Filters of countable type

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Following Mauldin Preiss & Weiszäcker 83, I work through the theory of ‘filters of countable type’ (1A). Results which may be new are that filters of countable type have the Bolzano-Weierstrass property (3C), the density filter is not of countable type (3D), there are c isomorphism classes of filters of countable type (5D), and every filter of countable type has coincidence at most d (6A).

1 Basics

1A Definitions (a) A filter on a set X is of countable type if it belongs to the smallest class of filters on X containing the principal ultrafilters and closed under the operations of countable intersection and increasing countable union.

(b) If X is a set, \( \langle F_i \rangle_{i \in I} \) is a non-empty family of filters on X, and \( F \) is a filter on I, I will write \( \lim_{n \to F} F_i \) for the filter \( \{ A : A \subseteq X, \{ i : i \in I, A \in F_i \} \in F \} \).

Note that if \( X = I = \mathbb{N} \) and \( F_n \) is the principal filter generated by \( \{ n \} \) for each n, then \( \lim_{n \to F} F_n = F \).

(c) Let \( F \) be a filter on a set X. For \( Y \subseteq X \), set

\[ F|Y = \{ A \cap Y : A \in F \} = \{ B : B \subseteq Y, B \cup (X \setminus Y) \in F \} \]

I will call \( F|Y \) the trace of \( F \) on \( Y \). If \( X \setminus Y \in F \), then \( F|Y = \emptyset \); otherwise, \( F|Y \) is a filter on \( Y \).

(d) I will write \( F_{\text{Fr}} \) for the Fréchet filter on \( \mathbb{N} \), the smallest free filter on \( \mathbb{N} \).

1B Proposition Let \( \mathcal{F} \) be a family of filters on a set X. Then the following are equiveridical:

(i) \( \bigcap_{n \in \mathbb{N}} F_n \in \mathcal{F} \) for every sequence \( \langle F_n \rangle_{n \in \mathbb{N}} \) in \( \mathcal{F} \), and \( \bigcup_{n \in \mathbb{N}} F_n \in \mathcal{F} \) for every non-decreasing sequence \( \langle F_n \rangle_{n \in \mathbb{N}} \) in \( \mathcal{F} \);

(ii) \( \lim_{n \to F} F_n \in \mathcal{F} \) whenever \( \langle F_n \rangle_{n \in \mathbb{N}} \) is a sequence in \( \mathcal{F} \);

(iii) \( \lim_{n \to \mathcal{F}} F_n \in \mathcal{F} \) whenever \( \langle F_n \rangle_{n \in \mathbb{N}} \) is a sequence in \( \mathcal{F} \) and \( \mathcal{F} \) is a filter of countable type on \( \mathbb{N} \).

proof (i)⇒(ii) Assume (i). Let \( \mathcal{G} \) be the set of filters \( \mathcal{G} \) on \( \mathbb{N} \) such that \( \lim_{n \to \mathcal{G}} F_n \in \mathcal{F} \).

(a) If \( \mathcal{G} = \{ m \} \), then \( \lim_{n \to \mathcal{G}} F_n = F_m \in \mathcal{F} \); so \( \mathcal{G} \in \mathcal{G} \).

(b) If \( \langle \mathcal{G}_m \rangle_{m \in \mathbb{N}} \) is a sequence in \( \mathcal{G} \) with intersection \( \mathcal{G} \), then

\[ \lim_{n \to \mathcal{G}} F_n = \{ A : \{ n : A \in F_n \} \in \mathcal{G} \} = \bigcap_{m \in \mathbb{N}} \{ A : \{ n : A \in F_n \} \in \mathcal{G}_m \} = \bigcap_{m \in \mathbb{N}} \lim_{n \to \mathcal{G}_m} F_n \]

is the intersection of a sequence in \( \mathcal{F} \), so belongs to \( \mathcal{F} \), and \( \mathcal{G} \in \mathcal{G} \).

(c) If \( \langle \mathcal{G}_m \rangle_{m \in \mathbb{N}} \) is a non-decreasing sequence in \( \mathcal{G} \) with union \( \mathcal{G} \), set \( F'_m = \lim_{n \to \mathcal{G}_m} F_n \in \mathcal{F} \) for each m. If \( k \leq m \) then

\[ F'_m = \{ A : \{ n : A \in F_n \} \in \mathcal{G}_k \} \subseteq \{ A : \{ n : A \in F_n \} \in \mathcal{G}_m \} = F'_m, \]

so \( \langle F'_m \rangle_{m \in \mathbb{N}} \) is a non-decreasing sequence in \( \mathcal{F} \) and its union belongs to \( \mathcal{F} \). Now

\[ \lim_{n \to \mathcal{G}} F_n = \{ A : \{ n : A \in F_n \} \in \mathcal{G} \} = \bigcup_{m \in \mathbb{N}} \{ A : \{ n : A \in F_n \} \in \mathcal{G}_m \} = \bigcup_{m \in \mathbb{N}} F'_m \in \mathcal{F} \].

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Then F of countable type on I.

We can induce on ct.

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Proof

set of principal ultrafilters on X.

So both clauses of (i) are true.

1C Countable-type levels

Let X be any non-empty set. Define (F_\xi)_{\xi \in \text{Ord}} as follows. F_0 is the set of principal ultrafilters on X. For ordinals \( \xi > 0 \), F_\xi is the set of filters on X expressible in the form

\[
\lim_{n \to F_{n \xi}} \mathcal{F}_{k_n} = \{ A : n : A \in F_{k_n} \} \in F_{n \xi} \} = \{ A : \exists m \in N, A \in F_{k_n} \text{ for every } n \geq m \}
\]

\[
\{ A : \exists m \in N, A \in F_n \text{ for every } n \geq m \} = \bigcap_{n \in N} F_n.
\]

(b) If \( (F_n)_{n \in N} \) is non-decreasing, then F contains

\[
\lim_{n \to F_n} F_n = \{ A : n : A \in F_n \} \in F_{n \xi}
\]

\[
= \{ A : \exists m \in N, A \in F_n \text{ for every } n \geq m \} = \bigcup_{n \in N} F_n.
\]

So both clauses of (i) are true.

1D Lemma

If \( F, G \) are filters on X, Y respectively, F is of countable type and G \( \leq_{\text{RK}} F \), then G is of countable type and ct F \( \leq_{\text{ct}} F \).

Proof

We can induce on ct F, because if \( (F_n)_{n \in N} \) is any sequence of filters on X, F is any filter on N, and g : X \to Y is a function, then

\[
g(\lim_{n \to F_n} F_n) = \{ A : n : g^{-1}[A] \in F_n \} \in F
\]

\[
= \{ A : n : g(\lim F_n) \} = \lim_{n \to F_n} g[F_n] = \lim F_n.
\]

1E Proposition

Let \( (F_i)_{i \in I} \) be a non-empty family of filters of countable type on a set X, and F a filter of countable type on I. Then \( \lim_{i \to F} F_i \) is a filter of countable type.

Proof

We can induce on ct F, because if \( (G_n)_{n \in N} \) is any sequence of filters on I and F = lim_{n \to F_n} G_n, then

\[
\lim_{i \to F} F_i = \{ A : \{ i : A \in F_i \} \in F \}
\]

\[
= \{ A : n : \{ i : A \in F_i \} \in G_n \} \in F_{n \xi}
\]

\[
= \{ A : \{ n : A \in \lim F_i \} \in F_{n \xi} \} = \lim_{n \to F_n, 1 \to G_n} F_i.
\]

1F Proposition

(a) Suppose that \( (F_n)_{n \in N} \) is a sequence of filters on a set X, and Y \( \subset X \) is such that \( X \setminus Y \) does not belong to \( F = \lim_{n \to F_n} F_n \). Let \( (n_k)_{k \in N} \) enumerate the infinite set \( D = \{ n : X \setminus Y \notin F_n \} \).

Then \( F[Y] = \lim_{k \to F_{n_k}} F_{n_k}[Y] \).
(b) Let $\mathcal{F}$ be a filter on a set $X$ and $Y$ a subset of $X$ such that $X \setminus Y \notin \mathcal{F}$. If $\mathcal{F}$ is of countable type, so is $\mathcal{F}|Y$, and $\text{ct}(\mathcal{F}|Y) \leq \text{ct} \mathcal{F}$.

**proof (a)** For $B \subseteq Y$,

$$B \in \mathcal{F}|Y \iff B \cup (X \setminus Y) \in \mathcal{F}$$

$$\iff \text{there is an } m \in \mathbb{N} \text{ such that } B \cup (X \setminus Y) \in \mathcal{F}_n \text{ for every } n \geq m$$

$$\iff \text{there is an } m \in \mathbb{N} \text{ such that } B \cup (X \setminus Y) \in \mathcal{F}_n$$

$$\text{whenever } n \in D \text{ and } n \geq m$$

$$\iff \text{there is an } m \in \mathbb{N} \text{ such that } B \cup (X \setminus Y) \in \mathcal{F}_{nk} \text{ for every } k \geq m$$

$$\iff \text{there is an } m \in \mathbb{N} \text{ such that } B \in \mathcal{F}_{nk}|Y \text{ for every } k \geq m$$

$$\iff B \in \lim_{k \to \mathcal{F}_{nk}} \mathcal{F}_{nk}|Y. \quad \Box$$

(b) The result is now an easy induction on $\text{ct} \mathcal{F}$.

1G **Proposition** Let $X$ be a countably compact topological space, $I$ a non-empty set and $\mathcal{F}$ a filter on $I$ of countable type. Then $f[[\mathcal{F}]]$ has a cluster point in $X$ for every $f : I \to X$.

**proof (a)** If $f : I \to X$ is a function, $(\mathcal{F}_n)_{n \in \mathbb{N}}$ is a sequence of filters on $I$, and $x_n$ is a cluster point of $f[[\mathcal{F}_n]]$ for every $n \in \mathbb{N}$, then any cluster point $x$ of $(x_n)_{n \in \mathbb{N}}$ will also be a cluster point of $f[[\lim_{n \to \mathcal{F}_n} \mathcal{F}_n]]$.

**P** If $G$ is an open set containing $x$ and $A \in f[[\lim_{n \to \mathcal{F}_n} \mathcal{F}_n]]$, then there is an $m \in \mathbb{N}$ such that $f^{-1}[A] \in \mathcal{F}_n$ for every $n \geq m$. Now there is an $n \geq m$ such that $x_n \in G$, in which case $f(f^{-1}[A]) \in f[[\mathcal{F}_n]]$ and

$$\emptyset \notin G \cap f[f^{-1}[A]] \subseteq G \cap A.$$ 

As $G$ and $A$ are arbitrary, $x$ is a cluster point of $f[[\lim_{n \to \mathcal{F}_n} \mathcal{F}_n]]$.

(b) Inducing on $\text{ct} \mathcal{F}$ we get the result.

1H **Theorem** Let $X$ be a topological space and $A$ a subset of $X$.

(a) If $\xi \in \omega_1$ and $x \in A_\xi^\omega$, then there are an $f : \mathbb{N} \to A$ and a filter $\mathcal{F}$ on $\mathbb{N}$, of countable-type level at most $\xi$, such that $f[[\mathcal{F}]] \to x$.

