

# CONVERGENCE IN VAN DER WAERDEN AND HINDMAN SPACES

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ABSTRACT. We consider four classes of topological spaces which are defined with the aid of convergence with respect to ideals on  $\mathbb{N}$ . All these classes are subclasses of countably compact spaces, and two of them are also subclasses of sequentially compact spaces. In the first part of the paper (Sections 1 and 2) we prove some properties of these classes. In the second part of the paper (Sections 3 and 4) we focus on spaces defined by two particular ideals connected with well known theorems in combinatorics, namely van der Waerden's theorem and Hindman's theorem. The main aim of this part of the paper is to show that two classes of the considered spaces coincide for the van der Waerden ideal and Hindman ideal respectively.

## 1. BW-SPACES

Throughout this paper, all topological spaces are assumed to be Hausdorff.

By an *ideal on a set  $X$*  we mean a nonempty family of subsets of  $X$  closed under taking finite unions and subsets of its elements. By  $\text{Fin}(X)$  we denote the ideal of all finite subsets of  $X$  (for  $X = \mathbb{N}$  we write  $\text{Fin} = \text{Fin}(\mathbb{N})$ ). Moreover, we always assume that ideals are *proper* (i.e.  $X \notin \mathcal{I}$ ) and contains all finite subsets of  $X$  (i.e.  $\text{Fin}(X) \subseteq \mathcal{I}$ ). We say that an ideal  $\mathcal{I}$  is a *P-ideal* if for every countable family  $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{I}$  there exists  $A \in \mathcal{I}$  such that  $A_n \setminus A$  is finite for any  $n \in \mathbb{N}$ .

By  $Y^X$  we denote the set of all functions from  $X$  into  $Y$ , and in the case of  $X \subseteq \mathbb{N}$  the set  $Y^X$  is the set of all sequences  $(y_n)_{n \in X}$  with  $y_n \in Y$  for  $n \in X$ .

Let  $X$  be a topological space,  $\mathcal{I}$  be an ideal on  $\mathbb{N}$  and  $A \subseteq \mathbb{N}$ . We say that a sequence  $(x_n)_{n \in A} \in X^A$  is  *$\mathcal{I}$ -convergent* to  $x \in X$  if

$$\{n \in A : x_n \notin U\} \in \mathcal{I}$$

for every open neighborhood  $U \subseteq X$ ,  $x \in U$ . For  $\mathcal{I} = \text{Fin}$  we obtain the ordinary convergence.

We say that the pair  $(X, \mathcal{I})$  has

- (1) *BW property* if every sequence  $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$  has an  $\mathcal{I}$ -convergent subsequence  $(x_n)_{n \in A}$  with  $A \notin \mathcal{I}$ ;
- (2) *FinBW property* if every sequence  $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$  has a Fin-convergent subsequence  $(x_n)_{n \in A}$  with  $A \notin \mathcal{I}$ ;

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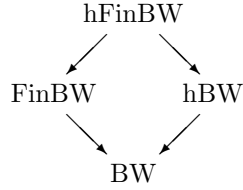
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- (3) *hBW property* if every sequence  $(x_n)_{n \in A} \in X^A$  with  $A \notin \mathcal{I}$  has an  $\mathcal{I}$ -convergent subsequence  $(x_n)_{n \in B}$  with  $B \subseteq A$  and  $B \notin \mathcal{I}$ ;
- (4) *hFinBW property* if every sequence  $(x_n)_{n \in A} \in X^A$  with  $A \notin \mathcal{I}$  has a Fin-convergent subsequence  $(x_n)_{n \in B}$  with  $B \subseteq A$  and  $B \notin \mathcal{I}$ .

We write  $(X, \mathcal{I}) \in \text{BW}$  if the pair  $(X, \mathcal{I})$  has the BW property; we say that an ideal  $\mathcal{I}$  has the *BW property* ( $\mathcal{I} \in \text{BW}$ , in short) if the pair  $([0, 1], \mathcal{I}) \in \text{BW}$  (and the same for properties FinBW, hBW and hFinBW). For examples and properties of ideals with(out) the BW-like properties see [6] where these definitions were introduced.

Clearly, the following diagram holds. By “ $\mathcal{A} \rightarrow \mathcal{B}$ ” we mean “if  $(X, \mathcal{I}) \in \mathcal{A}$  then  $(X, \mathcal{I}) \in \mathcal{B}$ ”. If there is no arrow in some direction between  $\mathcal{A}$  and  $\mathcal{B}$ , then it means that, in general,  $(X, \mathcal{I}) \in \mathcal{A}$  does not imply  $(X, \mathcal{I}) \in \mathcal{B}$  (see [6] for examples of ideals showing that no other implications hold).



The following theorems give us plenty of examples of spaces and ideals with BW-like properties.

**Theorem 1.1** ([7, Corollary 5.7]). *Suppose that a topological space  $X$  satisfies the following condition:*

- (\*) *the closure of every countable set in  $X$  is compact and first countable.*

If

- (1)  *$\mathcal{I}$  can be extended to an  $F_\sigma$  ideal (e.g.  $\mathcal{I}$  is an analytic  $P$ -ideal with the BW property), or*
- (2)  *$\mathcal{I}$  can be extended to a maximal  $P$ -ideal, or*
- (3)  *$\mathcal{I}$  is a maximal ideal,*

then  $(X, \mathcal{I}) \in \text{BW}$  (In the case (1) and (2),  $(X, \mathcal{I}) \in \text{FinBW}$ ).

**Theorem 1.2** ([1, Theorem 3.3], see also [19, Theorem 4.9]). (1) *If  $X$  is a compact space, then  $(X, \mathcal{I}) \in \text{hBW}$  for every maximal ideal  $\mathcal{I}$ .*

- (2) *If  $X$  is  $T_3$ -space, then  $(X, \mathcal{I}) \in \text{hBW}$  for every maximal ideal  $\mathcal{I}$  if and only if  $X$  is  $\omega$ -bounded (i.e. every countably subset of  $X$  has compact closure).*

It is known (see e.g. [11]) that the class of spaces which satisfy (\*) includes all compact metric spaces, all compact linearly ordered topological spaces and every limit ordinal of uncountable cofinality.

It is not difficult to see that if  $(X, \mathcal{I}) \in \text{BW}$ , then  $X$  is countably compact (i.e. every sequence in  $X$  has a cluster point). The reverse implication does not hold for the ideal

$$\mathcal{I}_d = \left\{ A \subseteq \mathbb{N} : \limsup_{n \rightarrow \infty} \frac{|A \cap \{0, 1, \dots, n-1\}|}{n} = 0 \right\}$$

of sets of asymptotic density zero does not have the BW property, i.e.  $([0, 1], \mathcal{I}_d) \notin \text{BW}$  ([9]). (For more examples of ideals without the BW property see [6].)

It is obvious that if  $(X, \mathcal{I}) \in \text{FinBW}$  then  $X$  is sequentially compact (i.e. every sequence in  $X$  has a convergent subsequence). The example  $([0, 1], \mathcal{I}_d) \notin \text{BW}$  shows that the reverse implication does not hold. Moreover,  $(X, \mathcal{I}) \in \text{hBW}$  does not imply that  $X$  is sequentially compact. Indeed, if we take any compact spaces that is not sequentially compact and any maximal ideal  $\mathcal{I}$  then, by Theorem 1.2,  $(X, \mathcal{I}) \in \text{hBW}$ .

In general, there is no relation between “ $(X, \mathcal{I}) \in \text{BW}$ ” and “ $X$  is compact”. The lack of one implication is given by the example  $([0, 1], \mathcal{I}_d) \notin \text{BW}$ . On the other hand, by Example 1.3, if  $X = \omega_1$  then  $(X, \mathcal{I}) \in \text{hFinBW}$  for every  $P$ -ideal  $\mathcal{I}$  and  $X$  is not compact.

**Example 1.3.** Let  $X = \omega_1$  with the order topology.

- (1)  $X$  is not compact.
- (2)  $(X, \mathcal{I}) \in \text{hBW}$  for every ideal  $\mathcal{I}$ .
- (3)  $X$  is sequentially compact.
- (4)  $(X, \mathcal{I}) \in \text{hFinBW}$  for every  $P$ -ideal  $\mathcal{I}$ .

*Proof.* (1) Let  $U_\alpha = [0, \alpha)$  for  $\alpha < \omega_1$ . Then  $\{U_\alpha : \alpha < \omega_1\}$  is a cover of  $X$  without finite subcover.

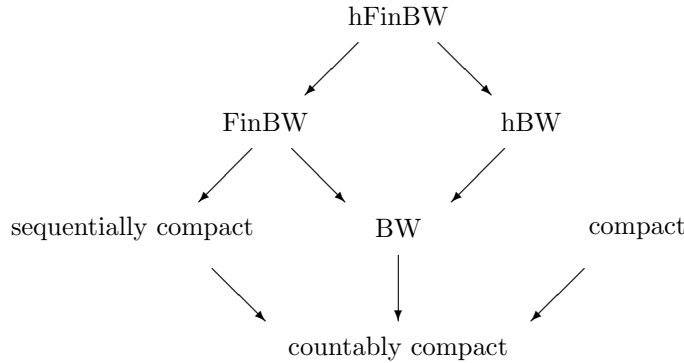
- (2) Let  $A \notin \mathcal{I}$  and  $x_n \in X$  for  $n \in A$ . We have two cases:
  - There is  $x \in X$  such that  $B = \{n \in A : x_n = x\} \notin \mathcal{I}$ ;
  - For every  $x \in X$  we have  $\{n \in A : x_n = x\} \in \mathcal{I}$ .

