REPRESENTATION OF IDEAL CONVERGENCE AS A UNION AND INTERSECTION OF MATRIX SUMMABILITY METHODS

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Abstract. We characterize ideals on $\mathbb{N}$ for which the ideal limit function is representable as an intersection (union resp.) of matrix summability methods. More specifically, we characterize ideals $\mathcal{I}$ for which there is a family $\mathcal{M}$ of matrices such that a sequence x is $\mathcal{I}$-convergent to $L \iff x$ is A-summable to L for every (some resp.) $A \in \mathcal{M}$. We consider separately cases of all sequences and bounded sequences. We also consider $\mathcal{I}^*$-convergence.

1. Introduction

All the notions and notations used in the introduction are defined in Section 2.

The following theorem of Mazur (see [1, p. 71–72] or [2, p. 44–45]) is a representation theorem of continuous linear functionals defined on separable subspaces of ℓ^∞.

Theorem 1.1 (Mazur). *Let $V \subseteq \ell^\infty$ be a separable linear subspace of ℓ^∞ with* sup *norm. For every continuous linear functional $\phi : V \to \mathbb{R}$ there is an infinite matrix $A = (a_{i,k})_{i,k\in\mathbb{N}}$ such that*

1. $a_{i,k} = 0$ *for all but finitely many k and every i,*
2. $\sum_{k=1}^\infty |a_{i,k}| \leq \|\phi\|$ *for every i,*
3. $\lim_{i\to\infty} \sum_{k=1}^\infty |a_{i,k}| = \|\phi\|$,
4. *for every $x \in V$,*

$$\phi(x) = \lim_{i\to\infty} \sum_{k=1}^\infty a_{i,k} x_k = \lim^A x.$$

The above theorem means that every continuous linear functional on a separable subspace of ℓ^∞ is equal to some matrix summability method on that subspace.

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ which contains an infinite set. Let $c^\mathcal{I}$ be the set of all $\mathcal{I}$-convergent sequences. It is not difficult to show that $V = c^\mathcal{I} \cap \ell^\infty$ is a nonseparable subspace of ℓ^∞ with sup norm[1]. Now we can define a continuous linear functional

$$\lim^\mathcal{I} : V \to \mathbb{R}$$

Date: August 17, 2019.

2010 *Mathematics Subject Classification.* Primary: 40C05, 40A35. Secondary: 26A03, 03E05, 11B05.

Key words and phrases. matrix summability method, matrix density, ideal, matrix ideal, ideal convergence, nonpathological submeasure, logarithmic density, exponential density, uniform density, Banach density.

[1] Indeed, take an infinite set $A \in \mathcal{I}$ and notice that balls $B(\mathbf{1}_B; 1/2)$ of the radius $1/2$ and the center $\mathbf{1}_B$ are pairwise disjoint for distinct $B \subseteq A$, there is uncountable many distinct subsets of A, and for each $B \subseteq A$ we have $B(\mathbf{1}_B; 1/2) \cap V \neq \emptyset$

mapping a sequence x to the $\mathcal{I}$-limit of x. In general, this functional does not have a matrix representation similar to one given by Mazur. Indeed, if $\mathcal{I}$ is a maximal ideal, then it is known that every bounded sequence is $\mathcal{I}$-convergent (see e.g. [19, Lemma 5.2]). On the other hand, Steinhaus [30, p. 122] proved that for every regular matrix A there is a sequence $x \in \ell^\infty$ which is not A-summable. Thus, $\lim^{\mathcal{I}} \neq \lim^A$ for every A.

For the ideal $\mathcal{I}_d$ of all sets of the asymptotic density zero, the question whether the functional $\lim^{\mathcal{I}_d}$ has a representation in terms of matrix summability methods was posed by Mazur as Problem 5 in "The Scottish Book" ([25, Problem 5, p. 55] or [24, Problem 5, p. 69]). This problem was solved by Khan and Orhan [18, Theorem 2.2] (see also Theorem 4.3 in this paper, and see [10] for more information on the problem). They proved that for every matrix ideal $\mathcal{I}$ (in particularly for $\mathcal{I}_d$) there is a matrix A such that

$$\lim{}^{\mathcal{I}} = \lim{}^A$$

in the realm of bounded sequences.