(b) If $X$ is sequentially compact, $I$ is a non-empty set, $f : I \to A$ is a function, and $\mathcal{F}$ is a filter on $I$ of countable-type level $\xi$, then

(i) there is a filter $\mathcal{G}$ on $I$, of countable-type level at most $\xi$, such that $\mathcal{G} \supseteq \mathcal{F}$ and $f[[\mathcal{G}]]$ has a limit; 

(ii) if $f[[\mathcal{F}]] \to x$ and $X$ is Hausdorff then $x \in A_\xi^\omega$.

**proof (a)** Induce on $\xi$. If $\xi = 0$, then $x \in A$ and we can take $f$ to be the constant function with value $x$ and $\mathcal{F}$ the principal ultrafilter generated by $\{0\}$. For the inductive step to $\xi > 0$, let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\bigcup_{n < \xi} A_n^\omega$ converging to $x$. For each $n \in \mathbb{N}$ let $f_n : \mathbb{N} \to A$ and $\mathcal{F}_n$ be such that $\text{ct} \mathcal{F}_n < \xi$ and $f_n[[\mathcal{F}_n]] \to x_n; \text{ set } g_n(i) = 2^n(2i + 1) - 1 \text{ for } i \in \mathbb{N}; \text{ set } \mathcal{G}_n = g_n[[\mathcal{F}_n]], \text{ so that } \text{ct} \mathcal{G}_n < \xi$. Set $\mathcal{F} = \lim_{n \to \mathcal{F}_n} \mathcal{G}_n$, so that $\text{ct} \mathcal{F} \leq \xi$; define $f : \mathbb{N} \to A$ by setting $f(g_n(i)) = f_n(i)$ for $n, i \in \mathbb{N}$.

For each $n \in \mathbb{N}$, $f[[\mathcal{G}_n]] = f_n[[\mathcal{F}_n]] \to x_n$. So if $H$ is any open set containing $x$,

$$\{n : f^{-1}[H] \in \mathcal{G}_n\} \supseteq \{n : x_n \in H\} \in \mathcal{F}_H,$$

and $f^{-1}[H] \in \mathcal{F}$. Accordingly $f[[\mathcal{F}]] \to x$ and the induction proceeds.

(b) Again induce on $\xi$. If $\xi = 0$ then $\mathcal{F}$ is the principal ultrafilter on $\mathbb{N}$ generated by $\{i\}$ for some $i$; take $\mathcal{G} = \mathcal{F}$. Then $f[[\mathcal{G}]]$ is the principal ultrafilter on $X$ generated by $f(i)$, and converges to $f(i)$. If the topology on $X$ is $T_1$ and $f[[\mathcal{F}]] \to x$, then $x = f(i) \in A = A_\xi^\omega$. So the induction starts.

For the inductive step to $\xi > 0$, express $\mathcal{F}$ as $\lim_{n \to \mathcal{F}_n} \mathcal{F}_n$ where $\text{ct} \mathcal{F}_n < \xi$ for every $n$. For each $n \in \mathbb{N}$ we have a filter $\mathcal{G}_n \supseteq \mathcal{F}_n$ such that $\text{ct} \mathcal{G}_n < \xi$ and $f[[\mathcal{G}_n]]$ has a limit $x_n$. Say $\langle x_n \rangle_{n \in \mathbb{N}}$ has a convergent subsequence; suppose that $(\mathcal{G}_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence in $\mathcal{N}$ such that $(\mathcal{G}_n)_{n \in \mathbb{N}}$ has a limit $z$ in $X$. Set $\mathcal{G} = \lim_{n \to \mathcal{F}_n} \mathcal{G}_n$; then $\mathcal{G} \leq \xi$. If $J \in \mathcal{F}$, then there is an $m \in \mathbb{N}$ such that $J \in \mathcal{F}_n$ for every $n \geq m$, and now
\[ \{k : J \in G_n \} \supseteq \{k : J \in F_n \} \supseteq \{k : n_k \geq m \} \in F_{\mathcal{I}}, \]

so \( J \in \mathcal{G} \). Thus \( \mathcal{F} \subseteq \mathcal{G} \). If \( H \) is an open set including \( z \), then

\[ \{k : f^{-1}[H] \in G_n \} \supseteq \{k : x_{n_k} \in H \} \in F_{\mathcal{I}}, \]

so \( f^{-1}[H] \in \mathcal{G} \); thus \( \mathcal{G} \rightarrow z \).

This deals with (i). As for (ii), if \( f[\mathcal{F}] \rightarrow x \) then \( f[\mathcal{G}] \rightarrow x \); as \( X \) is Hausdorff, \( x = z = \lim_{k \to \infty} x_{n_k} \); also the inductive hypothesis now tells us that

\[ x_{n_k} = \lim f[\mathcal{G}_{n_k}] \in \bigcup_{\eta < \xi} A_\eta^z \]

for every \( k \), so \( x \in A_\xi^z \) and the induction proceeds.

### 2 Borel filters and the Fatou property

#### 2A Proposition

If \( I \) is a non-empty set and \( \mathcal{F} \) is a filter of countable type on \( I \), then \( \mathcal{F} \) is a Baire subset of \( \mathcal{P}I \), therefore a Borel subset of \( \mathcal{P}I \).

#### 2B Proposition

Let \( \mathcal{F} \) be a filter on \( \mathbb{N} \) which is a Borel subset of \( \mathcal{P} \mathbb{N} \), \( X \) a set and \( \Sigma \) a \( \sigma \)-algebra of subsets of \( X \).

(a) If \( \{f_n\}_{n \in \mathbb{N}} \) is a sequence of \( \Sigma \)-measurable functions from \( X \) to \([\mathbb{R}, \infty)\), then \( \liminf_{n \to \mathcal{F}} f_n \) is \( \Sigma \)-measurable.

(b) If \( Y \) is a Polish space and \( \{f_n\}_{n \in \mathbb{N}} \) is a sequence of \( \Sigma \)-measurable functions from \( X \) to \( Y \), then \( \lim_{n \to \mathcal{F}} f_n \) is \( \Sigma \)-measurable and has domain in \( \Sigma \).

**proof (a)**  
Set \( f = \liminf_{n \to \mathcal{F}} f_n \). For \( x \in X \), \( \alpha \in \mathbb{R} \) set \( J(x, \alpha) = \{n : x(n) \geq \alpha\} \). Then, for given \( \alpha \), \( x \mapsto J(x, \alpha) : X \rightarrow \mathcal{P} \mathbb{N} \) is \( \Sigma \)-measurable, so \( F_\alpha = \{x : J(x, \alpha) \in \mathcal{F}\} \) belongs to \( \Sigma \). Now

\[
\{x : f(x) > \alpha\} = \{x : \text{there is an } I \in \mathcal{F} \text{ such that } \inf_{n \in I} f_n(x) > \alpha\} \\
= \{x : \text{there are an } I \in \mathcal{F} \text{ and a rational } q > \alpha \text{ such that } I \subseteq J(x, q)\} \\
= \{x : \text{there is a rational } q > \alpha \text{ such that } J(x, q) \in \mathcal{F}\} \\
= \bigcup_{q \in \mathbb{Q}, q > \alpha} F_q \in \Sigma.
\]

As \( \alpha \) is arbitrary, \( f \) is measurable.

(b) Let \( \rho \) be a complete metric on \( Y \) inducing the topology of \( Y \). Let \( \langle y_i \rangle_{i \in \mathbb{N}} \) run over a dense subset of \( Y \). (I am passing over the trivial case \( X = Y = \emptyset \). For \( i, j \in \mathbb{N} \), \( F_{ij} = \{x : \limsup_{n \to \mathcal{F}} \rho(f_n(x), y_i) \leq 2^{-j}\} \) belongs to \( \Sigma \), by (a), inverted. Setting \( f = \lim_{n \to \mathcal{F}} f_n \), \( \text{dom } f = \bigcap_{j \in \mathbb{N}} \bigcup_{i \in \mathbb{N}} F_{ij} \in \Sigma \). Moreover, for \( i, j \in \mathbb{N} \), \( \{x : \rho(f(x), y_i) \leq 2^{-j}\} = F_{ij} \cap \text{dom } f \) belongs to \( \Sigma \), so \( f \) is measurable.

#### 2B Definition

I will say that a filter \( \mathcal{F} \) on \( \mathbb{N} \) has the **Fatou property** if whenever \( (X, \Sigma, \mu) \) is a measure space and \( \{f_n\}_{n \in \mathbb{N}} \) is a sequence of non-negative functions defined on \( X \), then \( \int \liminf_{n \to \mathcal{F}} f_n d\mu \leq \liminf_{n \to \mathcal{F}} \int f_n d\mu \).

#### 2C Lemma

If \( \mathcal{F} \) is a filter on \( \mathbb{N} \), the following are equiveridical:

(i) \( \mathcal{F} \) has the Fatou property;

(ii) whenever \( (X, \Sigma, \mu) \) is a probability space, \( \langle E_n \rangle_{n \in \mathbb{N}} \) is a sequence in \( \Sigma \), and \( X = \bigcup_{i \in \mathcal{F}} \bigcap_{n \in I} E_n \), then \( \lim_{n \to \mathcal{F}} \mu E_n = 1 \).

**proof**  
(i)\(\Rightarrow\)(ii) is elementary. In the reverse direction, suppose that \( \mathcal{F} \) does not have the Fatou property. Let \( (X, \Sigma, \mu) \) be a measure space and \( \{f_n\}_{n \in \mathbb{N}} \) a sequence of non-negative functions defined on \( X \) such that \( \int \liminf_{n \to \mathcal{F}} f_n d\mu > \liminf_{n \to \mathcal{F}} \int f_n d\mu \). Take \( \alpha \) such that \( \int \liminf_{n \to \mathcal{F}} f_n d\mu > \alpha > \liminf_{n \to \mathcal{F}} \int f_n d\mu \); set \( J = \{n : \int f_n d\mu \leq \alpha\} \); then \( J \) meets every member of \( \mathcal{F} \). For \( n \in J \), let \( g_n : X \rightarrow [0, \infty] \) be a \( \Sigma \)-measurable function such that \( f_n \leq a.e., g_n \) and \( \int g_n d\mu \leq \alpha \). Then \( G = \{x : \sup_{n \in J} g_n(x) > 0\} \) is a countable union of sets of finite measure, and there is a negligible set \( H \in \Sigma \) such that \( f_n(x) \leq g_n(x) \) whenever \( x \in X \setminus H \) and \( n \in J \). Note that this means that \( \liminf_{n \to \mathcal{F}} f_n(x) = 0 \) whenever \( x \notin G \cup H \), so that \( \int_{G \cup H} \liminf_{n \to \mathcal{F}} f_n d\mu > \alpha \).
Let $\mathcal{G}$ be the filter $\{I \cap J : I \in \mathcal{F}\}$ on $J$. Then
\[
\liminf_{n \to \mathcal{G}} f_n(x) = \sup_{I \in \mathcal{F}} \inf_{n \in I \cap J} f_n(x) \geq \liminf_{n \to \mathcal{F}} f(x)
\]
for every $x$, so
\[
\int_{G \cup H} \liminf_{n \to \mathcal{G}} g_n \, d\mu \geq \int_{G \cup H} \liminf_{n \to \mathcal{G}} f_n \, d\mu \geq \int_{G \cup H} \liminf_{n \to \mathcal{F}} f_n \, d\mu > \alpha.
\]
Let $\lambda$ be the product measure on $(G \cup H) \times \mathbb{R}$, and consider the ordinate sets $W_n = \{(x, \alpha) : x \in G \cup H, \ 0 \leq \alpha < g_n(x)\}$ for $n \in J$. Set
\[
W = \bigcup_{I \in \mathcal{G}} \bigcap_{n \in I} W_n = \{(x, \alpha) : x \in G \cup H, \ \alpha \geq 0, \ \{n : n \in J, \ \alpha < g_n(x)\} \in \mathcal{G}\};
\]
setting $g = \liminf_{n \to \mathcal{G}} g_n$,
\[
\{(x, \alpha) : x \in G \cup H, \ 0 \leq \alpha < g(x)\} \subseteq W \subseteq \{(x, \alpha) : x \in G \cup H, \ 0 \leq \alpha \leq g(x)\}.
\]
So
\[
\lambda^* W = \int_{G \cup H} \liminf_{n \to \mathcal{G}} g_n \, d\mu > \alpha
\]
(Fremlin 01, 252Yh).

