frown\gamma$, then $m \le n$ and $\beta \le \gamma$, so

$$A_{\alpha,<n>^\frown\gamma} = \bigcap_{k\le n} A^{(k)}_{\alpha\gamma} \subseteq \bigcap_{k\le n} A^{(k)}_{\alpha\beta} \subseteq \bigcap_{k\le m} A^{(k)}_{\alpha\beta} = A_{\alpha,<m>^\frown\beta}.$$

If $A \in \mathcal{F}$ and $\alpha \in \mathbb{N}^{\mathbb{N}}$ are such that $f(A) \le \alpha$, then there is an $n \in \mathbb{N}$ such that $A \in \mathcal{F}_n$ and $f_n(A) \le \alpha$. Let $\beta \in \mathbb{N}$ be such that $A^{(n)}_{\alpha\beta} \subseteq A$; then

$$A_{\alpha,<n>^\frown\beta} \subseteq A^{(n)}_{\alpha\beta} \subseteq A.$$

So $f$ and $\langle A_{\alpha\beta}\rangle_{\alpha,\beta\in\mathbb{N}^{\mathbb{N}}}$ witness that $\mathcal{F} \in \mathfrak{F}$. **Q**

**(d)** If $\langle \mathcal{F}_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathfrak{F}$, then $\mathcal{F} = \bigcap_{n\in\mathbb{N}} \mathcal{F}_n$ belongs to $\mathfrak{F}$. **P** Let $f_n$, $\langle A^{(n)}_{\alpha\beta}\rangle_{\alpha,\beta\in\mathbb{N}^{\mathbb{N}}}$ witness that $\mathcal{F}_n \in \mathfrak{F}$. Define $f : \mathcal{F} \to \mathbb{N}^{\mathbb{N}}$ by setting

$$f(A)(i) = \max_{n\le i} f_n(A)(i)$$

for $A \in \mathcal{F}$ and $i \in \mathbb{N}$. For $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$ set $(\alpha \vee k)(i) = \max(\alpha(i), k)$ for every $i \in \mathbb{N}$. For $\alpha$, $\gamma \in \mathbb{N}^{\mathbb{N}}$ and sequences $\langle \beta_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$, set

$$C(\alpha, \gamma, \langle \beta_n\rangle_{n\in\mathbb{N}}) = \bigcup_{n\in\mathbb{N}} \bigcap_{k\le\gamma(n)} A_{\alpha\vee k,\beta_n} \in \mathcal{F}.$$

If $\alpha$, $\gamma$, $\gamma' \in \mathbb{N}^{\mathbb{N}}$ and two sequences $\langle \beta_n\rangle_{n\in\mathbb{N}}$, $\langle \beta'_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$ are such that $\gamma \le \gamma'$ and $\beta_n \le \beta'_n$ for every $n$, then

$$C(\alpha, \gamma', \langle \beta'_n\rangle_{n\in\mathbb{N}}) = \bigcup_{n\in\mathbb{N}} \bigcap_{k\le\gamma'(n)} A_{\alpha\vee k,\beta'_n} \subseteq \bigcup_{n\in\mathbb{N}} \bigcap_{k\le\gamma(n)} A_{\alpha\vee k,\beta'_n}$$
$$\subseteq \bigcup_{n\in\mathbb{N}} \bigcap_{k\le\gamma(n)} A_{\alpha\vee k,\beta_n} = C(\alpha, \gamma, \langle \beta_n\rangle_{n\in\mathbb{N}}).$$

If $A \in \mathcal{F}$ and $\alpha \in \mathbb{N}^{\mathbb{N}}$ are such that $f(A) \le \alpha$, then for each $n \in \mathbb{N}$ we have a $\gamma(n) \in \mathbb{N}$ such that $f_n(A) \le \alpha \vee \gamma(n)$, and now there is a $\beta_n \in \mathbb{N}^{\mathbb{N}}$ such that $A \supseteq A_{\alpha\vee\gamma(n),\beta_n}$; so that

$$A \supseteq \bigcup_{n\in\mathbb{N}} A_{\alpha\vee\gamma(n),\beta_n} \supseteq \bigcup_{n\in\mathbb{N}} \bigcap_{k\le\gamma(n)} A_{\alpha\vee k,\beta_n} = C(\alpha, \gamma, \langle \beta_n\rangle_{n\in\mathbb{N}}).$$

Since $\mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ is isomorphic, as partially ordered set, to $\mathbb{N}^{\mathbb{N}}$, this means that $f$ and $C$, suitably re-coded, witness that $\mathcal{F} \in \mathfrak{F}$.