In the first case  $(x_n)_{n \in B}$  is  $\mathcal{I}$ -convergent to  $x$ . Now we consider the second case. Let  $x = \inf\{\alpha \in \omega_1 : \{n \in A : x_n \leq \alpha\} \notin \mathcal{I}\}$ . It is not difficult to check that  $x$  is a limit ordinal. Let  $B = \{n \in A : x_n \leq x\}$ . Then  $B \notin \mathcal{I}$  and  $(x_n)_{n \in B}$  is  $\mathcal{I}$ -convergent to  $x$ .

- (3) Take  $\mathcal{I} = \text{Fin}$  and apply (2).

(4) Let  $A \notin \mathcal{I}$  and  $x_n \in X$  for  $n \in A$ . By (2) there is  $B \subseteq A$ ,  $B \notin \mathcal{I}$  such that  $(x_n)_{n \in B}$  is  $\mathcal{I}$ -convergent to  $x \in \omega_1$ . Let  $\{\alpha_n < x : n \in \mathbb{N}\}$  be a sequence such that  $\sup\{\alpha_n + 1 : n \in \mathbb{N}\} = x$ . Let  $U_n = (\alpha_n, x + 1)$  for  $n \in \mathbb{N}$ . Let  $C_n = \{n \in B : x_n \notin U_n\}$  for  $n \in \mathbb{N}$ . Since  $C_n \in \mathcal{I}$  for every  $n \in \mathbb{N}$  and  $\mathcal{I}$  is a  $P$ -ideal, so there is  $C \in \mathcal{I}$  such that  $C_n \setminus C$  is finite for every  $n \in \mathbb{N}$ . Now it is not difficult to show that  $(x_n)_{n \in B \setminus C}$  is  $\text{Fin}$ -convergent to  $x$ .  $\square$

The following diagram summarize the relationship between BW-like spaces and compactness properties (if there is no arrow in some direction, it means that, in general, there is no implication between those notions).



Below we provide some results showing that BW-like properties are closed under some topological operations.

**Proposition 1.4** ([7, Proposition 2.6]). *Let  $\mathcal{A} \in \{BW, FinBW, hBW, hFinBW\}$ .*

- (1) *If  $(X, \mathcal{I}) \in \mathcal{A}$  and  $Y \subseteq X$  is closed, then  $(Y, \mathcal{I}) \in \mathcal{A}$ .*
- (2) *If  $(X, \mathcal{I}) \in \mathcal{A}$  and  $f : X \rightarrow Y$  is continuous, then  $(f[X], \mathcal{I}) \in \mathcal{A}$ .*

**Proposition 1.5.** *Let  $\mathcal{A} \in \{hBW, hFinBW\}$ .*

- (1) *If  $S$  is finite and  $(X_s, \mathcal{I}) \in \mathcal{A}$  for  $s \in S$ , then  $(\prod_{s \in S} X_s, \mathcal{I}) \in \mathcal{A}$ .*
- (2) *Suppose that  $X_s \neq \emptyset$  for every  $s \in S$ . Then  $(\bigoplus_{s \in S} X_s, \mathcal{I}) \in \mathcal{A}$  if and only if  $(X_s, \mathcal{I}) \in \mathcal{A}$  for every  $s \in S$  and  $S$  is finite.*

*Proof.* Straightforward. □

In Proposition 1.7 we show that in Proposition 1.5(1) we cannot take an infinite  $S$ . In Example 1.8 we show that in Proposition 1.5 we cannot take  $\mathcal{A} = BW$ . On the other hand, Propositions 1.9, 1.10 and 1.11 show that for some classes of ideals or spaces we can extend Proposition 1.5(1) for  $S$  infinite or  $\mathcal{A} = BW$ .

An ideal  $\mathcal{I}$  is *analytic* if it is an analytic subset of  $\mathcal{P}(\mathbb{N})$  with the product topology (here we identify  $\mathcal{P}(\mathbb{N})$  with the Cantor space  $\{0, 1\}^{\mathbb{N}}$ ). A function  $\phi : \mathcal{P}(\mathbb{N}) \rightarrow [0, \infty]$  is called *submeasure* if  $\phi(\emptyset) = 0$  and  $\phi(A) \leq \phi(B) + \phi(C)$  for any  $A \subseteq B \cup C$ . A submeasure  $\phi$  is *lower semicontinuous* (*lsc* for short) if  $\phi(A) = \lim_{n \rightarrow \infty} \phi(A \cap \{0, 1, \dots, n\})$  for any  $A \subseteq \mathbb{N}$ . For a lsc submeasure  $\phi$  we define  $\|A\| = \lim_{n \rightarrow \infty} \phi(A \setminus \{0, 1, \dots, n\})$ . It is not difficult to show that  $\|\cdot\|$  is a submeasure. Moreover we define  $\text{Exh}(\phi) = \{A \subseteq \mathbb{N} : \|A\| = 0\}$ . In [16] the author proved that  $\mathcal{I}$  is an analytic  $P$ -ideal if and only if  $\mathcal{I} = \text{Exh}(\phi)$  for some lsc submeasure  $\phi$ . In [6] the authors proved the following characterization of analytic  $P$ -ideals with the BW property.

**Theorem 1.6** ([6, Theorem 3.6]). *Let  $\mathcal{I} = \text{Exh}(\phi)$  be an analytic  $P$ -ideal. The following conditions are equivalent.*

- (1)  $\mathcal{I} \in BW$ .
- (2) *There is  $\delta > 0$  such that for any partition  $A_1, A_2, \dots, A_N$  of  $\mathbb{N}$  there exists  $i \leq N$  with  $\|A_i\| \geq \delta$ .*

**Proposition 1.7.** *Let  $\mathcal{I} = \text{Exh}(\phi)$  be an analytic  $P$ -ideal without the BW property. Let  $X_i = \omega_1$  for every  $i \in \mathbb{N}$ . Then  $(X_i, \mathcal{I}) \in hFinBW$  for every  $i \in \mathbb{N}$ , but  $(\prod_{i \in \mathbb{N}} X_i, \mathcal{I}) \notin BW$ .*

*Proof.* Since  $\mathcal{I}$  is a  $P$ -ideal, so, by Example 1.3,  $(X_i, \mathcal{I}) \in hFinBW$  for every  $i \in \mathbb{N}$ .

Now we will show that  $(\prod_{i \in \mathbb{N}} X_i, \mathcal{I}) \notin BW$ . By Theorem 1.6, for every  $i \in \mathbb{N}$  there is a partition  $\{A_k^i : k < N_i\}$  of  $\mathbb{N}$  such that

$$(\star) \quad \|A_k^i\| < \frac{1}{i}$$

for every  $k < N_i$ . We define the sequence  $f_n \in \prod_{i \in \mathbb{N}} X_i$  by

$$f_n(i) = k \iff n \in A_k^i,$$

where  $n, i \in \mathbb{N}$  and  $k < N_i$ .

Let  $A \notin \mathcal{I}$ . We will show that  $(f_n)_{n \in A}$  is not  $\mathcal{I}$ -convergent.

Let  $\varepsilon = \|A\| > 0$ . Let  $i_0 \in \mathbb{N}$  be such that  $1/i_0 < \varepsilon$ . Since  $\{A_k^{i_0} : k < N_{i_0}\}$  is a partition of  $\mathbb{N}$  with  $(\star)$  property, so there are distinct  $k_0, k_1 < N_{i_0}$  such that

$\|A \cap A_{k_0}^{i_0}\| > 0$  and  $\|A \cap A_{k_1}^{i_0}\| > 0$ . Hence  $A \cap A_{k_0}^{i_0}, A \cap A_{k_1}^{i_0} \notin \mathcal{I}$ . Let  $U_0 = \{k_0\}$  and  $U_1 = X_{i_0} \setminus \{k_0\}$ . Then  $X_{i_0} = U_0 \cup U_1$ ,  $U_0, U_1$  are open in  $X_{i_0}$  and

$$\{n \in A : f_n(i_0) \in U_j\} \supseteq A \cap A_{k_j}^{i_0} \notin \mathcal{I}$$

for  $j = 0, 1$ . Thus  $(f_n(i_0))_{n \in A}$  is not  $\mathcal{I}$ -convergent, so  $(f_n)_{n \in A}$  is not  $\mathcal{I}$ -convergent.  $\square$

*Remark.* The ideal  $\mathcal{I}_d$  of asymptotic density zero sets is an example of an analytic  $P$ -ideal (see e.g. [3]) without the BW property ([9]).