In Sections 3 and 4 we summarize what is known about representation of ideal convergence as a matrix summability method. We do it (there and in the farther sections) both in the realm of bounded sequences and unbounded sequences. Moreover, beside ideal convergence, we also consider $\mathcal{I}^*$-convergence. The reason for considering this additional kind of convergence follows from the fact that Mazur defined $\mathcal{I}_d$-convergence as $\mathcal{I}_d^*$-convergence (we know that they are the same), but in general $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence are distinct.

Before Khan and Orhan proved the above-mentioned result, Fridy and Miller [12, Theorem 4] proved that the ideal limit function generated by a matrix ideal $\mathcal{I}$ is equal to an intersection of some matrix summability methods in the realm of all bounded sequences, more specifically they showed that there is a family of matrices $\mathcal{M}$ such that

$$\forall x \in \ell^\infty \left(\lim{}^{\mathcal{I}} x = L \iff \forall A \in \mathcal{M} \left(\lim{}^A x = L\right)\right)$$

or putting it in other words

$$\lim{}^{\mathcal{I}} \restriction \ell^\infty = \bigcap \{\lim{}^A \restriction \ell^\infty : A \in \mathcal{M}\}.$$

Later, Gogola, Mačaj and Visnyai [14, Theorem 4.4] proved that the similar result holds for another family of ideals and they asked ([14, Problem 4.6]) whether the same holds for every ideal $\mathcal{I}$. In [10], we showed that the answer is negative.

In Section 5 we characterize ideals for which ideal limits can be represented as an intersections of matrix summability methods. Next, using these characterizations, we show in Subsection 5.1 that ideal limit functions generated by ideals defined with the aid of nonpathological submeasures always have representations as intersections of matrix summability methods, and in Subsection 5.2 we answer a question of Visnyai about representations of ideal limit functions generated by ideals of logarithmic and exponential density zero sets.

Fridy and Miller [12, Theorem 3] also proved that the $\mathcal{I}_d$-limit function is equal to a union of some matrix summability methods in the realm of all sequences, more specifically there exists a family of matrices $\mathcal{W}$ such that

$$\forall x \in \mathbb{R}^{\mathbb{N}} \left(\lim{}^{\mathcal{I}_d} x = L \iff \exists A \in \mathcal{W} \left(\lim{}^A x = L\right)\right)$$

or putting it in other words

$$\lim^{\mathcal{I}_d} = \bigcup\{\lim^A : A \in \mathcal{W}\}.$$

In Section 6 we characterize ideals for which ideal limits can be represented as a unions of matrix summability methods.

2. Preliminaries

We denote by $\mathbb{N}$ the set of positive natural numbers, and by $\mathbb{R}^{\mathbb{N}}$ the family of all real sequences $x = (x_n)_{n\in\mathbb{N}}$. Let $\ell^\infty = \{x \in \mathbb{R}^{\mathbb{N}} : x \text{ is bounded}\}$ and $c = \{x \in \mathbb{R}^{\mathbb{N}} : x \text{ is convergent}\}$. We denote by $\mathbf{1}_A$ the characteristic function of a set A. Let $[A]^{\aleph_0} = \{B \subseteq A : |B| = \aleph_0\}$.

2.1. **Summability methods.**

Definition 2.1. Let $D \subseteq \mathbb{R}^{\mathbb{N}}$. Any function $\Lambda : D \to \mathbb{R}$ is called a *summability method.*

Example 2.2. The ordinary limit function $\lim : c \to \mathbb{R}$ is a summability method.

Definition 2.3. A summability method Λ is *regular* if $\Lambda \restriction c = \lim$ (i.e. $c \subseteq \operatorname{dom}(\Lambda)$ and $\Lambda(x) = \lim x$ for every $x \in c$).