There is therefore a set $V \subseteq (G \cup H) \times \mathbb{R}$ such that $\lambda V < \infty$ and $\lambda^*(V \cap W) > \alpha$. Let $\nu$ be the subspace measure on $V \cap W$. Set
\[
V_n = V \cap W \cap W_n \text{ if } n \in J
\]
\[
= V \cap W \text{ if } n \in \mathbb{N} \setminus J.
\]
Then
\[
\liminf_{n \to \mathcal{F}} \nu V_n = \sup_{I \in \mathcal{F}} \inf_{n \in I} \nu V_n \leq \sup_{n \in J} \nu V_n
\]
\[
\leq \sup_{n \in J} \lambda W_n = \sup_{n \in J} \int g_n \, d\mu \leq \alpha.
\]
On the other hand,
\[
\bigcup_{I \in \mathcal{F}} \bigcap_{n \in I} V_n = \bigcup_{I \in \mathcal{F}} \bigcap_{n \in I \cap J} V \cap W \cap W_n = V \cap W \cap \bigcup_{I \in \mathcal{G}} \bigcap_{n \in I} W_n = V \cap W
\]
and $\nu(V \cap W) = \lambda^*(V \cap W) > \alpha$. Moving to a normalization of $\nu$, we see that (ii) is false.

2D Proposition Let $\langle \mathcal{F}_n \rangle_{n \in \mathbb{N}}$ be a sequence of filters on $\mathbb{N}$ with the Fatou property, and $\mathcal{F}$ a filter with the Fatou property. Then $\mathcal{G} = \liminf_{n \to \mathcal{F}} \mathcal{F}_n$ has the Fatou property.

proof (a) The point is that if $\langle t_n \rangle_{n \in \mathbb{N}}$ is any sequence in $[0, \infty]$ then
\[
\liminf_{n \to \mathcal{G}} t_n = \liminf_{n \to \mathcal{F}} \liminf_{i \to \mathcal{G}} t_i.
\]

P For any $\alpha \in [0, \infty[$,
\[
\liminf_{n \to \mathcal{G}} t_n > \alpha \iff \exists \beta > \alpha, \ \{n : t_n \geq \beta\} \in \mathcal{G}
\]
\[
\iff \exists \beta > \alpha, \ A \in \mathcal{F}, \ \{i : t_i \geq \beta\} \in \mathcal{F}_n \forall n \in A
\]
\[
\iff \exists \beta > \alpha, \ A \in \mathcal{F}, \ \liminf_{i \to \mathcal{G}_n} t_i \geq \beta \forall n \in A
\]
\[
\iff \liminf_{n \to \mathcal{F}} \liminf_{i \to \mathcal{G}} t_i > \alpha. \ \Box
\]

(b) So if we have a measure space $(X, \Sigma, \mu)$ and a sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ of non-negative functions defined on $X$,
\[
\int \liminf_{n \to \mathcal{G}} f_n = \int \liminf_{n \to \mathcal{F}} \liminf_{i \to \mathcal{G}_n} f_i \leq \liminf_{n \to \mathcal{F}} \int \liminf_{i \to \mathcal{G}} f_i
\]
\[
\leq \liminf_{i \to \mathcal{G}} \int f_i = \liminf_{n \to \mathcal{G}} \int f_n.
\]
2E Corollary A filter on $\mathbb{N}$ of countable type has the Fatou property.

**proof** Principal ultrafilters on $\mathbb{N}$, and the Fréchet filter, have the Fatou property; use 1B(ii).

2E Example There is a filter $F$ on $\mathbb{N}$, an $F_\sigma$ subset of $\mathcal{P}\mathbb{N}$, which does not have the Fatou property.

**construction** Let $\mu$ be the usual measure on $X = \{0, 1\}^\mathbb{N}$. Let $(E_n)_{n \in \mathbb{N}}$ enumerate the family of sets of the form $\{x : x \in X, x(k) \in J\}$ where $k \in \mathbb{N}$ and $J \subseteq \{0, 1\}^k$ has $k + 1$ members. Then $\lim_{n \to \infty} \mu E_n = 1$. For $x \in X$ set $\phi(x) = \{n : x \in E_n\}$; then $\phi : X \to \mathcal{P}\mathbb{N}$ is continuous so $K = \phi[X]$ is compact. Set $F = \bigcup_{k \in \mathbb{N}} \{a \cup (b_0 \cap \ldots \cap b_k \setminus k) : a \subseteq \mathbb{N}, b_0, \ldots, b_k \in K\}$; then $F$ is an $F_\sigma$ set. If $x_0, \ldots, x_k \in X$, there is an $n \geq k$ such that $x_0, \ldots, x_k$ all belong to $E_n$; accordingly $\emptyset \notin F$ and $F$ is a filter on $\mathbb{N}$.

Since $F$ contains all cofinite subsets of $\mathbb{N}$, $\lim_{n \to \infty} \mu E_n = \lim_{n \to \infty} \mu E_n = 0$.

If $x \in X$, then $\phi(x) \in F$ and $x \in \bigcap_{n \in \phi(x)} E_n \subseteq \bigcup_{I \in F} \bigcap_{n \in I} E_n$.

So 2C(ii) is false and $F$ does not have the Fatou property.

3 The Bolzano-Weierstrass property

3A Definition (Filipów Mrożek Recław & Szuca 07) A filter $F$ on a set $X$ has the Bolzano-Weierstrass property if for every $g : X \to [0, 1]$ there is an $I \subseteq X$, meeting every member of $F$, such that $(g[I])[F[I]]$ is convergent.

3B Basic facts

(a) Any filter which is not free has the Bolzano-Weierstrass property. $F_{Fr}$ has the Bolzano-Weierstrass property. If $F$ is a filter on $X$ with the Bolzano-Weierstrass property, $Z$ is a compact metrizable space and $g : X \to Z$ is a function, there is an $I \subseteq X$, meeting every member of $F$, such that $(g[I])[F[I]]$ is convergent (Filipów Mrożek Recław & Szuca 07, §2.3).

(b) (Fridy 93, or Filipów Mrożek Recław & Szuca 07, §3) Let $F_d$ be the filter of subsets of $\mathbb{N}$ with asymptotic density 1. Then $F_d$ does not have the Bolzano-Weierstrass property. P Let $g = (g(n))_{n \in \mathbb{N}}$ be a sequence in $[0, 1]$ which is equidistributed for Lebesgue measure $\mu$. Then $d^*(I) \leq \mu[I]$ for every $I \subseteq \mathbb{N}$. If $I \subseteq \mathbb{N}$ meets every member of $F_d$, then $d^*(I) > 0$. Let $m \geq 1$ be such that $\frac{1}{m} < d^*(I)$, and for $k \leq m$ set $J_k = \{n : n \in I, \frac{k}{m} \leq g(n) < \frac{k+1}{m}\}$.

Then $d^*(I \cap J_k) \leq \frac{\mu[I \cap J_k]}{\mu[I]} \leq \frac{2}{m} < d^*(I)$ so $I \cap J_k \notin F_d[I]$ for every $k$. But this means that $(g[I])[F_d[I]]$ cannot contain any interval of the form $[0, 1] \cap \left(\frac{k-1}{m}, \frac{k+1}{m}\right]$ and cannot be convergent. Q

(c) For $I \subseteq \mathbb{N}$ write $d^*_n(I)$ for its Banach upper density, that is,

$$d^*_n(I) = \inf_{m \geq 1} \frac{1}{m} \#(I \cap [k, k + m]) = \lim_{m \to \infty} \sup_{k \in \mathbb{N}} \frac{1}{m} \#(I \cap [k, k + m])$$

(Fremlin 05). Let $F_r$ be the Banach density filter, that is, the filter of sets $I \subseteq \mathbb{N}$ such that $d^*_n(N \setminus N) = 0$. Then $F_r$ does not have the Bolzano-Weierstrass property. P Argue as in (b), but with a well-distributed sequence. Q

3C Theorem Let $G$ be a filter on a set $X$, and suppose that there is a filter $F$ on $X$, of countable type, such that $G \subseteq F$. Then $G$ has the Bolzano-Weierstrass property.

**proof** Induce on the countable-type level $ct[F]$ of $F$. 

(a) If $\mathcal{F}$ is the principal filter generated by $\{x\}$, then $\mathcal{G}$ is not free, so has the Bolzano-Weierstrass property.

(b) If $ct\mathcal{F}=\xi>0$, there is a sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of filters on $X$, all of countable-type level less than $\xi$, such that $\mathcal{F} = \lim_{n\to\infty} \mathcal{F}_n$. Let $g : X \to [0, 1]$ be any function.

(i) Suppose that there are a set $I \subseteq X$ and an $n \in \mathbb{N}$ such that $I$ meets every member of $\mathcal{F}_n$ and $\mathcal{G}[I] \subseteq \mathcal{F}_n[I]$. In this case, $\mathcal{G}[I] \neq \mathcal{F}[I]$, so $\mathcal{G}[I]$ is a filter, while $ct(\mathcal{F}_n[I]) \leq ct\mathcal{F}_n < \xi$ (1Fb). So $\mathcal{G}[I]$ has the Bolzano-Weierstrass property, by the inductive hypothesis. Applying this to the function $g[I]$, we see that there is a $J \subseteq I$, meeting every member of $\mathcal{G}[I]$, such that $(g[J])[\mathcal{G}[I]](J)$ converges; but in this case $J$ meets every member of $\mathcal{G}$, so $\mathcal{G}[J] = \mathcal{G}[I][J]$ and $(g[J])[\mathcal{G}[I]](J)$ converges.