**Example 1.8.** There exist spaces  $X, Y$  and an ideal  $\mathcal{I}$  such that  $(X, \mathcal{I}) \in \text{BW}$  and  $(Y, \mathcal{I}) \in \text{BW}$ , but  $(X \times Y, \mathcal{I}) \notin \text{BW}$  and  $(X \oplus Y, \mathcal{I}) \notin \text{BW}$ .

*Proof.* Let  $X$  be a sequentially compact space and  $\mathcal{J}_1$  be a maximal ideal such that  $(X, \mathcal{J}_1) \notin \text{BW}$  (such a space is constructed in [15]). Let  $Y$  be a compact space which is not sequentially compact (say  $Y = \beta\mathbb{N}$ ) and  $\mathcal{J}_2 = \text{Fin}$ .

Let  $\mathcal{I} = \mathcal{J}_1 \oplus \mathcal{J}_2$  be an ideal on  $\mathbb{N} \times \{0, 1\}$  defined by

$$A \in \mathcal{J}_1 \oplus \mathcal{J}_2 \iff \begin{aligned} \{n \in \mathbb{N} : (n, 0) \in A\} &\in \mathcal{J}_1 \text{ and} \\ \{n \in \mathbb{N} : (n, 1) \in A\} &\in \mathcal{J}_2. \end{aligned}$$

(Using any bijection between  $\mathbb{N} \times \{0, 1\}$  and  $\mathbb{N}$  we can see  $\mathcal{I}$  as an ideal on  $\mathbb{N}$ .)

It is not difficult to see that for any space  $Z$  we have

$$(Z, \mathcal{J}_1 \oplus \mathcal{J}_2) \in \text{BW} \iff (Z, \mathcal{J}_1) \in \text{BW} \text{ or } (Z, \mathcal{J}_2) \in \text{BW}.$$

Since  $X$  is sequentially compact, so  $(X, \mathcal{I}) \in \text{BW}$ . And by Theorem 1.2,  $(Y, \mathcal{I}) \in \text{BW}$ .

Below we show that  $(X \oplus Y, \mathcal{I}) \notin \text{BW}$ . (It is not difficult to show that if  $(X \oplus Y, \mathcal{I}) \notin \text{BW}$ , then  $(X \times Y, \mathcal{I}) \notin \text{BW}$ .)

Since  $(X, \mathcal{J}_1) \notin \text{BW}$ , so there is a sequence  $(x_n)_{n \in \mathbb{N}}$ ,  $x_n \in X$ , such that  $(x_n)_{n \in A}$  is not  $\mathcal{J}_1$ -convergent for any  $A \notin \mathcal{J}_1$ . Since  $Y$  is not sequentially compact, so there is a sequence  $(y_n)_{n \in \mathbb{N}}$ ,  $y_n \in Y$ , such that  $(y_n)_{n \in B}$  is not  $\text{Fin}$ -convergent for any  $B \notin \text{Fin} = \mathcal{J}_2$ . Let  $z_{n,0} = x_n$  and  $z_{n,1} = y_n$  for every  $n \in \mathbb{N}$ . Then  $(z_{n,i})_{(n,i) \in A}$  is not  $\mathcal{I}$ -convergent for any  $A \notin \mathcal{I}$ .  $\square$

**Question 1.** Do there exist topological spaces  $X, Y$  and an ideal  $\mathcal{I}$  such that  $(X, \mathcal{I}) \in \text{FinBW}$  and  $(Y, \mathcal{I}) \in \text{FinBW}$ , but  $(X \times Y, \mathcal{I}) \notin \text{FinBW}$  or  $(X \oplus Y, \mathcal{I}) \notin \text{FinBW}$ ?

**Proposition 1.9** ([7, Proposition 2.7]). *Let  $\mathcal{A} \in \{BW, \text{FinBW}, hBW, h\text{FinBW}\}$ . If  $([0, 1], \mathcal{I}) \in \mathcal{A}$ , then  $(\prod_{n \in \mathbb{N}} X_n, \mathcal{I}) \in \mathcal{A}$  for any compact metric spaces  $X_n$ .*

**Proposition 1.10** ([1, Theorem 4.2], see also [19, Theorem 4.7]). *Let  $\mathcal{I}$  be a maximal ideal. Let  $\mathcal{A} \in \{BW, hBW\}$ . If  $(X_s, \mathcal{I}) \in \mathcal{A}$  for every  $s \in S$ , then  $(\prod_{s \in S} X_s, \mathcal{I}) \in \mathcal{A}$ .*

If  $\mathcal{I}$  is an ideal, then  $\mathcal{I}^+ = \mathcal{P}(\mathbb{N}) \setminus \mathcal{I}$  is called a *coideal*. We say that  $\mathcal{I}^+$  is a  $P(\mathcal{I})$ -coideal ( $P$ -coideal, resp.) if for every sets  $A_n \in \mathcal{I}^+$  such that  $A_n \supseteq A_{n+1}$  for every  $n \in \mathbb{N}$ , there exists  $A \in \mathcal{I}^+$  with  $A \setminus A_n \in \mathcal{I}$  ( $A \setminus A_n \in \text{Fin}$ , resp.) for every  $n \in \mathbb{N}$ . For examples and properties of  $P$ -coideals see e.g. [2, 17].  $P(\mathcal{I})$ -coideals are considered in [7]. For instance, if  $\mathcal{I}$  is an  $F_\sigma$  ideal then  $\mathcal{I}^+$  is a  $P$ -coideal (see [7, Proposition 5.1]).

**Proposition 1.11.** (1) If  $\mathcal{I}^+$  is a  $P(\mathcal{I})$ -coideal and  $(X_n, \mathcal{I}) \in hBW$  for  $n \in \mathbb{N}$ , then  $(\prod_{n \in \mathbb{N}} X_n, \mathcal{I}) \in hBW$ .  
(2) If  $\mathcal{I}^+$  is a  $P$ -coideal and  $(X_n, \mathcal{I}) \in hFinBW$  for  $n \in \mathbb{N}$ , then  $(\prod_{n \in \mathbb{N}} X_n, \mathcal{I}) \in hFinBW$ .

*Proof.* We show (1) and the proof of (2) is an easy modification of (1). Let  $f_n \in \prod_{i \in \mathbb{N}} X_i$  for  $n \in B \notin \mathcal{I}$ . Since  $(X_i, \mathcal{I}) \in hBW$  for every  $i \in \mathbb{N}$ , so there are sets  $A_i \notin \mathcal{I}$ ,  $i \in \mathbb{N}$ , such that  $A_0 \subseteq B$ ,  $A_i \supseteq A_{i+1}$  for every  $i$  and  $(f_n(i))_{n \in A_i}$  is  $\mathcal{I}$ -convergent to some  $y_i \in X_i$  for every  $i \in \mathbb{N}$ . Let  $f \in \prod_{i \in \mathbb{N}} X_i$  be given by  $f(i) = y_i$ . Since  $\mathcal{I}^+$  is a  $P(\mathcal{I})$ -coideal, so there is  $A \notin \mathcal{I}$  such that  $A \setminus A_i \in \mathcal{I}$  for every  $i \in \mathbb{N}$ . Without loss of generality we can assume that  $A \subseteq B$ . Now we claim that  $(f_n)_{n \in A}$  is  $\mathcal{I}$ -convergent to  $f$ . Let  $U = \prod_{i \in \mathbb{N}} U_i \subseteq \prod_{i \in \mathbb{N}} X_i$  be an open neighborhood of  $f$ . Let  $i_0 \in \mathbb{N}$  such that  $U_i = X_i$  for every  $i \geq i_0$ . Then

$$\{n \in A : f_n \notin U\} = \bigcup_{i < i_0} \{n \in A : f_n(i) \notin U_i\} \subseteq$$

$$\bigcup_{i < i_0} ((A \setminus A_i) \cup \{n \in A_i : f_n(i) \notin U_i\}) \in \mathcal{I}.$$

□

## 2. SPACES DEFINED BY MAD FAMILIES

We say that sets  $A, B$  are *almost disjoint* if  $A \cap B$  is finite.

Let  $\mathcal{A}$  be a pairwise almost disjoint family of infinite subsets of  $\mathbb{N}$ . Define a topological space  $\Psi(\mathcal{A})$  as follows: the underlying set of  $\Psi(\mathcal{A})$  is  $\mathbb{N} \cup \mathcal{A}$ , the points of  $\mathbb{N}$  are isolated and a basic neighborhood of  $A \in \mathcal{A}$  has the form  $\{A\} \cup (A \setminus F)$ , with  $F$  finite. (The space  $\Psi(\mathcal{A})$  was introduced in [14].) It is known that the space  $\Psi(\mathcal{A})$  is Hausdorff, regular, locally compact, first countable and separable; if  $\mathcal{A}$  is infinite then  $\Psi(\mathcal{A})$  is not compact (see [14] or [18, Section 11]). Moreover, it is easy to see that

- $A \cup \{A\}$  is compact in  $\Psi(\mathcal{A})$  for every  $A \in \mathcal{A}$ ;
- $K \cap \mathcal{A}$  is finite for every compact  $K \subseteq \Psi(\mathcal{A})$ ;
- $(K \setminus \bigcup (K \cap \mathcal{A})) \cap \mathbb{N}$  is finite for every compact  $K \subseteq \Psi(\mathcal{A})$ .