Definition 2.4. Let Λ_1 and Λ_2 be two summability methods. We say that

- Λ_1 and Λ_2 are *equal* if $\operatorname{dom}(\Lambda_1) = \operatorname{dom}(\Lambda_2)$ and $\Lambda_1(x) = \Lambda_2(x)$ for every $x \in \operatorname{dom}(\Lambda_1)$ (i.e. $\Lambda_1 = \Lambda_2$ as relations);
- Λ_1 is *contained in* Λ_2 (or Λ_2 *contains* Λ_1) if $\operatorname{dom}(\Lambda_1) \subseteq \operatorname{dom}(\Lambda_2)$ and $\Lambda_1(x) = \Lambda_2(x)$ for every $x \in \operatorname{dom}(\Lambda_1)$ (i.e. $\Lambda_1 \subseteq \Lambda_2$ as relations).

2.2. **Matrix summability methods.**

Definition 2.5. Let $A = (a_{i,k})_{i,k\in\mathbb{N}}$ be an infinite matrix of reals. We say that $x \in \mathbb{R}^{\mathbb{N}}$ is *A-summable* if

1. the series $A_i(x) = \sum_{k\in\mathbb{N}} a_{i,k}x_k$ is convergent for every $i \in \mathbb{N}$, and
2. the sequence $(A_i(x))_{i\in\mathbb{N}}$ is convergent.

The real $\lim_{i\to\infty} A_i(x)$ is called the *A-limit of x*. We denote by c^A the family of all A-summable sequences. Finally, the *matrix summability generated by A* (for short, *A-summability method*) is the function $\lim^A : c^A \to \mathbb{R}$ given by $\lim^A(x) = \lim_{i\to\infty} A_i(x)$. We write $\lim^A x$ instead of $\lim^A(x)$.

Example 2.6. For the identity matrix $I = (a_{i,k})$ where $a_{i,i} = 1$ and $a_{i,k} = 0$ for $i \neq k$, the matrix summability method is equal to the ordinary limit i.e. $\lim^I = \lim$ (i.e. $c^I = c$ and $\lim^I x = \lim x$ for every $x \in c$).

Example 2.7. For the Cesàro matrix $C = (a_{i,k})$ where $a_{i,k} = 1/i$ for $k \leq i$ and $a_{i,k} = 0$ for $k > i$, the matrix summability method is regular, and $\lim^C x = \lim \frac{x_1+\cdots+x_n}{n}$ for every $x \in c^C$. In this case, C-summability method is called the *Cesáro summability method.*

Definition 2.8. We say that a matrix $A = (a_{i,k})$ is *regular* if the matrix summability method generated by A is regular. The matrix A is *nonnegative* if $a_{i,k} \geq 0$ for every $i, k \in \mathbb{N}$. We denote by $\mathcal{NRM}$ the family of all nonnegative regular matrices.

All regular matrices are characterized by the following theorem.

Theorem 2.9 (Silverman [28], Toeplitz [32])**.** *The matrix summability method generated by a matrix A is regular if and only if*

1. $\lim_{i\to\infty} a_{i,k} = 0$ *for every* $k \in \mathbb{N}$,
2. $\sup_i \sum_{k\in\mathbb{N}} |a_{i,k}| < \infty$,
3. $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} a_{i,k} = 1$.

2.3. **Ideals.**

Definition 2.10. A family $\mathcal{I} \subseteq \mathcal{P}(X)$ is called an *ideal on X* if

1. $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$,
2. $A \subseteq B \wedge B \in \mathcal{I} \implies A \in \mathcal{I}$,
3. $\mathcal{I}$ contains all finite subsets of X,
4. $X \notin \mathcal{I}$.

For an ideal $\mathcal{I}$ on X, we write $\mathcal{I}^* = \{X \setminus A : A \in \mathcal{I}\}$ and call it the *filter dual to $\mathcal{I}$*.

Example 2.11. The ideal of all finite subsets of an infinite set X is denoted by $\mathrm{Fin}(X)$ (or just Fin if X is clear from the context).

Definition 2.12. Ideals $\mathcal{I}$ and $\mathcal{J}$ on X and Y respectively are *isomorphic* (in short $\mathcal{I} \approx \mathcal{J}$) if there exists a bijection $\phi : X \to Y$ such that $A \in \mathcal{I} \iff \phi[A] \in \mathcal{J}$ for every $A \subseteq X$.

Definition 2.13. An ideal $\mathcal{I}$ on X is a *P-ideal* if for every countable family $\mathcal{A} \subseteq \mathcal{I}$ there is $B \in \mathcal{I}$ such that $A \setminus B$ is finite for every $A \in \mathcal{A}$.