(ii) Otherwise, choose $(I_k)_{k\in\mathbb{N}}$ and $(\eta_k)_{k\in\mathbb{N}}$ inductively, as follows. $I_0 = \mathbb{N}$. Given that $I_k$ meets every set in $\mathcal{F}$, there must be an $n_k$, greater than $n_i$ for any $i < k$, such that $I_k$ meets every member of $\mathcal{F}_{n_k}$. In this case, since (i) is false, there is a $J_k \in (\mathcal{G}[I_k] \setminus \mathcal{F}_{n_k}[I_k])$. As $J_k \cup (X \setminus I_k) \in \mathcal{G} \subseteq \mathcal{F}$, $J_k = I_k \cap (J_k \cup (X \setminus I_k))$ meets every member of $\mathcal{F}$; let $I_{k+1} \subseteq J_k$ be a set meeting every member of $\mathcal{F}$ such that $\text{diam } g[I_{k+1}] \leq 2^{-k-1}$. Continue.

At the end of the induction, set $I = \bigcup_{k\in\mathbb{N}} I_k \setminus J_k$. As $I_{k+1} \subseteq J_k \subseteq I$ for every $k$, $I \cap I_{k+1} = (I \cap I_k) \setminus (I_k \setminus J_k)$ for every $k$. Now, for each $k$, $J_k \notin \mathcal{F}_n[I_k]$ so $I_k \setminus J_k$ meets every member of $\mathcal{F}_{n_k}[I_k]$ and $I$ meets every member of $\mathcal{F}_{n_k}$. Consequently $I$ meets every member of $\mathcal{F} \supseteq \mathcal{G}$. On the other hand, $\mathbb{N} \setminus (I_k \setminus J_k) \in \mathcal{G}$ for every $k$, so (inducing on $k$) $I \cap I_k \in \mathcal{G}[I]$ for every $k$. Since $\text{diam } g[I_k] \leq 2^{-k}$ for every $k$, $(g[I])[\mathcal{G}[I]]$ is Cauchy, therefore convergent.

(iii) Thus in either case the condition of Definition 3A is satisfied. As $g$ is arbitrary, $\mathcal{G}$ has the Bolzano-Weierstrass property, and the proposition continues.

3D Corollary Neither the density filter nor the Banach density filter is of countable type.

3E Proposition $\mathcal{F}_d$ has the Fatou property.

**proof (a)** Let $(X, \Sigma, \mu)$ be a probability space and $(E_n)_{n\in\mathbb{N}}$ a sequence in $\Sigma$ such that $\bigcup_{I \in \mathcal{F}_d} \bigcap_{n \in \mathbb{N}} E_n = X$; set $\alpha = \lim inf_{n\to\infty} \mu E_n$. Take any $\epsilon > 0$. Set $I = \{ n : \mu E_n \leq \alpha + \epsilon \}$; then $d^*(I) > 0$. Take any $\eta \in ]0, \frac{1}{4} d^*(I) [ ]$. Then $\mathcal{H}_k = \{ n \in \mathbb{N} : \#(\{ n \in I \cap m, x \in E_n \}) \geq m(d^*(I) - 2\eta) \}$. Then $(\mathcal{H}_k)_{k\in\mathbb{N}}$ is a non-decreasing sequence in $\Sigma$ and $\bigcup_{k\in\mathbb{N}} \mathcal{H}_k = X$. If $x \in X$, there is a $J \in \mathcal{F}_d$ such that $x \in E_n$ for every $n \in J$. Now $d^*(\mathbb{N} \setminus J) = 0$ so there is a $k \in \mathbb{N}$ such that $\#(\{ m \setminus J \}) \leq \eta m$ for every $m \geq k$. If $m \in \mathbb{N}$ and $m \geq k$, then

$$\#(\{ n : n \in I \cap m, x \in E_n \}) \geq \#(\{ n \in I \cap J \}) \geq \#(\{ n \cap I \} - \eta m \geq m(d^*(I) - 2\eta),$$

so $x \in \mathcal{H}_k$. Q

(b) There is therefore an $m$ in $\mathbb{N}$ such that $\mu \mathcal{H}_m \geq 1 - \epsilon$ and $\#(I \cap m) \leq m(d^*(I) + \eta)$. In this case

$$m(1 - \epsilon)(d^*(I) - 2\eta) \leq m(d^*(I) - 2\eta)\mu \mathcal{H}_m \leq \int \#(\{ n : n \in I \cap m, x \in E_n \}) \mu(dx)$$

$$= \sum_{n \in \mathbb{N} \cap m} \mu E_n \leq (\alpha + \epsilon)\#(I \cap m) \leq m(\alpha + \epsilon)(d^*(I) + \eta).$$

So $(1 - \epsilon)(d^*(I) - 2\eta) \leq (\alpha + \epsilon)(d^*(I) + \eta)$. As $\eta$ and $\epsilon$ are arbitrary, and $d^*(I) > 0$, $\alpha = 1$. By 2C, $\mathcal{F}$ has the Fatou property.

3F Proposition $\mathcal{F}_3$ has the Fatou property.

**proof (a)** Let $(X, \Sigma, \mu)$ be a probability space and $(E_n)_{n\in\mathbb{N}}$ a sequence in $\Sigma$ such that $\bigcup_{I \in \mathcal{F}_3} \bigcap_{n \in \mathbb{N}} E_n = X$; set $\alpha = \lim inf_{n\to\infty} \mu E_n$. Take any $\epsilon > 0$. Set $I = \{ n : \mu E_n \leq \alpha + \epsilon \}$; then $d^*(I) > 0$. Take any $\eta \in ]0, \frac{1}{4} d^*(I) [ ]$. Then there is a sequence $(k_m)_{m\in\mathbb{N}}$ such that $k_m + m \leq k_{m+1}$ and $\#(I \cap [k_m, k_m + m]) \geq m(d^*_3(I) - \eta)$ for every $m \in \mathbb{N}$; set $K_m = [k_m, k_m + m]$ for each $m$. For $I \in \mathbb{N}$ set
\[ H_l = \bigcap_{m \geq l} \{ x : x \in X, \#(\{ n : n \in I \cap K_m, x \in E_n \}) \geq m(d^*_n(I) - 2\eta) \}. \]

Then \( H_l \) is a non-decreasing sequence in \( \Sigma \) and \( \bigcup_{l \in \mathbb{N}} H_l = X. \) If \( x \in X \), there is a \( J \in \mathcal{F}_x \) such that \( x \in E_n \) for every \( n \in J \). Now \( d^*_n(\mathbb{N} \setminus J) = 0 \) so there is an \( l \in \mathbb{N} \) such that \( \#(K_m \setminus J) \leq \eta m \) for every \( m \geq l. \) If \( m \geq l \), then
\[
\#(\{ n : n \in I \cap K_m, x \in E_n \}) \geq \#(I \cap J \cap K_m) \geq \#(I \cap K_m) - \eta m \geq m(d^*_n(I) - 2\eta),
\]
so \( x \in H_l. \)

**Q**

(b) There is therefore an \( m \geq 1 \) such that \( \mu H_m \geq 1 - \epsilon \) and \( \#(I \cap K_m) \leq m(d^*_n(I) + \eta) \). In this case
\[
\mu H_m \geq (1 - \epsilon)(d^*_n(I) - 2\eta) \leq m\mu H_m \leq \int \#(\{ n : n \in I \cap K_m, x \in E_n \}) \mu(dx) = \sum_{n \in I \cap K_m} \mu E_n \leq (\alpha + \epsilon)\#(I \cap K_m) \leq m(\alpha + \epsilon)(d^*_n(I) + \eta).
\]

So \( (1 - \epsilon)(d^*_n(I) - 2\eta) \leq (\alpha + \epsilon)(d^*_n(I) + \eta) \). As \( \eta \) and \( \epsilon \) are arbitrary, and \( d^*_n(I) > 0, \alpha = 1. \) By 2C, \( \mathcal{F}_x \) has the Fatou property.

**4 Medial limits**

4A Recall that a **medial limit** is a non-negative additive functional \( \theta \) on PN such that \( \int w \, d\theta = \lim_{n \to \infty} w(n) \) whenever \( w \in \mathbb{R}^N \) is a convergent sequence and \( \bigint f_n(x) \mu(dx) \theta(dn) \) is defined and equal to \( \bigint f_n(x) \mu(dx) \theta(dn) \) whenever \( (X, \Sigma, \mu) \) is a probability space and \( (f_n)_{n \in \mathbb{N}} \) is a uniformly bounded sequence of integrable real-valued functions on \( X \); here \( \theta, \mu \) is defined as in FREMLIN 02, 363L. See FREMLIN N02 or HOFFMAN-JORGENSEN 78 for the basic theory of medial limits, in particular, for the proof that if \( \mathfrak{p} = \mathfrak{c} \) then there is a medial limit.

If \( \mathcal{F} \) is a filter on \( \mathbb{N} \) a medial limit \( \theta \) **refines** \( \mathcal{F} \) if \( \liminf_{n \to \mathcal{F}} w(n) \leq \int w \, d\theta \) for every \( w \in \ell^\infty. \)

4B **Proposition** Let \( \mathfrak{F} \) be the set of filters \( \mathcal{F} \) on \( \mathbb{N} \) such that there is an additive functional \( \theta : \mathcal{P}\mathbb{N} \to [0, 1] \) such that
\[
\bigint f_n(x) \theta(dn) \mu(dx) = \bigint f_n(x) \mu(dx) \theta(dn) = \bigint f_n(x) \theta(n) \mu(dx) \theta(dn) = \bigint f_n(x) \mu(dx) \theta(n) \mu(dx).
\]

**proof (a)** Write \( \mathcal{G} \) for \( \lim_{n \to \mathcal{F}} \mathcal{F}_n. \) For each \( n \in \mathbb{N} \) let \( \theta_n \) witness that \( \mathcal{F}_n \in \mathfrak{F}, \) and let \( \theta \) witness that \( \mathcal{F} \in \mathfrak{F}. \) Set \( \theta^* \) for \( \mathcal{F} \) and \( \mathcal{F}_n \) is a sequence in \( \mathfrak{F} \) and \( \mathcal{F} \in \mathfrak{F}. \)

**proof (b)** \( \theta^* \) has the property (\( \dagger \)). If \( (X, \Sigma, \mu) \) is a probability space and \( (f_i)_{i \in \mathbb{N}} \) is a uniformly bounded sequence of integrable real-valued functions on \( X, \)
\[
\bigint f_i(x) \theta^*(di) \mu(dx) = \bigint f_i(x) \theta_n(di) \theta(dn) \mu(dx) = \bigint f_i(x) \theta_n(di) \mu(dx) \theta(dn) = \bigint f_i(x) \mu(dx) \theta^*(di).
\]

(c) For any \( w \in \ell^\infty, \)
\[
\liminf_{i \to \mathcal{G}} w(i) \leq \liminf_{n \to \mathcal{F}} \liminf_{i \to \mathcal{F}_n} w(i).
\]

**P** Suppose that \( \gamma > \liminf_{n \to \mathcal{F}} \liminf_{i \to \mathcal{F}_n} w(i), \) and that \( A \in \mathcal{G}. \) Then \( B = \{ n : A \in \mathcal{F}_n \} \) belongs to \( \mathcal{F}, \)
so there is an \( n \in B \) such that \( \liminf_{i \to \mathcal{F}_n} w(i) \leq \gamma. \) Now \( A \in \mathcal{F}_n, \) so
\[
\liminf_{i \in A} w(i) \leq \liminf_{i \to \mathcal{F}_n} w(i) \leq \gamma.
\]
As \( A \) is arbitrary, \( \liminf_{i \to \mathcal{G}} w(i) \leq \gamma. \)
Accordingly,
\[
\liminf_{i \to \mathcal{G}} w(i) \leq \liminf_{n \to \mathcal{F}} \liminf_{i \to \mathcal{F}_n} w(i) \leq \int \liminf_{i \to \mathcal{F}_n} w(i) \theta(dn)
\]
\[
\leq \iint w \, d\theta n \theta(dn) = \int w \, d\theta^*.
\]
So \(\theta^*\) has property (\(\dagger\)), and witnesses that \(\mathcal{G} \in \mathfrak{F}\).