**(e)** Thus every filter on $\mathbb{N}$ of countable type belongs to $\mathfrak{F}$. On the other hand, every filter in $\mathfrak{F}$ has coinitiality at most $\mathfrak{d} = \operatorname{cf} \mathbb{N}^{\mathbb{N}}$. **P** Let $D \subseteq \mathbb{N}^{\mathbb{N}}$ be a cofinal set of cardinal $\mathfrak{d}$. If $\mathcal{F} \in \mathfrak{F}$, let $f$ and $\langle A_{\alpha\beta}\rangle_{\alpha,\beta\in\mathbb{N}^{\mathbb{N}}}$ witness this. If $A \in \mathcal{F}$, there is an $\alpha \in D$ such that $f(A) \le \alpha$; now there are a $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $A \supseteq A_{\alpha\beta}$ and a $\gamma \in D$ such that $\beta \le \gamma$, in which case $A \supseteq A_{\alpha\gamma}$. So $\{A_{\alpha\gamma} : \alpha, \gamma \in D\}$ is coinitial with $\mathcal{F}$ and witnesses that $\operatorname{ci} \mathcal{F} \le \mathfrak{d}$. **Q**

Accordingly every filter on $\mathbb{N}$ of countable type has coinitiality at most $\mathfrak{d}$, as claimed.

**6B Remark** In the notation of Fremlin 08?, §522, the sequential composition $(\mathbb{N}^{\mathbb{N}}, \le, \mathbb{N}^{\mathbb{N}}) \ltimes (\mathbb{N}^{\mathbb{N}}, \le, \mathbb{N}^{\mathbb{N}})$ is the supported relation $(\mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}^{\mathbb{N}}}, R, \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}})$, where $R$ is the relation

$$\{((\alpha, h), (\beta, \gamma)) : \alpha,\ \beta,\ \gamma \in \mathbb{N}^{\mathbb{N}},\ h \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}^{\mathbb{N}}},\ \alpha \le \beta \text{ and } h(\beta) \le \gamma\}.$$

Suppose that $\mathcal{F}$ is a filter in the family $\mathfrak{F}$ of the proof of 6A, and $\mathcal{I}$ is the corresponding ideal $\{\mathbb{N}\setminus A : A \in \mathcal{F}\}$. Let $f$, $\langle A_{\beta\gamma}\rangle_{\beta,\gamma\in\mathbb{N}^{\mathbb{N}}}$ witness that $\mathcal{F} \in \mathfrak{F}$. For $E \in \mathcal{I}$, choose $h_E \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}^{\mathbb{N}}}$ such that $\mathbb{N}\setminus E \supseteq A_{\beta,h_E(\beta)}$ whenever $f(\mathbb{N} \setminus E) \le \beta \in \mathbb{N}^{\mathbb{N}}$. Define $\phi : \mathcal{I} \to \mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}^{\mathbb{N}}}$ and $\psi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathcal{I}$ by setting

$$\phi(E) = (f(\mathbb{N} \setminus E), h_E), \quad \psi(\beta, \gamma) = \mathbb{N} \setminus A_{\beta\gamma}$$

for $E \in \mathcal{I}$ and $\beta,\gamma \in \mathbb{N}^{\mathbb{N}}$. Now suppose that $E \in \mathcal{I}$ and $\beta,\gamma \in \mathbb{N}^{\mathbb{N}}$ are such that $(\phi(E),(\beta,\gamma)) \in R$. Then $f(\mathbb{N}\setminus E) \le \beta$ and $h_E(\beta) \le \gamma$, so

$$\mathbb{N}\setminus E \supseteq A_{\beta,h_E(\beta)} \supseteq A_{\beta\gamma}$$

and $E \subseteq \psi(\beta,\gamma)$. But this means that $(\phi,\psi)$ is a Galois-Tukey connection from $(\mathcal{I},\subseteq,\mathcal{I})$ to $(\mathbb{N}^{\mathbb{N}},\le,\mathbb{N}^{\mathbb{N}}) \ltimes (\mathbb{N}^{\mathbb{N}},\le,\mathbb{N}^{\mathbb{N}})$. This implies in particular that

$$\begin{aligned}\operatorname{ci}\mathcal{F} = \operatorname{cf}\mathcal{I} &= \operatorname{cov}(\mathcal{I},\subseteq,\mathcal{I}) \le \operatorname{cov}((\mathbb{N}^{\mathbb{N}},\le,\mathbb{N}^{\mathbb{N}}) \ltimes (\mathbb{N}^{\mathbb{N}},\le,\mathbb{N}^{\mathbb{N}}))\\ &= \operatorname{cov}(\mathbb{N}^{\mathbb{N}},\le,\mathbb{N}^{\mathbb{N}})\cdot\operatorname{cov}(\mathbb{N}^{\mathbb{N}},\le,\mathbb{N}^{\mathbb{N}})\end{aligned}$$

(the cardinal product)

$$= \operatorname{cf}\mathbb{N}^{\mathbb{N}}\cdot\operatorname{cf}\mathbb{N}^{\mathbb{N}} = \operatorname{cf}\mathbb{N}^{\mathbb{N}}.$$

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