Remark. While considering an ideal on an infinite countable set we can always see it as an ideal on $\mathbb{N}$ by identifying this set with $\mathbb{N}$ via a fixed bijection.

Definition 2.14. For ideals $\mathcal{I}, \mathcal{J}$ and $A \notin \mathcal{I}$ we define the following new ideals:

1. $\mathcal{I} \restriction A = \{B \subseteq A : B \in \mathcal{I}\} = \{B \cap A : B \in \mathcal{I}\}$,
2. $\mathcal{I} \oplus \mathcal{J} = \{A \subseteq \mathbb{N} \times \{0,1\} : \{n : (n,0) \in A\} \in \mathcal{I} \wedge \{n : (n,1) \in A\} \in \mathcal{J}\}$,
3. $\mathcal{I} \oplus \mathcal{P}(\mathbb{N}) = \{A \subseteq \mathbb{N} \times \{0,1\} : \{n : (n,0) \in A\} \in \mathcal{I}\}$,
4. $\mathcal{I} \otimes \mathcal{J} = \{A \subseteq \mathbb{N} \times \mathbb{N} : \{n : \{k : (n,k) \in A\} \notin \mathcal{J}\} \in \mathcal{I}\}$,
5. $\emptyset \otimes \mathcal{J} = \{A \subseteq \mathbb{N} \times \mathbb{N} : \forall n \in \mathbb{N} (\{k : (n,k) \in A\} \in \mathcal{J})\}$,
6. $\mathcal{I} \otimes \emptyset = \{A \subseteq \mathbb{N} \times \mathbb{N} : \{n : \{k : (n,k) \in A\} \neq \emptyset\} \in \mathcal{I}\}$.

2.3.1. *F_σ ideals and analytic P-ideals.* By identifying sets of natural numbers with their characteristic functions, we equip $\mathcal{P}(\mathbb{N})$ with the topology of the Cantor space $\{0,1\}^{\mathbb{N}}$ and therefore we can assign topological complexity to ideals on $\mathbb{N}$. In particular, an ideal $\mathcal{I}$ is F_σ ($F_{\sigma\delta}$, analytic resp.) if it is an F_σ ($F_{\sigma\delta}$, analytic resp.) subset of the Cantor space.

Example 2.15. The family $\mathrm{Fin} = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$ is an F_σ P-ideal.

Definition 2.16. For a set $A \subseteq \mathbb{N}$ we define the *asymptotic density of A* by

$$d(A) = \lim_{i\to\infty} d_i(A),$$

where $d_i(A) = |A \cap \{1, 2, \ldots, n\}|/n$, provided that the limit exists.

Definition 2.17. The family $\mathcal{I}_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$ of all sets of asymptotic density zero is an $F_{\sigma\delta}$ P-ideal (see e.g. [9, Example 1.2.3(d)]).

Definition 2.18. A map $\Phi : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is a *submeasure* on $\mathbb{N}$ if

1. $\Phi(\emptyset) = 0$,

(2) if $A \subseteq B$ then $\Phi(A) \leq \Phi(B)$,
(3) $\Phi(A \cup B) \leq \Phi(A) + \Phi(B)$.

A submeasure Φ is

- *lower semicontinuous* if $\Phi(A) = \lim_{n\to\infty} \Phi(A \cap \{1, \ldots, n\})$ for every $A \subseteq \mathbb{N}$;
- *nonpathological* if
$$\Phi(A) = \sup\{\mu(A) : \mu \leq \Phi, \mu \text{ is a measure on } \mathcal{P}(\mathbb{N})\}$$
for each A.

Definition 2.19. For a submeasure Φ we define $\mathrm{Fin}(\Phi) = \{A \subseteq \mathbb{N} : \Phi(A) < \infty\}$. If $\Phi(\mathbb{N}) = \infty$ and $\Phi(\{n\}) < \infty$ for every $n \in \mathbb{N}$, then $\mathrm{Fin}(\Phi)$ is an ideal.

All F_σ ideals are characterized by the following theorem.