**4C Corollary** If there is any medial limit, and \(\mathcal{F}\) is a free filter on \(\mathbb{N}\) of countable type, then there is a medial limit refining \(\mathcal{F}\).

**proof** In 4B, it is easy to check that \(\mathfrak{F}\) contains all principal ultrafilters on \(\mathbb{N}\); if \(\mathcal{F}\) is generated by \(\{k\}\), set \(\theta_a = \chi_a(k)\) for \(a \subseteq \mathbb{N}\); then \(\theta\) witnesses that \(\mathcal{F} \in \mathfrak{F}\). By definition, a medial limit is a witness that \(\mathcal{F}_{F_\mathfrak{F}} \in \mathfrak{F}\).

So an induction on the countable-type level of \(\mathcal{F}\) shows that every filter on \(\mathbb{N}\) of countable type belongs to \(\mathfrak{F}\). If \(\mathcal{F}\) is a free filter and belongs to \(\mathfrak{F}\), then a witness that \(\mathcal{F} \in \mathfrak{F}\) is a medial limit refining \(\mathcal{F}\).

**4D Proposition** Suppose that \(\mathcal{F}\) is a filter on \(\mathbb{N}\) such that whenever \(I \subseteq \mathbb{N}\) meets every member of \(\mathcal{F}\) there is a medial limit refining \(\mathcal{F} \cup \{I\}\). Then \(\mathcal{F}\) has the Fatou property.

**proof** Let \(J\) be any set meeting every member of \(\mathcal{F}\), and \(\theta\) a medial limit refining \(\mathcal{F} \cup \{J\}\). Let \((X, \Sigma, \mu)\) be a probability space and \((E_n)_{n \in \mathbb{N}}\) a sequence in \(\Sigma\) such that \(X = \bigcup_{I \in \mathcal{F}} \bigcap_{n \in I} E_n\), that is, \(\liminf_{n \to \mathcal{F}} \chi E_n(x) = 1\) for every \(x \in X\). Then

\[
1 = \int \liminf_{n \to \mathcal{F}} \chi E_n(x) \mu(dx) \leq \iint \chi E_n(x) \theta(dn) \mu(dx)
\]

(since \(\theta\) also refines \(\mathcal{F}\))

\[
= \iint \chi E_n(x) \mu(dx) \theta(dn) = \mu \int E_n \theta(dn) \leq \limsup_{n \to \mathcal{F} \cup \{J\}} \mu E_n \leq \sup_{n \in J} \mu E_n.
\]

As \(J\) is arbitrary, \(\liminf_{n \to \mathcal{F}} \mu E_n = 1\). As \((X, \Sigma, \mu)\) and \((E_n)_{n \in \mathbb{N}}\) are arbitrary, \(\mathcal{F}\) has the Fatou property.

**4E Remark** The filter \(\mathcal{F}\) of Example 2E is a Borel subset of \(\mathcal{P}\mathbb{N}\) but is not refined by any medial limit.

**P?** Otherwise, follow the argument of 4D with \(J = \mathbb{N}\); we should get \(1 \leq \limsup_{n \to \mathcal{F}} \mu E_n = 0\). XQ

**4F Proposition** If there is a medial limit, there is a medial limit refining the density filter.

**proof** Let \(\theta\) be a medial limit, and define \(\theta^*\) by setting

\[
\theta^*(a) = \int \frac{1}{n+1} \mathfrak{d}(a \cap (n+1)) \theta(dn)
\]

for \(a \subseteq \mathbb{N}\). Then

\[
\int w \, d\theta^* = \int \frac{1}{n+1} \sum_{i=0}^n w(i) \theta(dn)
\]

for \(w \in \ell^\infty\). So if \((X, \Sigma, \mu)\) is a probability space and \((f_n)_{n \in \mathbb{N}}\) is a uniformly bounded sequence of real-valued functions on \(X\),

\[
\iint f_n(x) \theta^*(dn) \mu(dx) = \iint \frac{1}{n+1} \sum_{i=0}^n f_i(x) \theta(dn) \mu(dx) = \iint \frac{1}{n+1} \sum_{i=0}^n f_i(x) \mu(dx) \theta(dn)
\]

\[
= \iint \frac{1}{n+1} \sum_{i=0}^n f_i(x) \mu(dx) \theta(dn) = \iint f_n(x) \mu(dx) \theta(dn).
\]
Thus $\theta^*$ is a medial limit. Now $\theta^*(a) \leq d^*(a)$ for every $a \subseteq \mathbb{N}$; in particular, $\theta^*(a) = 0$ whenever $d^*(a) = 0$ and $\theta^*(a) = 1$ for every $a \in \mathcal{F}_d$; it follows that $\liminf_{n \to \mathcal{F}_d} w(n) \leq \int w \, d\theta^*$ for every $w \in \ell^\infty$, and $\theta^*$ refines $\mathcal{F}_d$.

5 Isomorphism classes of filters of countable type

5A Homogeneous and critical filters (a) I will say that a filter $\mathcal{F}$ on a set $X$ is homogeneous if $\mathcal{F}$ is isomorphic to its trace $\mathcal{F}|Y$ whenever $Y \subseteq X$ and $X \setminus Y \notin \mathcal{F}$. Observe that in this case $\mathcal{F}$ must contain every set $A \subseteq X$ such that $\#(X \setminus A) < \#(X)$; in particular, $\mathcal{F}$ must be free, unless $X$ is a singleton. An ultrafilter on $X$ is homogeneous iff it is uniform.

(b) I will say that a filter $\mathcal{F}$ of countable type on a set $I$ is critical if there are a sequentially compact Hausdorff space $X$ and a function $f : I \to X$ such that $f|\mathcal{F}$ converges to a point of $X \setminus \bigcup_{\eta \in \mathcal{F}} f[I]_\eta$.

5B Lemma (a) Let $X$ be a discrete space. Then there is a locally countable sequentially compact Hausdorff space $Y$ with an open set homeomorphic to $X$.

(b) Let $(X, \Xi)$ be a Hausdorff space, and suppose that whenever $(A_i)_{i \in I}$ is a countable family of countably infinite subsets of $X$ and $x \in X$ then either there are an $i \in I$ and a sequence of distinct points in $A_i$ convergent to a point of $X$ or there are distinct $i, j \in I$ such that $A_i \cap A_j$ is infinite or there is a disjoint family $(G_i)_{i \in I}$ of open sets such that $A_i \setminus G_i$ is finite for every $i$ and $x \notin \bigcup_{i \in I} G_i$.

Suppose that we are also given a sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ with no convergent subsequence in $X$. Then $X$ can be embedded as an open set in a sequentially compact Hausdorff space $Z$ such that $(x_n)_{n \in \mathbb{N}}$ is convergent in $Z$ to a point $z^*$ such that no sequence in $Z \setminus \{z^* \cup \{x_n : n \in \mathbb{N}\}\}$ converges to $z^*$.

Proof (a)(i) We can suppose that $X$ is a cardinal $\kappa$; let $\lambda > \kappa$ be a cardinal of uncountable cofinality such that the cardinal power $\lambda^\omega$ is equal to $\lambda$, and let $(A_\xi)_{\xi \leq \kappa \lambda}$ enumerate $[\lambda]^\omega$. Define $(B_\xi)_{\xi \leq \lambda}$ inductively by setting

$$B_\xi = A_\xi \text{ if } \kappa \leq \xi < \lambda \text{ and } A_\xi \cap B_\eta \text{ is finite for every } \eta < \xi = \emptyset \text{ otherwise.}$$

Observe that $B_\xi \cap B_\eta$ is finite for all distinct $\xi, \eta < \kappa$, and that if $C \in [\lambda]^\omega$ there is a $\xi < \lambda$ such that $C \cap B_\xi$ is infinite.

On $Y = \lambda$, let $\Xi$ be the topology

$$\{G : G \subseteq Y, B_\xi \setminus G \text{ is finite for every } \xi \in G\}.$$

(ii) Since $B_\xi = \emptyset$ for $\xi < \kappa$, every subset of $X$ is an open subset of $Y$, and $X$, with its discrete topology, is a subspace of $Y$.

(iii) If $(\xi_n)_{n \in \mathbb{N}}$ is a sequence in $Y$, then either it has a constant subsequence, which is surely convergent, or $C = \{\xi_n : n \in \mathbb{N}\}$ is infinite. In this case, let $\xi < \lambda$ be such that $B_\xi \cap C$ is infinite; then $(\xi_n)_{n \in \mathbb{N}}$ has a subsequence converging to $\xi$.

Thus $Y$ is sequentially compact.

(iv) If $\alpha, \beta$ are distinct points of $Y$, enumerate $\mathbb{N} \times \mathbb{N}$ as $(i_n, j_n)_{n \in \mathbb{N}}$ in such a way that $i_n \leq n$ for every $n$, and define $(U_n, V_n)_{n \in \mathbb{N}}$ as follows. $U_0 = \{\alpha\}$, $V_0 = \{\beta\}$. Given that $U_n$ and $V_n$ are disjoint countable sets such that $B_\xi \cap U_n$ is finite for every $\xi \in \lambda \setminus U_n$, while $B_\xi \cap V_n$ is finite for every $\xi \in \lambda \setminus V_n$, let $(\xi_n)_{n \in \mathbb{N}}$ run over $U_n$ and $(\eta_n)_{n \in \mathbb{N}}$ run over $V_n$. Set

$$U_{n+1} = U_n \cup (B_{i_n, j_n} \setminus V_n), \quad V_{n+1} = V_n \cup (B_{i_n, j_n} \setminus U_{n+1})$$

and continue. At the end of the induction, set $G = \bigcup_{n \in \mathbb{N}} U_n$ and $H = \bigcup_{n \in \mathbb{N}} V_n$; then $G = \{\xi_{ij} : i, j \in \mathbb{N}\}$ and $H = \{\eta_{ij} : i, j \in \mathbb{N}\}$ are disjoint and open, $\alpha \in G$ and $\beta \in H$.

Thus $Y$ is Hausdorff.

(v) Note that in the construction of (iv), both $G$ and $H$ are necessarily countable; it follows at once that $Y$ is locally countable.
(b)(i) Let $C$ be the family of all infinite subsets $C$ of $X$ such that there is no convergent sequence of distinct points of $C$. Note that $A^* = \{x_n : n \in \mathbb{N}\}$ belongs to $C$. Let $A \subseteq C$ be a maximal set, containing $A^*$, such that $A \cap B$ is finite for all distinct $A, B \in A$. Let $\langle z_A \rangle_{A \in A}$ be a family of distinct points not in $X$, and set $Z_0 = \{z_A : A \in A\}$.