Let  $\Phi(\mathcal{A}) = \Psi(\mathcal{A}) \cup \{\infty\}$  be the one-point compactification of  $\Psi(\mathcal{A})$ . (Recall that open neighborhoods of  $\infty$  are of the form  $\Phi(\mathcal{A}) \setminus K$  for compact sets  $K \subseteq \Psi(\mathcal{A})$ .) Thus,  $\Phi(\mathcal{A})$  is Hausdorff and compact. It is not difficult to show that  $\Phi(\mathcal{A})$  is separable and first countable at every point of  $\Phi(\mathcal{A}) \setminus \{\infty\}$ ; if  $\mathcal{A}$  is infinite then  $\Phi(\mathcal{A})$  is not first countable at the point  $\infty$ . If  $\mathcal{A}$  is a *mad family* on  $\mathbb{N}$  (i.e. infinite maximal pairwise almost disjoint family of infinite subsets of  $\mathbb{N}$ ), then  $\Phi(\mathcal{A})$  is sequentially compact (see [11, Theorem 6]).

The following theorem shows that in [4, Theorem 2.2] one does not have to assume that  $\mathcal{A} \subseteq \mathcal{I}$  and BW can be replaced by hBW. The proof of this generalization is a slight modification of the original one but we provide it for the sake of completeness.

**Theorem 2.1** (Essentially [4, Theorem 2.2]). *If  $\mathcal{A}$  is a mad family on  $\mathbb{N}$ , then  $(\Phi(\mathcal{A}), \mathcal{I}) \in hBW$  for every ideal  $\mathcal{I}$ .*

*Proof.* Let  $Y = \Psi(\mathcal{A})$ ,  $X = \Phi(\mathcal{A})$ ,  $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ ,  $B \notin \mathcal{I}$ . We have 3 cases:

- (1)  $C = \{n \in B : x_n = \infty\} \notin \mathcal{I}$ ;

(2)  $C = \{n \in B : x_n \in \mathcal{A}\} \notin \mathcal{I}$ ;

(3)  $C = \{n \in B : x_n \in \mathbb{N}\} \notin \mathcal{I}$ .

In case (1) the subsequence  $(x_n)_{n \in C}$  is  $\mathcal{I}$ -convergent to  $\infty$  and  $C \notin \mathcal{I}$ .

In case (2) we have two subcases:

(2a) there is  $A \in \mathcal{A}$  with  $D = \{n \in C : x_n = A\} \notin \mathcal{I}$ ;

(2b)  $\{n \in C : x_n = A\} \in \mathcal{I}$  for every  $A \in \mathcal{A}$ .

In case (2a) the subsequence  $(x_n)_{n \in D}$  is  $\mathcal{I}$ -convergent to  $A$  and  $D \notin \mathcal{I}$ .

In case (2b) the sequence  $(x_n)_{n \in C}$  is  $\mathcal{I}$ -convergent to  $\infty$  and  $C \notin \mathcal{I}$ . Indeed, let  $K \subseteq Y$  be a compact subset of  $Y$ . We will show that  $\{n \in C : x_n \notin X \setminus K\} \in \mathcal{I}$ . Let  $K \cap \mathcal{A} = \{A_1, \dots, A_m\}$ . Let  $F_i = \{n \in C : x_n = A_i\}$  ( $i = 1, \dots, m$ ). Since  $F_i \in \mathcal{I}$  for every  $i = 1, \dots, m$ , so  $\{n \in C : x_n \notin X \setminus K\} = F_1 \cup \dots \cup F_m \in \mathcal{I}$ .

In case (3) we have 3 subcases:

(3a) there is  $r \in \mathbb{N}$  with  $D = \{n \in C : x_n = r\} \notin \mathcal{I}$ ;

(3b)  $\{n \in C : x_n = r\} \in \mathcal{I}$  for every  $r \in \mathbb{N}$  and  $D = \{n \in C : x_n \in A\} \notin \mathcal{I}$  for some  $A \in \mathcal{A}$ ;

(3c)  $\{n \in C : x_n = r\} \in \mathcal{I}$  for every  $r \in \mathbb{N}$  and  $\{n \in C : x_n \in A\} \in \mathcal{I}$  for every  $A \in \mathcal{A}$ .

In case (3a) the subsequence  $(x_n)_{n \in D}$  is  $\mathcal{I}$ -convergent to  $r$  and  $D \notin \mathcal{I}$ .

In case (3b) the subsequence  $(x_n)_{n \in D}$  is  $\mathcal{I}$ -convergent to  $A$  and  $D \notin \mathcal{I}$ . Indeed, let  $F \subseteq A$  be a finite set. We will show that  $\{n \in D : x_n \notin \{A\} \cup (A \setminus F)\} \in \mathcal{I}$ . Let  $F = \{r_1, \dots, r_m\}$ . Let  $F_i = \{n \in C : x_n = r_i\}$  ( $i = 1, \dots, m$ ). Since  $F_i \in \mathcal{I}$  for every  $i = 1, \dots, m$ , so  $\{n \in D : x_n \notin \{A\} \cup (A \setminus F)\} \subseteq F_1 \cup \dots \cup F_m \in \mathcal{I}$ .

In case (3c) the subsequence  $(x_n)_{n \in C}$  is  $\mathcal{I}$ -convergent to  $\infty$  and  $C \notin \mathcal{I}$ . Indeed, let  $K \subseteq Y$  be a compact subset of  $Y$ . We will show that  $\{n \in C : x_n \notin X \setminus K\} \in \mathcal{I}$ . Let  $K \cap \mathcal{A} = \{A_1, \dots, A_m\}$ . Let  $F_i = \{n \in C : x_n \in A_i\}$  ( $i = 1, \dots, m$ ). Let  $F = (K \setminus (A_1 \cup \dots \cup A_m)) \cap \mathbb{N} = \{r_1, \dots, r_k\}$ . Let  $G_j = \{n \in C : x_n = r_j\}$  ( $j = 1, \dots, k$ ). Since  $F_i, G_j \in \mathcal{I}$  for every  $i = 1, \dots, m, j = 1, \dots, k$ , so

$$\{n \in C : x_n \notin X \setminus K\} \subseteq F_1 \cup \dots \cup F_m \cup G_1 \cup \dots \cup G_k \in \mathcal{I}.$$

□

An ideal  $\mathcal{I}$  on  $\mathbb{N}$  is *dense* if for every infinite  $A \subseteq \mathbb{N}$  there is an infinite  $B \subseteq A$  such that  $B \in \mathcal{I}$ .

**Proposition 2.2.** *An ideal  $\mathcal{I}$  on  $\mathbb{N}$  is dense if and only if there is a mad family  $\mathcal{A}$  on  $\mathbb{N}$  such that  $\mathcal{A} \subseteq \mathcal{I}$ .*

*Proof.* ( $\Rightarrow$ ) Let  $\mathcal{A}$  be a maximal element of the family

$$\mathcal{F} = \{\mathcal{B} : \mathcal{B} \subseteq \mathcal{I} \text{ and } \mathcal{B} \text{ is almost disjoint family consisting of infinite sets}\}.$$

(There is one by Zorn's lemma.) Now it is enough to show that  $\mathcal{A}$  is also a maximal element of the family

$$\mathcal{G} = \{\mathcal{C} : \mathcal{C} \text{ is almost disjoint family consisting of infinite sets}\}.$$

Let  $C$  be an infinite subset of  $\mathbb{N}$ . Since  $\mathcal{I}$  is dense, so there is an infinite  $C' \subseteq C$  such that  $C' \in \mathcal{I}$ . Since  $\mathcal{A}$  is maximal in  $\mathcal{F}$ , so there is  $A \in \mathcal{A}$  such that  $A \cap C'$  is infinite. Then  $A \cap C$  is infinite as well, hence  $\mathcal{A}$  is maximal in  $\mathcal{G}$ .

( $\Leftarrow$ ) Let  $\mathcal{A}$  be a mad family on  $\mathbb{N}$  such that  $\mathcal{A} \subseteq \mathcal{I}$ . Let  $B$  be an infinite subset of  $\mathbb{N}$ . Since  $\mathcal{A}$  is mad, so there is  $A \in \mathcal{A}$  such that  $A \cap B$  is infinite. But  $A \cap B \subseteq A \in \mathcal{A} \subseteq \mathcal{I}$ , so  $A \cap B \in \mathcal{I}$ .

□

**Proposition 2.3.** *Let  $\mathcal{I}$  be a dense ideal on  $\mathbb{N}$ . If  $\mathcal{A}$  is a mad family on  $\mathbb{N}$  such that  $\mathcal{A} \subseteq \mathcal{I}$ , then  $(\Phi(\mathcal{A}), \mathcal{I}) \notin \text{FinBW}$ .*

*Proof.* Let  $x_n = n$  for  $n \in \mathbb{N}$ . Suppose that there is  $B \notin \mathcal{I}$  such that  $(x_n)_{n \in B}$  is Fin-convergent to some  $x \in X$ . It is obvious that  $x \notin \mathbb{N}$ . Thus we have two cases:

(1)  $x = A$  for some  $A \in \mathcal{A}$  or (2)  $x = \infty$ .