(ii) By (a), we can construct a locally countable sequentially compact Hausdorff space $(Z_1, \mathcal{T}_1)$, disjoint from $X$, such that $Z_0 \subseteq Z_1$ and the subspace topology on $Z_0$ is discrete. Set $Z = Z_1 \cup X$, and give $Z$ the topology

\[ \mathcal{G} = \{G : G \subseteq Z, G \cap Z_1 \in \mathcal{T}_1, G \cap X \in \mathcal{T}\}, \]

\[ A \setminus G \] is finite whenever $A \in A$ and $z_A \in G$.

(iii) $X$ is an open subset of $Z$ and $\mathcal{T}$ is the subspace topology induced by $\mathcal{G}$ on $X$.

(iv) If $(t_n)_{n \in \mathbb{N}}$ is a sequence in $Z$, then

- either it has a constant subsequence, which is surely convergent;
- or it has a subsequence in $Z_1$, which must in turn have a subsequence converging in $Z_1$ and in $Z$;
- or it has a convergent subsequence in $X$, which will converge in $Z$;
- or $C = \{t_n : n \in \mathbb{N}\} \cap X$ belongs to $C$, and there is an $A \in A$ such that $C \cap A$ is infinite, in which case $\langle t_n \rangle_{n \in \mathbb{N}}$ has a subsequence convergent to $z_A$.

So $Z$ is sequentially compact.

(v) $Z$ is Hausdorff. Suppose that $w, z$ are distinct points of $Z$.

- If $w, z$ both belong to $X$, they are separated by disjoint open sets in $X$, which are still open in $Z$.

- If $w \in X$ and $z \in Z_1$, let $V$ be a countable neighbourhood of $z$ in $Z_1$, and consider the countable set $D = \{A : A \in A, z_A \in V\}$. By the hypothesis on $X$, there is a family $\langle H_A \rangle_{A \in D}$ of open sets in $X$ such that $A \setminus H_A$ is finite for every $A \in D$ and $w \notin \bigcup_{A \in D} H_A$. Set $G = X \setminus \bigcup_{A \in D} H_A$, $H = V \cup \bigcup_{A \in D} H_A$; then $G$ and $H$ are open subsets of $Z$ separating $w$ from $z$.

- Similarly, $w$ and $z$ can be separated if $w \in Z_1$ and $z \in X$.

- Suppose that $w, z$ both belong to $Z_1$. Let $U, V \subseteq Z_1$ be disjoint countable sets, both open in $Z_1$, containing $w, z$ respectively. Set $D = \{A : A \in A, z_A \in U \cup V\}$; let $\langle G_A \rangle_{A \in D}$ be a disjoint family of open subsets in $X$ such that $A \setminus G_A$ is finite for every $A \in D$; set $G = U \cup \{G_A : A \in D, z_A \in U\}$, $H = V \cup \{G_A : A \in D, z_A \in V\}$. Then $G$ and $H$ are open sets in $Z$ separating $w$ from $z$.

(vi) Setting $z^* = z_{A^*}$, we see that $A^* \setminus G$ is finite for every neighbourhood $G$ of $z^*$, so that $z^* = \lim_{n \to \infty} x_n$. The construction ensured that $\{z^*\}$ is an open set in $Z_1$, so that $X \cup \{z^*\}$ is an open set in $Z$, as required. If $\langle z_n \rangle_{n \in \mathbb{N}}$ is a sequence in $Z \setminus \{\{z^*\} \cup \{x_n : n \in \mathbb{N}\}\}$, then

- either it has a subsequence in $Z_1$, which cannot converge to $z^*$ because $z^*$ is isolated in $Z_1$,
- or it has a subsequence which is a convergent sequence in $X$, and does not converge to $z^*$,
- or $C = \{z_n : n \in \mathbb{N}\} \cap X$ belongs to $C$, so meets some $A \in A$ in an infinite set; as $A \neq A^*$, $\langle z_n \rangle_{n \in \mathbb{N}}$ has a subsequence converging to some $z_A \neq z^*$.

In any case, we see that $\langle z_n \rangle_{n \in \mathbb{N}}$ does not converge to $z^*$, which is what we needed to know.

5C Lemma Suppose that $\langle F_n \rangle_{n \in \mathbb{N}}$ is a sequence of critical filters of countable type on $\mathbb{N}$ such that $\langle \text{ct} F_n \rangle_{n \in \mathbb{N}}$ is non-decreasing. Let $(I_n)_{n \in \mathbb{N}}$ be a partition of $\mathbb{N}$ into infinite sets, and $g_n : \mathbb{N} \to I_n$ a bijection for each $n$; let $F$ be $\lim_{n \to \infty} F_n \cap \big[\mathcal{F}_n\big]$]. Then $F$ is a critical filter, and $\text{ct} F = \sup_{n \in \mathbb{N}} (\text{ct} F_n + 1)$.

proof (a) Setting $\xi = \sup_{n \in \mathbb{N}} (\text{ct} F_n + 1)$, then of course

$$\text{ct} F \leq \sup_{n \in \mathbb{N}} (\text{ct} g_n\big[\mathcal{F}_n\big] + 1) \leq \xi$$

by 1D.
(b) For each \( n \in \mathbb{N} \), let \( X_n \) be a sequentially compact Hausdorff space and \( f_n : \mathbb{N} \to X_n \) a function such that \( \lim f_n[\mathcal{F}_n] \) is defined and does not belong to \( \bigcup_{\eta \in \text{ct} \mathcal{F}_n} f[\mathcal{N}^*_\eta] \). Set \( Z_n = X_n \times \{ n \} \) for each \( n \), \( X = \bigcup_{n \in \mathbb{N}} X_n \) with the disjoint union topology. Then \( X \) satisfies the conditions of 5Bb. \( \mathbf{P} \) Let \( (A_i)_{i \in I} \) be a countable family of countably infinite subsets of \( X \) such that \( A_i \cap A_j \) is finite for all \( i \neq j \) and there is no non-trivial convergent sequence made up of points from any single \( A_i \). As every \( X_n \) is sequentially compact, this means that \( A_i \cap Z_n \) is finite for every \( i \). Take any \( (x, k) \in X \).

If \( I \) is finite, there is an \( n > 1 \) such that \( A_i \cap A_j \) does not meet \( Z_m \) for any \( m \geq n \) and any \( i \neq j \). In this case, for \( m \geq n \), \( \bigcup_{i \in I} A_i \cap Z_m \) is finite, so we can find disjoint open sets \( G_{im} \subseteq X_m \) such that \( A_i \cap Z_m \subseteq G_{im} \times \{ m \} \) for each \( i \); setting \( G_i = \bigcup_{m \geq n} G_{im} \times \{ m \} \), we have a disjoint family \( (G_i)_{i \in I} \) of open sets in \( X \) such that \( A_i \cap G_i \) is finite for every \( i \), and \( (x, k) \notin \bigcup_{i \in I} G_i \).

If \( I \) is infinite, we may take it that \( I = \mathbb{N} \). In this case, let \( \langle n_i \rangle_{i \in \mathbb{N}} \) be a strictly increasing sequence, starting with \( n_0 > 1 \), such that \( A_i \cap A_j \cap Z_m \) is empty whenever \( i < j \) and \( m \geq n_i \); set \( A_i = A_i \setminus \bigcup_{m < n_i} Z_m \). Then \( A_i \setminus A_i^* \) is finite and \( A_i \cap A_j \cap Z_m \) is empty whenever \( i < j \) in \( \mathbb{N} \). This time, \( \bigcup_{i \in \mathbb{N}} A_i \cap Z_m \) is finite for every \( m \in \mathbb{N} \), so we can use the same method as before to find \( G_{im} \), for \( i, m \in \mathbb{N} \), and \( G_i \), for \( m \in \mathbb{N} \), such that \( A_i \subseteq G_i \) for every \( i \) and \( (G_i)_{i \in \mathbb{N}} \) is a disjoint family of open sets. If we take the elementary precaution of setting \( G_{ik} = \emptyset \) for every \( i \), then \( (x, k) \notin \bigcup_{i \in \mathbb{N}} G_i \).

(c) By Lemma 5Bb, we can embed \( X \) in a sequentially compact Hausdorff space \( Z \) such that \( \langle (x_n, n) \rangle_{n \in \mathbb{N}} \) converges to a point \( z^* \) of \( Z \) and no sequence in \( Z \setminus (\{ z^* \} \cup \{ (x_n, n) : n \in \mathbb{N} \}) \) converges to \( z^* \). Let \( f : \mathbb{N} \to Z \) be such that \( f(g_n(i)) = (f_n(i), n) \) for all \( i, n \in \mathbb{N} \). Then \( f[\mathcal{F}_n] \to z^* \). \( \mathbf{P} \) If \( G \) is any neighbourhood of \( z^* \), then

\[
\{ n : \{ j : f(j) \in G \} \in \mathcal{F}_n[\mathcal{F}_n] \} = \{ n : \{ i : f(g(i)) \in G \} \in \mathcal{F}_n \} = \{ n : \{ i : (f_n(i), n) \in G \} \in \mathcal{F}_n \} \supseteq \{ n : (x, i) \in G \} \in \mathcal{F}_{\mathbb{N}} \}
\]

so \( f^{-1}[G] \in \mathcal{F}_n \).

If \( n \in \mathbb{N} \) and \( \eta < \text{ct} \mathcal{F}_n \), then \((x_n, n) \notin f[\mathcal{N}^*_\eta] \). \( \mathbf{P} \) Because \( Z_n \) is open in \( X \) and \( Z \),

\[
Z_n \cap f[\mathcal{N}^*_\eta] = (Z_n \cap f[\mathcal{N}^*_\eta]) = f[I^*_\eta] = (f_n[I^*_\eta]) = (f_n[I] \times \{ n \}) = f_n[\mathcal{N}^*_\eta] \times \{ n \}
\]

(calculating \( f_n[\mathcal{N}^*_\eta] \) in \( X_n \), of course); as \( x_n \notin f_n[\mathcal{N}^*_\eta] \), \((x_n, n) \notin f[\mathcal{N}^*_\eta] \).

If \( \zeta < \xi \) and \( z^* \in f[\mathcal{N}^*_\zeta] \), then \( (z^* \) surely does not belong to \( f[\mathcal{N}] \) there is a sequence \( \langle z_i \rangle_{i \in \mathbb{N}} \) in \( \bigcup_{\eta < \xi} f[\mathcal{N}^*_\eta] \) converging to \( z^* \), with no \( z_i \) equal to \( z^* \). Now \( \langle z_i \rangle_{i \in \mathbb{N}} \) must have a subsequence in common with \( \langle (x_n, n) \rangle_{n \in \mathbb{N}} \), so there must be infinitely many \( n \) such that \( (x_n, n) \in \bigcup_{\eta < \xi} f[\mathcal{N}^*_\eta] \). However, there is an \( m \in \mathbb{N} \) such that \( \zeta < \text{ct} \mathcal{F}_n \) for every \( n \geq m \), in which case \( (x_n, n) \notin \bigcup_{\eta < \xi} f_n[\mathcal{N}^*_\eta] \); which is impossible.

Thus we know that \( \text{ct} \mathcal{F} \leq \xi \), that \( f[\mathcal{F}] \to z^* \), and that \( z^* \notin \bigcup_{\zeta < \xi} f[\mathcal{N}^*_\zeta] \); we conclude that \( \text{ct} \mathcal{F} = \xi \) and that \( f : \mathbb{N} \to Z \) witnesses that \( \mathcal{F} \) is critical.

5D Theorem For every \( \xi < \omega_1 \) there is a critical filter on \( \mathbb{N} \) with countable-type level \( \xi \).

**proof** Induce on \( \xi \). For \( \xi = 0 \), take a principal ultrafilter. For the step to \( \xi > 0 \), let \( \langle \xi_n \rangle_{n \in \mathbb{N}} \) be a non-decreasing sequence such that \( \xi = \text{sup}_{n \in \mathbb{N}}(\xi_n + 1) \). For each \( n \in \mathbb{N} \), there is a critical filter \( \mathcal{F}_n \) on \( \mathbb{N} \) such that \( \text{ct} \mathcal{F}_n = \xi_n \); now the construction of Lemma 5C gives a critical filter with countable-type level \( \xi \).

5E Lemma For every \( n \in \mathbb{N} \) there is a homogeneous critical filter \( \mathcal{H}_n \) on \( \mathbb{N} \) such that \( \text{ct} \mathcal{H}_n = n + 1 \).

**proof** Induce on \( n \).

(a) Start with \( \mathcal{H}_0 = \mathcal{F}_{\mathbb{N}} \). To see that \( \mathcal{F}_{\mathbb{N}} \) is critical, take \( X = \mathbb{N} \cup \{ \infty \} \) to be the one-point compactification of \( \mathbb{N} \), and \( f : \mathbb{N} \to X \) the identity function.

(b) For the inductive step to \( n + 1 \), let \( \langle I_k \rangle_{k \in \mathbb{N}} \) be a partition of \( \mathbb{N} \) into infinite sets, and let \( g_k \) be the increasing enumeration of \( I_k \) for each \( k \); set \( \mathcal{H}_{n+1} = \text{lim}_{k \to \mathcal{F}_{\mathbb{N}}} g_k[(\mathcal{H}_n)] \). By 5C, \( \mathcal{H}_{n+1} \) is a critical filter and \( \text{ct} \mathcal{H}_{n+1} = n + 2 \).
(c) (i) For each \( k \in \mathbb{N} \), set \( \mathcal{G}_k = g_k[\mathcal{H}_n][I_k] \), so that \( \mathcal{G}_k \) is isomorphic to \( \mathcal{H}_n \) and is homogeneous; observe that

\[ \mathcal{H}_{n+1} = \{ A : A \subseteq \mathbb{N}, A \cap I_k \in \mathcal{G}_k \} \text{ for all but finitely many } k. \]

(ii) Suppose that \( A \subseteq \mathbb{N} \) is infinite, and that \( Y \subseteq \bigcup_{k \in A} I_k \) is such that \( I_k \setminus Y \notin \mathcal{G}_k \) for every \( k \in A \). Of course \( \mathbb{N} \setminus Y \notin \mathcal{H}_{n+1} \). For \( k \in A \), \( \mathcal{G}_k \) is isomorphic to its trace \( \mathcal{G}_k' = \mathcal{G}_k[\langle Y \cap I_k \rangle] \); let \( h_k : I_k \to I_k \cap Y \) be a corresponding isomorphism. Enumerate \( A \) in ascending order as \( \langle k_i \rangle \in \mathbb{N} \), and let \( h : \mathbb{N} \to Y \) be the bijection defined by setting

\[ h(j) = h_{k_i}g_{k_i}^{-1}(j) \text{ if } i \in \mathbb{N} \text{ and } j \in I_i. \]

For \( B \subseteq \mathbb{N} \),

\[ B \in \mathcal{H}_{n+1} \iff B \cap I_i \in \mathcal{G}_i \text{ for all but finitely many } i \]
\[ \iff g_i^{-1}[B \cap I_i] \in \mathcal{H}_n \text{ for all but finitely many } i \]
\[ \iff g_k[g_i^{-1}[B \cap I_i]] \in \mathcal{G}_k \text{ for all but finitely many } i \]
\[ \iff h_k[g_i^{-1}[B \cap I_i]] \in \mathcal{G}_k' \text{ for all but finitely many } i \]
\[ \iff h[B] \cap I_k \in \mathcal{G}_k' \text{ for all but finitely many } i \]
\[ \iff h[B] \cap I_k \in \mathcal{G}_k \text{ for all but finitely many } k \in A \]
\[ \iff \exists C \subseteq \mathbb{N}, h[B] = Y \cap C, C \cap I_k \in \mathcal{G}_k \text{ for all but finitely many } k \in A \]
\[ \iff \exists C \subseteq \mathbb{N}, h[B] = Y \cap C, C \cap I_k \in \mathcal{G}_k \text{ for all but finitely many } k \in A \]
\[ \iff B \in \mathcal{H}_{n+1}[Y]. \]

So \( h \) is an isomorphism between \( \mathcal{H}_{n+1} \) and its trace on \( Y \).

(iii) Now let \( Y \) be any subset of \( \mathbb{N} \) such that \( \mathbb{N} \setminus Y \notin \mathcal{H}_{n+1} \); set \( A = \{ k : Y \cap I_k \notin \mathcal{G}_k \}, Y' = Y \cap \bigcup_{k \in A} I_k \) and \( Y'' = \mathbb{N} \setminus \min(I_k \cap Y) \). By (ii), \( \mathcal{H}_{n+1} \) is isomorphic to its trace on \( Y' \). But observe that \( J = Y \setminus Y'' \) is an infinite set such that \( \mathbb{N} \setminus J \) belongs to \( \mathcal{H}_{n+1} \), and that the same is true of \( J' = Y' \setminus Y'' \). So the traces of \( \mathcal{H}_{n+1} \) on \( Y \) and \( Y' \) are isomorphic, since one is mapped to the other by any bijection between \( J \) and \( J' \); and the trace of \( \mathcal{H}_{n+1} \) on \( Y \) is therefore isomorphic to \( \mathcal{H}_{n+1} \).

Thus \( \mathcal{H}_{n+1} \) is homogeneous.

**5F Lemma** Suppose that \( K \subseteq \mathbb{N} \). Then there is a filter \( \mathcal{F} \) on \( \mathbb{N} \) such that \( ct(\mathcal{F}) = \omega \) and

\[ K = \{ n : n \in \mathbb{N} \text{ and there is an } A \subseteq \mathbb{N} \text{ such that } \mathcal{F}[A] \text{ is a homogeneous filter and } ct(\mathcal{F}[A]) = n + 2 \}. \]

**proof** (a) Let \( \langle n(k) \rangle_{k \in \mathbb{N}} \) be an unbounded sequence in \( \mathbb{N} \) such that

\[ K = \{ n : n \in \mathbb{N}, \{ k : k \in \mathbb{N}, n(k) = n \} \text{ is infinite} \}. \]

Let \( \langle I_k \rangle_{k \in \mathbb{N}} \) be a partition of \( \mathbb{N} \) into infinite sets, and \( g_k : \mathbb{N} \to I_k \) a bijection for each \( k \); let \( \mathcal{H}_n \in \mathbb{N} \) be the sequence of filters constructed in Lemma 5E, and set \( \mathcal{F} = \lim_{k \to \mathcal{F}_n} g_k[\mathcal{H}_n]. \) Since

\[ ct g_k[\mathcal{H}_n] \leq ct \mathcal{H}_n(k) < \omega \]

for every \( k \), \( ct(\mathcal{F}) \leq \omega \).

(b) Suppose that \( \langle k_i \rangle_{i \in \mathbb{N}} \) is a strictly increasing sequence in \( \mathbb{N} \) such that \( \langle n(k_i) \rangle_{i \in \mathbb{N}} \) is non-decreasing. Set

\[ V = \bigcup_{k \in \mathbb{N}} I_k. \]

Then

\[ \mathcal{F}[V] = \lim_{k \to \mathcal{F}_n} g_k[[\mathcal{H}_n(k_i)][V] \]

(1Fa), so \( ct(\mathcal{F}[V]) = \sup_{k \in \mathbb{N}} n(k_i) + 1 \) (Lemma 5C, transferred to the partition \( \langle I_k \rangle_{i \in \mathbb{N}} \) of \( A \)). In particular, since \( \langle n(k) \rangle_{k \in \mathbb{N}} \) is unbounded, we can find a strictly increasing sequence \( \langle k_i \rangle_{i \in \mathbb{N}} \) such that \( \langle n(k_i) \rangle_{i \in \mathbb{N}} \) is also strictly increasing, in which case \( ct(\mathcal{F}) \geq ct(\mathcal{F}[V]) \geq \omega \), and \( ct(\mathcal{F}) \) must be exactly \( \omega \).
(c) If \( n \in K \), there is a strictly increasing sequence \( \langle k_i \rangle_{i \in \mathbb{N}} \) such that \( n(k_i) = n \) for every \( i \in \mathbb{N} \). In this case, setting \( V = \bigcup_{i \in \mathbb{N}} I_{k_i} \),

\[
\mathcal{F}[V] = \lim_{n \to \infty} g_{k_n}[[\mathcal{H}_n]][V] \cong \mathcal{H}_{n+1}
\]

by the construction in the proof of 5E. By 5E, \( \mathcal{F}[V] \) is homogeneous with countable-type level \( n + 2 \).

(d) Now suppose that \( A \subseteq \mathbb{N} \) is such that \( \mathbb{N} \setminus A \notin \mathcal{F} \) and \( \mathcal{F}[A] \) is homogeneous with finite countable-type level. Set

\[
J = \{ k : k \in \mathbb{N}, \mathbb{N} \setminus A \notin g_k[[\mathcal{H}_n(k_i)]] \},
\]

so that \( J \) is infinite; let \( \langle k_i \rangle_{i \in \mathbb{N}} \) be a strictly increasing sequence in \( J \) such that \( n(k_i) \) is strictly increasing or constant. For each \( i \in \mathbb{N} \), set \( G_i = g_{k_i}[[\mathcal{H}_n(k_i)]][I_{k_i}] \), so that \( G_i \cong \mathcal{H}_n(k_i) \) is homogeneous and \( I_{k_i} \setminus A \notin G_i \); let \( h_i : I_{k_i} \to I_{k_i} \cap A \) be an isomorphism between \( (I_{k_i}, G_i) \) and \( (I_{k_i} \cap A, G_i[I_{k_i} \cap A]) \). Set \( V = \bigcup_{i \in \mathbb{N}} I_{k_i} \), and define \( h : V \to V \cap A \) by setting \( h(i) = h_i(j) \) if \( i \in \mathbb{N} \) and \( j \in I_{k_i} \). Just as in part (c-ii) of the proof of 5E, \( h \) is an isomorphism between \( \mathcal{F}[V] \) and \( \mathcal{F}[(V \cap A)] \). But \( \mathcal{F}[V \cap A] \) must be isomorphic to \( \mathcal{F}[A] \), so \( \mathcal{F}[V] \) also is.