Case (1). Since  $A \in \mathcal{I}$  so  $C = B \setminus A$  is infinite. Since  $U = \{A\} \cup A$  is an open neighborhood of  $A$ , so  $\{n \in B : x_n \notin U\}$  is finite. On the other hand  $C \subseteq \{n \in B : x_n \notin U\}$ , a contradiction.

Case (2). Let  $A \in \mathcal{A}$  be such that  $C = A \cap B$  is infinite. Since  $U = X \setminus (A \cup \{A\})$  is an open neighborhood of  $\infty$ , so  $\{n \in B : x_n \notin U\}$  is finite. On the other hand,  $C \subseteq \{n \in B : x_n \notin U\}$ , a contradiction. □

In [8, see the note below the proof of Proposition 2.2] the author claims that for every  $F_\sigma$  ideal  $\mathcal{I}$  there exists a Hausdorff, compact, sequentially compact, separable space which is first-countable at all points but one, such that  $(X, \mathcal{I}) \notin \text{FinBW}$ . In Proposition 2.4 we show that this is not true for  $F_\sigma$  ideals which are not dense, and in Proposition 2.5 we show that this is true for all (not only  $F_\sigma$ ) dense ideals.

**Proposition 2.4.** *Let  $\mathcal{I}$  be an ideal on  $\mathbb{N}$  which is not dense. For every  $X$ ,  $X$  is sequentially compact  $\iff (X, \mathcal{I}) \in \text{FinBW}$ .*

*Proof.* The implication “ $\Leftarrow$ ” is obvious (and holds even for dense ideals). Now we show the opposite implication. Let  $A = \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{N}$  be an infinite set such that for every infinite  $B \subseteq A$  we have  $B \notin \mathcal{I}$ . Let  $x_n \in X$  ( $n \in \mathbb{N}$ ). Define  $y_n = x_{a_n}$  ( $n \in \mathbb{N}$ ). Since  $X$  is sequentially compact, so there is infinite  $C \subseteq \mathbb{N}$  such that  $(y_n)_{n \in C}$  is convergent to some  $x \in X$ . Then  $B = \{a_n : n \in C\} \notin \mathcal{I}$  and  $(x_n)_{n \in B}$  is convergent to  $x$ . □

**Proposition 2.5.** *Let  $\mathcal{I}$  be a dense ideal on  $\mathbb{N}$ . There exists a Hausdorff, compact, sequentially compact, separable space which is first-countable at all points but one, such that  $(X, \mathcal{I}) \notin \text{FinBW}$ .*

*Proof.* By Proposition 2.2 there is a mad family  $\mathcal{A}$  on  $\mathbb{N}$  such that  $\mathcal{A} \subseteq \mathcal{I}$ . Then by Proposition 2.3 the space  $X = \Phi(\mathcal{A})$  is the required one. □

### 3. THE VAN DER WAERDEN IDEAL

A set  $A \subseteq \mathbb{N}$  is called an *AP-set* if it contains arbitrarily long finite arithmetic progressions.

**Theorem 3.1** (van der Waerden). *If an AP-set is partitioned into finitely many parts, then one of the parts is an AP-set.*

Let  $\mathcal{W}$  denote the family of all non-AP-sets  $A \subseteq \mathbb{N}$ . Then by van der Waerden’s theorem  $\mathcal{W}$  is an ideal, and we call it the *van der Waerden ideal*. It is not difficult to see that  $\mathcal{W}$  is a dense ideal and it is not a  $P$ -ideal.

The BW-like properties with respect to the ideal  $\mathcal{W}$  were already considered in the literature [8, 11, 12], where the authors considered spaces  $X$  such that  $(X, \mathcal{W}) \in \text{FinBW}$  and called them *van der Waerden spaces*. For instance, in [11, Theorem 10] the author proved that  $(X, \mathcal{W}) \in \text{FinBW}$  for all compact metric spaces  $X$ . Moreover, in [11, Proposition 4] the author proved the following theorem.

**Theorem 3.2** (Kojman). *For every  $X$ ,  $(X, \mathcal{W}) \in \text{FinBW} \iff (X, \mathcal{W}) \in \text{hFinBW}$ .*

Below we show how to modify the proof of Theorem 3.2 to obtain the following theorem.

**Theorem 3.3.** *For every  $X$ ,  $(X, \mathcal{W}) \in \text{BW} \iff (X, \mathcal{W}) \in \text{hBW}$ .*

*Proof.* The implication “ $\Leftarrow$ ” is obvious, so we only prove “ $\Rightarrow$ ”. Let  $x_n \in X$  for  $n \in A \notin \mathcal{W}$ . By thinning out  $A$  we may assume that  $A = \bigcup_{n \in \mathbb{N}} C_n$ , where each  $C_n$  is an arithmetic progression,  $2 \cdot \max C_n < \min C_{n+1}$  and  $|C_{n+1}| > (\max C_n)^2$ .

Let  $h : \mathbb{N} \rightarrow A$  be the increasing enumeration of  $A$ . Since  $(X, \mathcal{W}) \in \text{BW}$ , there exists  $B \notin \mathcal{W}$  such that  $(x_{h(n)})_{n \in B}$  is  $\mathcal{W}$ -convergent to some  $x \in X$ .

We will show that  $(x_n)_{n \in h[B]}$  is  $\mathcal{W}$ -convergent to  $x$  and  $h[B] \notin \mathcal{W}$ .

First observe that for each  $C_n$  and arithmetic progression  $C \subseteq B$ ,  $h[C] \cap C_n$  is an arithmetic progression and for each arithmetic progression  $D \subseteq C_n$ ,  $h^{-1}[D]$  is also an arithmetic progression.

Second note that for each arithmetic progression  $D = \{a_0, a_0 + d, \dots, a_0 + (m-1)d\} \subseteq A$  of length  $m$ , at least  $m-1$  of its elements belong to a particular  $C_{n_0}$ . Indeed, let  $a_0 + ld \in C_{n_0}$  for some positive  $l \leq (m-1)$  and  $a_0 + (l-1)d \in C_n$  for some  $n < n_0$ . Then  $a_0 + (l-2)d$  would be below 0 since  $d > \max C_n$  in that situation.

Now we show that  $(x_n)_{n \in h[B]}$  is  $\mathcal{W}$ -convergent to  $x$ . Suppose that there is an open neighborhood of  $x$ ,  $U \subseteq X$ , such that  $A' = \{i \in h[B] : x_i \notin U\} \notin \mathcal{W}$ . Let  $D \subseteq A'$  be an arithmetic progression of length  $m$ . Then there exists  $n_0$  such that  $A' \cap C_{n_0}$  is an arithmetic progression of length at least  $m-1$ . Hence  $h^{-1}[A' \cap C_{n_0}]$  is also an arithmetic progression of length at least  $m-1$ , and  $h^{-1}[A' \cap C_{n_0}] \subseteq \{n \in B : x_{h(n)} \notin U\}$ . Thus  $\{n \in B : x_{h(n)} \notin U\} \notin \mathcal{W}$ , a contradiction.

Finally we show that  $h[B] \notin \mathcal{W}$ . The argument is the same as in the proof of Theorem 3.2, so we provide it only for the completeness.

Let  $k \in \mathbb{N}$  be arbitrary and  $C = \{a_0, a_0 + d, \dots, a_0 + (k-1)d\} \subseteq B$  be an arithmetic progression of length  $k$  so that  $k < a_0$ . Let  $n_0$  be such that  $h(a_0 + d) \in C_{n_0}$ . Clearly  $k, a_0$  and  $d$  are smaller than  $\max C_{n_0}$ , hence  $a_0 + (k-1)d \leq \max C_{n_0} + (\max C_{n_0} - 1)(\max C_{n_0}) = (\max C_{n_0})^2 \leq |C_{n_0+1}|$ . Since  $h$  is the increasing enumeration of  $\bigcup_{n \in \mathbb{N}} C_n$ , it holds that  $h(a_0 + (k-1)d) \in C_{n_0} \cup C_{n_0+1}$ , and consequently  $h[C \setminus \{a_0\}] \subseteq C_{n_0} \cup C_{n_0+1}$ . Thus either  $h[C] \cap C_{n_0}$  or  $h[C] \cap C_{n_0+1}$  is an arithmetic progression of length no less than  $(k-1)/2$ .  $\square$

**Proposition 3.4.** *There exists a Hausdorff, compact, sequentially compact, separable space  $X$  which is first-countable at all points but one such that  $(X, \mathcal{W}) \in \text{hBW}$  and  $(X, \mathcal{W}) \notin \text{FinBW}$ .*

*Proof.* Since  $\mathcal{W}$  is a dense ideal, so, by Proposition 2.2, there is a mad family on  $\mathbb{N}$  such that  $\mathcal{A} \subseteq \mathcal{W}$ . Let  $X = \Phi(\mathcal{A})$ . By Theorem 2.1  $(X, \mathcal{W}) \in \text{hBW}$  and by [11, Theorem 6]  $(X, \mathcal{W}) \notin \text{FinBW}$ .  $\square$

**Example 3.5.** There exists a sequentially compact space  $X$  which is not compact and  $(X, \mathcal{W}) \in \text{hFinBW}$ .