(e) Looking back at (b), we see that because \( \mathcal{F}[V] \) has finite countable-type level, \( n(k_i) \) must be bounded, and therefore constant, with value belonging to \( K \). In the latter case, (c) tells us that

\[
\text{ct}(\mathcal{F}[A]) = \text{ct}(\mathcal{F}[V]) = n + 2.
\]

(f) So

\[
K \subseteq \{ n : n \in \mathbb{N} \text{ and there is an } A \subseteq \mathbb{N} \text{ such that } \mathcal{F}[A] \text{ is a homogeneous filter and } \text{ct}(\mathcal{F}[A]) = n + 2 \}
\]

(by (c))

\[
\subseteq K
\]

by (e), and we’re done.

5G Theorem There are \( \mathfrak{c} \) non-isomorphic filters of countable-type level \( \omega \).

proof For every \( K \subseteq \mathbb{N} \) we have a corresponding filter \( \mathcal{F}_K \) as constructed in 5F. But the formula

\[
K = \{ n : n \in \mathbb{N} \text{ and there is an } A \subseteq \mathbb{N} \text{ such that } \mathcal{F}_K[A] \text{ is a homogeneous filter and } \text{ct}(\mathcal{F}_K[A]) = n + 2 \}
\]

tells us that the \( \mathcal{F}_K \) are all non-isomorphic, so there are at least \( \mathfrak{c} \) isomorphism classes of these filters. In the other direction, of course, \( \mathcal{P} \mathbb{N} \) has only \( \mathfrak{c} \) Borel sets, so there can be at most \( \mathfrak{c} \) filters on \( \mathbb{N} \) of countable type.

6 The coinitiality of filters of countable type

6A Theorem If \( \mathcal{F} \) is a filter on \( \mathbb{N} \) of countable type, \( \text{ci} \mathcal{F} \leq \text{cf} \mathbb{N} \).

proof (a) Let \( \mathfrak{F} \) be the set of filters \( \mathcal{F} \) on \( \mathbb{N} \) such that there are a function \( f : \mathcal{F} \to \mathbb{N}^\mathbb{N} \) and a family \( \{ A_{\alpha \beta} \}_{\alpha, \beta \in \mathbb{N}^\mathbb{N}} \) in \( \mathcal{F} \) such that

\[
A_{\alpha \gamma} \subseteq A_{\alpha \beta} \text{ whenever } \alpha, \beta, \gamma \in \mathbb{N}^\mathbb{N} \text{ and } \beta \leq \gamma,
\]

\[
\text{whenever } A \in \mathcal{F} \text{ and } f(A) \leq \alpha \text{ in } \mathbb{N}^\mathbb{N}, \text{ there is a } \beta \in \mathbb{N}^\mathbb{N} \text{ such that } A_{\alpha \beta} \subseteq A.
\]

(b) Principal ultrafilters belong to \( \mathfrak{F} \). If \( \mathcal{F} \) is generated by \( \{ n \} \), set \( f(A) = 0 \) for every \( A \in \mathcal{F} \) and \( A_{\alpha \beta} = \{ n \} \) for \( \alpha, \beta \in \mathbb{N}^\mathbb{N} \).

Proof For each \( n \in \mathbb{N} \), let \( f_n : \mathcal{A}(n) \to \mathbb{N}^\mathbb{N} \) witness that \( \mathcal{F}_n \in \mathfrak{F} \). If \( A \in \mathcal{F} \), set \( f(A) = f_n(A) \) where \( n = \min \{ i : A \in \mathcal{F}_i \} \). If \( \alpha, \beta \in \mathbb{N}^\mathbb{N} \) and \( n \in \mathbb{N} \), define \( <n>\beta \in \mathbb{N}^\mathbb{N} \) by setting

\[
\langle n \rangle \beta = \{ \langle n \rangle \alpha \beta \}.
\]
Suppose that $F$ is the supported relation ($F \in \gamma$ and a witness this. If $A$ then is an $n \in N$ such that $A \in \mathcal{F}_n$ and $f_n(A) \leq \alpha$. Let $\beta \in N$ be such that $A_{\alpha\beta} \subseteq A$; then

$$A_{\alpha,<n>\beta} \subseteq A_{\alpha\beta} \subseteq A.$$  

So $f$ and $(A_{\alpha\beta})_{\alpha,\beta \in N}$ witness that $\mathcal{F} \in \mathfrak{F}$. QED.

(d) If $(\mathcal{F}_n)_{n \in N}$ is any sequence in $\mathfrak{F}$, then $\mathcal{F} = \bigcap_{n \in N} \mathcal{F}_n$ belongs to $\mathfrak{F}$. Let $f_n$, $(A_{\alpha\beta})_{\alpha,\beta \in N}$ witness that $\mathcal{F}_n \in \mathfrak{F}$. Define $f : \mathcal{F} \to N^N$ by setting

$$f(A)(i) = \max_{n \leq i} f_n(A)(i)$$

for $A \in \mathcal{F}$ and $i \in N$. For $\alpha \in N^N$ and $k \in N$ set $(\alpha \vee k)(i) = \max(\alpha(i), k)$ for every $i \in N$. For $\alpha, \gamma \in N^N$ and sequences $(\beta_n)_{n \in N}$ in $N^N$, set

$$C(\alpha, \gamma, (\beta_n)_{n \in N}) = \bigcup_{n \in N} \bigcap_{k \leq \gamma(n)} A_{\alpha \vee k, \beta_n} \in \mathcal{F}.$$  

If $\alpha, \gamma, \gamma' \in N^N$ and two sequences $(\beta_n)_{n \in N}$, $(\gamma'_n)_{n \in N}$ in $N^N$ are such that $\gamma \leq \gamma'$ and $\beta_n \leq \beta'_n$ for every $n$, then

$$C(\alpha, \gamma', (\beta'_n)_{n \in N}) = \bigcup_{n \in N} \bigcap_{k \leq \gamma'(n)} A_{\alpha \vee k, \beta_n} \subseteq \bigcup_{n \in N} \bigcap_{k \leq \gamma(n)} A_{\alpha \vee k, \beta_n} = C(\alpha, \gamma, (\beta_n)_{n \in N}).$$  

If $A \in \mathcal{F}$ and $\alpha \in N^N$ are such that $f(A) \leq \alpha$, then for each $n \in N$ we have a $\gamma(n) \in N$ such that $f_n(A) \leq \alpha \vee \gamma(n)$, and now there is a $\beta_n \in N^N$ such that $A \supseteq A_{\alpha \vee \gamma(n), \beta_n}$; so that

$$A \supseteq \bigcup_{n \in N} A_{\alpha \vee \gamma(n), \beta_n} \supseteq \bigcup_{n \in N} \bigcap_{k \leq \gamma(n)} A_{\alpha \vee k, \beta_n} = C(\alpha, \gamma, (\beta_n)_{n \in N}).$$  

Since $N^N \times (N^N)^N$ is isomorphic, as partially ordered set, to $N^N$, this means that $f$ and $C$, suitably re-coded, witness that $\mathcal{F} \in \mathfrak{F}$.

(e) Thus every filter on $N$ of countable type belongs to $\mathfrak{G}$. On the other hand, every filter in $\mathfrak{G}$ has coinitiality at most $\mathfrak{d} = \text{cf}N$. Let $D \subseteq N^N$ be a colinal set of cardinal $\mathfrak{d}$. If $\mathcal{F} \in \mathfrak{G}$, let $f$ and $(A_{\alpha\beta})_{\alpha,\beta \in N}$ witness this. If $A \in \mathcal{F}$, there is an $\alpha \in D$ such that $f(A) \leq \alpha$; now there are a $\beta \in N^N$ such that $A \supseteq A_{\alpha,\beta}$ and $\alpha \in D$ such that $\beta \leq \gamma$, in which case $A \supseteq A_{\alpha,\gamma}$. So $(A_{\alpha,\gamma} : \alpha, \gamma \in D)$ is coinitial with $\mathcal{F}$ and witnesses that $\text{cf} \mathcal{F} \leq \mathfrak{d}$. QED.

Accordingly every filter on $N$ of countable type has coinitiality at most $\mathfrak{d}$, as claimed.

6B Remark In the notation of Fremlin 08?, §522, the sequential composition $(N^N, \leq, \leq, \leq, \leq)$ is the supported relation $(N^N \times (N^N)^N, R, N^N \times N^N)$, where $R$ is the relation

$$\{((\alpha, h), (\beta, \gamma)) : \alpha, \beta, \gamma \in N^N, h \in (N^N)^N, \alpha \leq \beta \text{ and } h(\beta) \leq \gamma\}.$$  

Suppose that $\mathcal{F}$ is a filter in the family $\mathfrak{G}$ of the proof of 6A, and $\mathcal{I}$ is the corresponding ideal $\{N \setminus A : A \in \mathcal{F}\}$. Let $f$, $(A_{\beta\gamma})_{\beta,\gamma \in N}$ witness that $\mathcal{F} \in \mathfrak{G}$. For $E \in \mathcal{I}$, choose $h_E \in (N^N)^N$ such that $N \setminus E \supseteq A_{\beta, h_E(\beta)}$ whenever $f(N \setminus E) \leq \beta \in N^N$. Define $\phi : \mathcal{I} \to N^N \times (N^N)^N$ and $\psi : N^N \times N^N \to \mathcal{I}$ by setting

$$\phi(E) = (f(N \setminus E), h_E), \quad \psi(\beta, \gamma) = N \setminus A_{\beta,\gamma}.$$
for $E \in \mathcal{I}$ and $\beta, \gamma \in \mathbb{N}^\mathbb{N}$. Now suppose that $E \in \mathcal{I}$ and $\beta, \gamma \in \mathbb{N}^\mathbb{N}$ are such that $(\phi(E), (\beta, \gamma)) \in R$. Then $f(N \setminus E) \leq \beta$ and $h_E(\beta) \leq \gamma$, so
\[ N \setminus E \supseteq A_{\beta, h_E(\beta)} \supseteq A_{\beta, \gamma} \]
and $E \subseteq \psi(\beta, \gamma)$. But this means that $(\phi, \psi)$ is a Galois-Tukey connection from $(\mathcal{I}, \subseteq, \mathcal{I})$ to $(\mathbb{N}^\mathbb{N}, \leq, \mathbb{N}^\mathbb{N}) \rtimes (\mathbb{N}^\mathbb{N}, \leq, \mathbb{N}^\mathbb{N})$. This implies in particular that
\[
\text{ci} \mathcal{F} = \text{cf} \mathcal{I} = \text{cov}(\mathcal{I}, \subseteq, \mathcal{I}) \leq \text{cov}((\mathbb{N}^\mathbb{N}, \leq, \mathbb{N}^\mathbb{N}) \times (\mathbb{N}^\mathbb{N}, \leq, \mathbb{N}^\mathbb{N}))
\[
= \text{cov}(\mathbb{N}^\mathbb{N}, \leq, \mathbb{N}^\mathbb{N}) \cdot \text{cov}(\mathbb{N}^\mathbb{N}, \leq, \mathbb{N}^\mathbb{N})
\]
(the cardinal product)
\[
= \text{cf} \mathbb{N}^\mathbb{N} \cdot \text{cf} \mathbb{N}^\mathbb{N} = \text{cf} \mathbb{N}^\mathbb{N}.
\]

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