*Proof.* Let  $X = \omega_1$  with the order topology. By Example 1.3,  $X$  is sequentially compact and not compact. By [11, Theorem 10 and Proposition 4],  $(X, \mathcal{W}) \in \text{hFinBW}$ .  $\square$

**Example 3.6.** There exists a compact space  $X$  which is not sequentially compact and  $(X, \mathcal{W}) \notin \text{BW}$ .

*Proof.* Let  $X = [0, 1]^{[0,1]}$  with the product topology. It is known that  $X$  is compact and not sequentially compact (see e.g. [13, Example 105]).

It is not difficult to see that for any  $A \notin \mathcal{W}$  there are disjoint sets  $B, C \notin \mathcal{W}$  such that  $A = B \cup C$ , so, by [5, Proposition 5.2],  $(X, \mathcal{W}) \notin \text{BW}$   $\square$

**Proposition 3.7.** *Let  $S$  be finite. Let  $\mathcal{A} \in \{\text{BW}, \text{FinBW}, \text{hBW}, \text{hFinBW}\}$ . If  $(X_s, \mathcal{W}) \in \mathcal{A}$  for every  $s \in S$ , then  $(\bigoplus_{s \in S} X_s, \mathcal{W}) \in \mathcal{A}$ .*

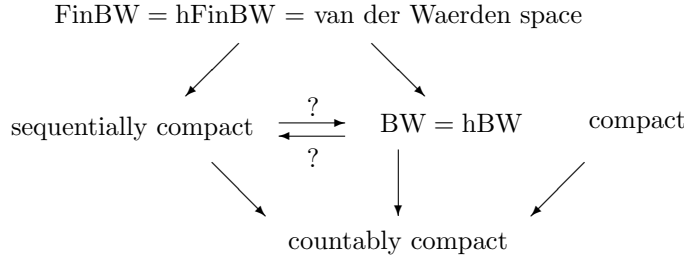
*Proof.* For  $\mathcal{A} \in \{\text{hBW}, \text{hFinBW}\}$  it is Proposition 1.5. For  $\mathcal{A} \in \{\text{BW}, \text{FinBW}\}$ , apply Proposition 1.5 and Theorem 3.3 or 3.2, respectively.  $\square$

**Proposition 3.8.** *Let  $S$  be countable. Let  $\mathcal{A} \in \{\text{BW}, \text{FinBW}, \text{hBW}, \text{hFinBW}\}$ . If  $(X_s, \mathcal{W}) \in \mathcal{A}$  for every  $s \in S$ , then  $(\prod_{s \in S} X_s, \mathcal{W}) \in \mathcal{A}$ .*

*Proof.* First we show that  $\mathcal{W}$  is a  $P$ -coideal (hence  $P(\mathcal{W})$ -coideal, as well). Let  $A_n \in \mathcal{W}^+$ ,  $A_n \supseteq A_{n+1}$  for  $n \in \mathbb{N}$ . For every  $n$  let  $F_n \subseteq A_n$  be an arithmetic progression of length  $n$ . Then  $A = \bigcup_n F_n \notin \mathcal{W}$  and  $A \setminus A_n \subseteq \bigcup_{i < n} F_i \in \text{Fin}$ .

Now it is enough to apply Proposition 1.11 and Theorem 3.3 or 3.2, respectively.  $\square$

The following diagram summarize the relationship between BW-like spaces with respect to the van der Waerden ideal  $\mathcal{W}$  and compactness properties (if there is no arrow in some direction, it means that there is no implication between those notions).



**Question 2.** Does there exist a sequentially compact space  $X$  such that  $(X, \mathcal{W}) \notin \text{BW}$ ?

**Question 3.** Does there exist a space  $X$  such that  $(X, \mathcal{W}) \in \text{BW}$ , but  $X$  is not sequentially compact?

#### 4. THE HINDMAN IDEAL

For  $A \subseteq \mathbb{N}$  we write

$$\text{FS}(A) = \left\{ \sum_{n \in F} n : F \subseteq A \text{ is nonempty and finite} \right\},$$

i.e.  $\text{FS}(A)$  is the set of all finite non-repeating sums of members of  $A$ . A set  $A \subseteq \mathbb{N}$  is called an *IP-set* if there is an infinite  $D \subseteq A$  with  $\text{FS}(D) \subseteq A$ .

**Theorem 4.1** (Hindman). *If an IP-set is partitioned into finitely many parts, then one of the parts is an IP-set.*

Let  $\mathcal{H}$  denote the family of all non-IP-sets  $A \subseteq \mathbb{N}$ . Then by Hindman's theorem  $\mathcal{H}$  is an ideal, and we call it the *Hindman ideal*.

Let  $D \subseteq \mathbb{N}$  be an infinite set and  $X$  be a topological space. A sequence  $(x_n)_{n \in \text{FS}(D)}$ ,  $x_n \in X$ , is *IP-convergent* to  $x \in X$  if for every open neighborhood  $U \subseteq X$ ,  $x \in U$ , there is  $m \in \omega$  such that

$$\{x_n : n \in \text{FS}(D \setminus \{0, 1, \dots, m-1\})\} \subseteq U.$$

A topological space  $X$  is called a *Hindman space* if for every sequence  $(x_n)_{n \in \mathbb{N}}$ ,  $x_n \in X$  there is an infinite set  $D \subseteq \mathbb{N}$  such that  $(x_n)_{n \in \text{FS}(D)}$  is IP-convergent to some  $x \in X$  ([10, Definition 4]). It is easy to see that every Hindman space is sequentially compact.

The BW-like properties with respect to the ideal  $\mathcal{H}$  and Hindman spaces were already considered in the literature [4, 8, 10, 12]. For instance, in [10, Theorem 11] the author proved that all compact metric spaces are Hindman, whereas in [4, Proposition 3.5] the author proved that if  $X$  is a Hindman space, then  $(X, \mathcal{H}) \in \text{BW}$  (hence  $(X, \mathcal{H}) \in \text{BW}$  for all compact metric spaces  $X$ ). Moreover in [4, Theorem 3.7] the author showed that there exists a Hausdorff, compact, sequentially compact, separable space  $X$  which is first-countable at all points but one such that  $(X, \mathcal{H}) \in \text{BW}$  but  $X$  is not Hindman. In [10, Theorem 3] the author proved the following theorem.

**Theorem 4.2** (Kojman). *For every  $X$ ,  $(X, \mathcal{H}) \in \text{FinBW} \iff (X, \mathcal{H}) \in \text{hFinBW} \iff X$  is finite.*

An infinite set  $D \subseteq \mathbb{N}$  is called *sparse* if for every  $x \in \text{FS}(D)$  there is a unique nonempty finite set  $F \subseteq D$  with  $x = \sum_{n \in F} n$ . It is not difficult to show that for every infinite  $D$  there is an infinite sparse  $E \subseteq D$ .

When  $D = \{d_1, d_2, \dots\} \subseteq \mathbb{N}$  is sparse and  $x \in \text{FS}(D)$ , there is a unique nonempty finite set  $F$  for which  $x = \sum_{i \in F} d_i$ . We denote that unique  $F$  by  $\alpha_D(x)$ .

Let  $D$  be a sparse set. A set  $D_1 \subseteq \text{FS}(D)$  is called *normal* in  $\text{FS}(D)$  if  $D_1$  is sparse and for all  $x, y \in D_1$ , if  $x \neq y$ , then  $\alpha_D(x) \cap \alpha_D(y) = \emptyset$ .

**Lemma 4.3.** *If  $A = \{a_1, a_2, \dots\}$  is sparse such that  $2 \sum_{i \leq n} a_i < a_{n+1}$  for each  $n \in \mathbb{N}$ , then for every  $\text{FS}(H) \subseteq \text{FS}(A)$ ,  $H$  is normal in  $\text{FS}(A)$ .*

*Proof.* Suppose  $H$  is not normal in  $\text{FS}(A)$ . We could find then such  $h_1, h_2 \in H$  that  $h_1 = a_k + \sum_{i \in \alpha_A(h_1) \setminus \{k\}} a_i$ ,  $h_2 = a_k + \sum_{i \in \alpha_A(h_2) \setminus \{k\}} a_i$ , for some  $k \in \mathbb{N}$ .

Let  $K = \max(\alpha_A(h_1) \cup \alpha_A(h_2))$ . Notice that since  $2 \sum_{i \leq K} a_i < a_{K+1}$ , it means that  $h_1 + h_2 < a_{K+1}$ . Now we have two possible cases:

(a) Should  $K \in \alpha_A(h_1) \cap \alpha_A(h_2)$  then  $a_{K+1} > h_1 + h_2 > \sum_{i \leq K} a_i$ , so a contradiction with  $h_1 + h_2 \in \text{FS}(H) \subseteq \text{FS}(A)$ .

(b) Should  $K \notin \alpha_A(h_1) \cap \alpha_A(h_2)$  then  $K \in \alpha_A(h_1 + h_2)$ . It is clear that  $h_1 + h_2 - a_K \in \text{FS}(A)$ . Repeat the above reasoning for  $h_1 + h_2 - a_K$  and  $K' = \max((\alpha_A(h_1) \cup \alpha_A(h_2)) \setminus \{K\})$ . Since  $k \in \alpha_A(h_1) \cap \alpha_A(h_2)$ , after at most  $K - k$  steps case (a) will hold. That gives a contradiction.  $\square$

**Lemma 4.4** ([10, Lemma 7]). *Let  $E = \{2^n : n \in \mathbb{N}\}$ . If  $B$  is sparse, then there exists  $B_1 \subseteq \text{FS}(B)$  which is normal both in  $\text{FS}(E)$  and  $\text{FS}(B)$  (hence  $\text{FS}(B_1) \subseteq \text{FS}(B)$ ).*

**Theorem 4.5.** *For every  $X$ ,  $(X, \mathcal{H}) \in BW \iff (X, \mathcal{H}) \in hBW$ .*

*Proof.* The implication “ $\Leftarrow$ ” is obvious, so we only prove “ $\Rightarrow$ ”. Let  $x_n \in X$  for  $n \in A' \notin \mathcal{H}$ . Without loss of generality we can assume that  $A' = \text{FS}(A)$  where  $A = \{a_n : n \in \mathbb{N}\}$  is a sparse set such that  $2 \sum_{i \leq n} a_i < a_{n+1}$  for each  $n \in \mathbb{N}$ .

Let  $E = \{2^n : n \in \mathbb{N}\}$ . Note that  $E$  is sparse and  $\text{FS}(E) = \mathbb{N}$ . We define  $g : \mathbb{N} \rightarrow \text{FS}(A)$  by

$$g(n) = g \left( \sum_{i \in \alpha_E(n)} 2^i \right) = \sum_{i \in \alpha_E(n)} a_i.$$

Let  $y_n = x_{g(n)}$  for  $n \in \mathbb{N}$ . Since  $(X, \mathcal{H}) \in BW$ , so there is an infinite  $B \subseteq \mathbb{N}$  and  $y \in X$  such that  $(y_n)_{n \in \text{FS}(B)}$  is  $\mathcal{H}$ -convergent to  $y$ . We can assume that  $B$  is sparse and, by Lemma 4.4, we can also assume that  $B$  is normal in  $\text{FS}(E)$ .

Let  $C = g[B]$ . Since  $B$  is normal in  $\text{FS}(E)$ , so  $C$  is normal in  $\text{FS}(A)$ . Thus  $\text{FS}(C) \subseteq \text{FS}(A)$ . We will show that  $(x_n)_{n \in \text{FS}(C)}$  is  $\mathcal{H}$ -convergent to  $y$  (and that will finish the proof). First we have to prove two claims.

*Claim.*  $g[\text{FS}(B)] = \text{FS}(g[B])$ ;

*Proof of Claim.* Let  $b_i \in B$  for  $i = 1, \dots, n$ . Since  $B$  is normal in  $\text{FS}(E)$ , so  $\alpha_E(b_i) \cap \alpha_E(b_j) = \emptyset$  for  $i \neq j$ . Since  $E$  is sparse, so  $\alpha_E(\sum_{i=1}^n b_i) = \bigcup_{i=1}^n \alpha_E(b_i)$ .

Finally,

$$\begin{aligned} g \left( \sum_{i=1}^n b_i \right) &= g \left( \sum_{j \in \alpha_E(\sum_{i=1}^n b_i)} 2^j \right) = \sum_{j \in \alpha_E(\sum_{i=1}^n b_i)} a_j = \\ &= \sum_{j \in \bigcup_{i=1}^n \alpha_E(b_i)} a_j = \sum_{i=1}^n \left( \sum_{j \in \alpha_E(b_i)} a_j \right) = \sum_{i=1}^n g(b_i). \end{aligned}$$

□

*Claim.* If  $D \subseteq \text{FS}(B)$ ,  $D \in \mathcal{H}$ , then  $g[D] \in \mathcal{H}$ .

*Proof of Claim.* Suppose that  $g[D] \notin \mathcal{H}$ . Let  $G \subseteq \mathbb{N}$  be an infinite set with  $\text{FS}(G) \subseteq g[D]$ . Since  $\text{FS}(G) \subseteq \text{FS}(A)$ , so, by Lemma 4.3,  $G$  is normal in  $\text{FS}(A)$ . Let  $D' = g^{-1}[G]$ . Since  $A$  is sparse,  $g$  is a bijection, thus  $D' \subseteq D$ . We will show that  $\text{FS}(D') \subseteq D$ , which would obviously be a contradiction with  $D \in \mathcal{H}$  (and that will finish the proof of the claim). First note that  $\alpha_A(g(n)) = \alpha_E(n)$  for every  $n \in \mathbb{N}$ . Suppose  $d_1, d_2 \in D'$ . Since  $G$  is normal in  $\text{FS}(A)$ , we know that  $\alpha_A(g(d_1)) \cap \alpha_A(g(d_2)) = \emptyset$ , which means that

$$\begin{aligned} g(d_1) + g(d_2) &= \sum_{i \in \alpha_A(g(d_1))} a_i + \sum_{i \in \alpha_A(g(d_2))} a_i = \sum_{i \in \alpha_E(d_1)} a_i + \sum_{i \in \alpha_E(d_2)} a_i = \\ &= \sum_{i \in \alpha_E(d_1) \cup \alpha_E(d_2)} a_i = \sum_{i \in \alpha_E(d_1 + d_2)} a_i = g(d_1 + d_2). \end{aligned}$$

Therefore,  $g(d_1 + d_2) \in \text{FS}(G) \subseteq g[D]$ , so  $d_1 + d_2 \in D$ . The case of arbitrary finite sum can be shown in the same manner. □

Now we are ready to show that  $(x_n)_{n \in \text{FS}(C)}$  is  $\mathcal{H}$ -convergent to  $y$ . Let  $U \subseteq X$  be an open neighborhood of  $y$ . Since  $\{n \in \text{FS}(B) : y_n \notin U\} \in \mathcal{H}$ , so  $g[\{n \in \text{FS}(B) : y_n \notin U\}] \in \mathcal{H}$ . On the other hand,  $g[\{n \in \text{FS}(B) : y_n \notin U\}] = \{g(n) \in g[\text{FS}(B)] : x_{g(n)} \notin U\} = \{k \in \text{FS}(g[B]) : x_k \notin U\} = \{k \in \text{FS}(C) : x_k \notin U\}$ .  $\square$

**Corollary 4.6.** *If  $X$  is a Hindman space then  $(X, \mathcal{H}) \in \text{hBW}$ .*

*Proof.* If  $X$  is a Hindman space, then by [4, Proposition 3.5]  $(X, \mathcal{H}) \in \text{BW}$ . Thus, by Theorem 4.5  $(X, \mathcal{H}) \in \text{hBW}$ .  $\square$

**Corollary 4.7.** *There exists a Hausdorff, compact, sequentially compact, separable space  $X$  which is first-countable at all points but one such that  $(X, \mathcal{H}) \in \text{hBW}$  but  $X$  is not Hindman.*

*Proof.* By [4, Theorem 3.7] there exists a Hausdorff, compact, sequentially compact, separable space  $X$  which is first-countable at all points but one such that  $(X, \mathcal{H}) \in \text{BW}$  and  $X$  is not Hindman. Thus, by Theorem 4.5  $(X, \mathcal{H}) \in \text{hBW}$ .  $\square$

**Example 4.8.** There exists a sequentially compact, Hindman space  $X$  which is not compact and  $(X, \mathcal{H}) \in \text{hBW}$ .

*Proof.* Let  $X = \omega_1$  with the order topology. By Example 1.3,  $X$  is sequentially compact, not compact and  $(X, \mathcal{H}) \in \text{hBW}$ . By [10, Theorem 11],  $X$  is Hindman.  $\square$

**Example 4.9.** There exists a compact space  $X$  which is not sequentially compact and  $(X, \mathcal{H}) \notin \text{BW}$ .

*Proof.* Let  $X = [0, 1]^{[0, 1]}$  with the product topology. It is known that  $X$  is compact and not sequentially compact (see e.g. [13, Example 105]).

It is not difficult to see that for any  $A \notin \mathcal{H}$  there are disjoint sets  $B, C \notin \mathcal{H}$  such that  $A = B \cup C$ , so, by [5, Proposition 5.2],  $(X, \mathcal{H}) \notin \text{BW}$ .  $\square$

**Proposition 4.10.** *Let  $S$  be finite. Let  $\mathcal{A} \in \{\text{BW}, \text{FinBW}, \text{hBW}, \text{hFinBW}\}$ . If  $(X_s, \mathcal{H}) \in \mathcal{A}$  for every  $s \in S$ , then  $(\prod_{s \in S} X_s, \mathcal{H}) \in \mathcal{A}$  and  $(\bigoplus_{s \in S} X_s, \mathcal{H}) \in \mathcal{A}$ .*

*Proof.* For  $\mathcal{A} \in \{\text{hBW}, \text{hFinBW}\}$  it is Proposition 1.5. For  $\mathcal{A} \in \{\text{BW}, \text{FinBW}\}$ , apply Proposition 1.5 and Theorem 4.5 or 4.2, respectively.  $\square$

In Proposition 4.10 one cannot take  $S$  infinite and  $\mathcal{A} \in \{\text{FinBW}, \text{hFinBW}\}$  for products of spaces (unless all but finitely many spaces consist of one point). Indeed, if  $X_i = \{0, 1\}$  for  $i \in \mathbb{N}$  then  $(X_i, \mathcal{H}) \in \text{hFinBW}$  for every  $i$ , whereas  $\prod_i X_i$  is infinite, so  $(\prod_i X_i, \mathcal{H}) \notin \text{FinBW}$ .

In Proposition 4.11 we show that in Proposition 4.10 one can take  $S$  infinite and  $\mathcal{A} \in \{\text{BW}, \text{hBW}\}$  for products of spaces. However the proof does not use Proposition 1.11 because we do not know if  $\mathcal{H}^+$  is a  $P(\mathcal{H})$ -coideal.

**Proposition 4.11.** *Let  $\mathcal{A} \in \{\text{BW}, \text{hBW}\}$ . If  $(X_n, \mathcal{H}) \in \mathcal{A}$  for  $n \in \mathbb{N}$ , then  $(\prod_{n \in \mathbb{N}} X_n, \mathcal{H}) \in \mathcal{A}$ .*

*Proof.* By Theorem 4.5 it is enough to consider the case  $\mathcal{A} = \text{hBW}$ . Let  $f_n \in \prod_{i \in \mathbb{N}} X_i$  for  $n \in \mathbb{N}$ . Without loss of generality we can assume that  $B' = \text{FS}(B)$  for some infinite sparse  $B = \{b_0, b_1, \dots\}$  such that  $2 \sum_{k \leq n} b_k < b_{n+1}$ .

First we construct infinite sets  $A_i = \{a_k^i : k \in \mathbb{N}\} \subseteq \mathbb{N}$ ,  $i \in \mathbb{N}$ , such that

- (1)  $\text{FS}(B) \supseteq \text{FS}(A_i) \supseteq \text{FS}(A_{i+1})$  for any  $i$ ,
- (2)  $(f_n(i))_{n \in \text{FS}(A_i)}$  is  $\mathcal{H}$ -convergent to some  $y_i \in X_i$ ,

(3)  $A_i$  is normal in  $\text{FS}(B)$ .

Since  $(X_0, \mathcal{H}) \in \text{hBW}$ , there is an infinite set  $A_0$ , such that  $\text{FS}(A_0) \subseteq \text{FS}(B)$  and  $(f_n(0))_{n \in \text{FS}(A_0)}$  is  $\mathcal{H}$ -convergent to some  $y_0 \in X_0$ . By Lemma 4.3  $A_0$  is normal in  $\text{FS}(B)$ .

Suppose that sets  $A_j$  have been constructed for  $j < i$ . Since  $(X_i, \mathcal{H}) \in \text{hBW}$ , there is an infinite set  $A_i$  such that  $\text{FS}(A_i) \subseteq \text{FS}(A_{i-1})$  and  $(f_n(i))_{n \in A_i}$  is  $\mathcal{H}$ -convergent to some  $y_i \in X_i$ . By Lemma 4.3  $A_i$  is normal in  $\text{FS}(B)$  and  $\text{FS}(A_j)$  for  $j < i$ . That finishes the construction of sets  $A_i$ . It is obvious that sets  $A_i$  have the required properties.

Now we construct an infinite set  $A = \{a_0, a_1, \dots\}$  such that

- (1)  $\text{FS}(A) \subseteq \text{FS}(B)$ ,
- (2)  $\text{FS}(A) \setminus \text{FS}(A_i) \in \mathcal{H}$  for each  $i \in \mathbb{N}$ .

Let  $a_0 = \min A_0$ . Suppose that we have already constructed  $a_k$  for  $k < i$ . Let

$$a_i = \min \left\{ m \in A_i : \min \alpha_B(m) > \max \alpha_B(a_{i-1}) \text{ and } 2 \sum_{k < i} a_k < a_i \right\}.$$

First note that the definition of  $a_i$  is correct. Indeed, since  $A_i$  is infinite and normal in  $\text{FS}(B)$ , so the minimum is taken from a nonempty set. Second we show that the set  $A$  has the required properties. Since  $\alpha_B(a_n) \cap \alpha_B(a_k) = \emptyset$  for  $n \neq k$ , so  $\text{FS}(A) \subseteq \text{FS}(B)$ . Suppose that there is an infinite set  $H$  and  $i \in \mathbb{N}$  such that  $\text{FS}(H) \subseteq \text{FS}(A) \setminus \text{FS}(A_i)$ . Since

$$H \subseteq \text{FS}(A) \setminus \text{FS}(A_i) \subseteq \text{FS}(\{a_k : k < i\}) \cup (\text{FS}(\{a_k : k < i\}) + \text{FS}(\{a_k : k \geq i\})),$$

$\text{FS}(\{a_k : k < i\})$  is finite and  $H$  is infinite, so there are distinct  $x, y \in H$  and  $k < i$  such that  $k \in \alpha_A(x) \cap \alpha_A(y)$ . But, by Lemma 4.3,  $H$  is normal in  $\text{FS}(A)$ , a contradiction.

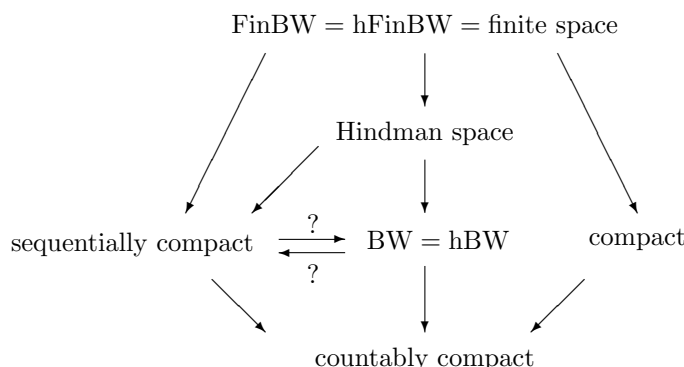
Let  $f \in \prod_{i \in \mathbb{N}} X_i$  be given by  $f(i) = y_i$ . We claim that  $(f_n)_{n \in \text{FS}(A)}$  is  $\mathcal{H}$ -convergent to  $f$ . Let  $U = \prod_{i \in \mathbb{N}} U_i \subseteq \prod_{i \in \mathbb{N}} X_i$  be an open neighborhood of  $f$ . Let  $i_0 \in \mathbb{N}$  such that  $U_i = X_i$  for every  $i \geq i_0$ . Then

$$\{n \in \text{FS}(A) : f_n \notin U\} = \bigcup_{i < i_0} \{n \in \text{FS}(A) : f_n(i) \notin U_i\} \subseteq$$

$$\bigcup_{i < i_0} ((\text{FS}(A) \setminus \text{FS}(A_i)) \cup \{n \in \text{FS}(A_i) : f_n(i) \notin U_i\}) \in \mathcal{H}.$$

□

The following diagram summarize the relationship between BW-like spaces with respect to the Hindman ideal  $\mathcal{H}$  and compactness properties (if there is no arrow in some direction, it means that there is no implication between those notions).



**Question 4.** Does there exist a sequentially compact space  $X$  such that  $(X, \mathcal{H}) \notin \text{BW}$ ?

**Question 5.** Does there exist a space  $X$  such that  $(X, \mathcal{H}) \in \text{BW}$ , but  $X$  is not sequentially compact?

5. THE VAN DER WAERDEN IDEAL VERSUS THE HINDMAN IDEAL

In [12] the authors proved that if we assume the Continuum Hypothesis then there exists a space  $X$  such that  $(X, \mathcal{W}) \in \text{FinBW}$  but  $X$  is not a Hindman space.

**Proposition 5.1.** *There exists a Hausdorff, compact, sequentially compact, separable space  $X$  which is first-countable at all points but one such that  $(X, \mathcal{W}) \notin \text{FinBW}$  but  $(X, \mathcal{H}) \in \text{hBW}$  and  $(X, \mathcal{W}) \in \text{hBW}$ .*

*Proof.* Let  $\mathcal{A}$  be a mad family on  $\mathbb{N}$  such that  $\mathcal{A} \subseteq \mathcal{W}$ . Then  $X = \Phi(\mathcal{A})$  is as required (by Theorem 2.1 and Proposition 2.3).  $\square$

It seems that other questions about relations between BW-like spaces for these two ideals are still open.

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