Theorem 2.20 (Mazur [26])**.** *$\mathcal{I}$ is an F_σ ideal $\iff$ $\mathcal{I} = \mathrm{Fin}(\Phi)$ for some lower semicontinuous submeasure Φ on $\mathbb{N}$ such that $\Phi(\mathbb{N}) = \infty$ and $\Phi(\{n\}) < \infty$ for every $n \in \mathbb{N}$.*

Definition 2.21. For a submeasure Φ we define $\|A\|_\Phi = \lim_{n\to\infty} \Phi(A \setminus \{1, \ldots, n\})$ and $\mathrm{Exh}(\Phi) = \{A \subseteq \mathbb{N} : \|A\|_\Phi = 0\}$. If $\|\mathbb{N}\|_\Phi \neq 0$, then $\mathrm{Exh}(\Phi)$ is an ideal (see e.g. [9]).

All F_σ P-ideals are characterized by the following theorem.

Theorem 2.22 (e.g. [9])**.** *$\mathcal{I}$ is an F_σ P-ideal $\iff$ $\mathcal{I} = \mathrm{Fin}(\Phi) = \mathrm{Exh}(\Phi)$ for some lower semicontinuous submeasure Φ on $\mathbb{N}$ such that $\|\mathbb{N}\|_\Phi \neq 0$, $\Phi(\mathbb{N}) = \infty$ and $\Phi(\{n\}) < \infty$ for every $n \in \mathbb{N}$.*

All $F_{\sigma\delta}$ P-ideals are characterized by the following theorem.

Theorem 2.23 (Solecki [29])**.** *The following conditions are equivalent.*

(1) *$\mathcal{I}$ is an analytic P-ideal.*
(2) *$\mathcal{I}$ is an $F_{\sigma\delta}$ P-ideal.*
(3) *$\mathcal{I} = \mathrm{Exh}(\Phi)$ for some lower semicontinuous submeasure Φ on $\mathbb{N}$ such that $\|\mathbb{N}\|_\Phi \neq 0$.*

2.3.2. *Tall and nowhere tall ideals.*

Definition 2.24. An ideal $\mathcal{I}$ is *tall* if for every infinite $B \subseteq \mathbb{N}$ there is an infinite $C \in \mathcal{I}$ such that $C \subseteq B$. It is easy to see that $\mathcal{I}$ is tall $\iff$ $\forall B \in [\mathbb{N}]^{\aleph_0}\, (\mathcal{I} \restriction B \neq \mathrm{Fin})$.

Remark. The name "tall ideal" was introduced by Mathias [22, 23], but later Todorčević [31] and Farah [9] used the name *dense ideal* as a tall ideal is dense in the poset $([\mathbb{N}]^{\aleph_0}, \subseteq)$, and now the two names are used in the literature.

Definition 2.25. We say that an ideal $\mathcal{I}$ is *nowhere tall* if $\mathcal{I} \restriction A$ is not tall for every $A \notin \mathcal{I}$. It is easy to see that $\mathcal{I}$ is nowhere tall $\iff$ $\forall A \notin \mathcal{I}\, \exists B \in [A]^{\aleph_0}\, (\mathcal{I} \restriction B = \mathrm{Fin})$.

Remark. The name "nowhere tall ideal" was introduced by Matet and Pawlikowski [21], but there are also other names used in the literature: a *nowhere dense* ideal [8], and a *locally fin* ideal [17].

It is obvious that Fin is nowhere tall and it is not difficult to see that $\emptyset \otimes \mathrm{Fin}$ and $\mathrm{Fin} \oplus \mathcal{P}(\mathbb{N})$ are nowhere tall as well. It is known that in the realm of analytic P-ideals they are the only three nowhere tall ideals.

Theorem 2.26 (see e.g. [9, Corollary 1.2.11])**.** *Let $\mathcal{I}$ be an analytic P-ideal. Then $\mathcal{I}$ is nowhere tall $\iff$ $\mathcal{I}$ is isomorphic to one of the three ideals:* Fin, $\mathrm{Fin}\oplus\mathcal{P}(\mathbb{N})$ *or* $\emptyset\otimes\mathrm{Fin}$.

In Proposition 2.27 we show how to produce more examples of nowhere tall ideals. Moreover, by [21, Proposition 2.1], every ideal generated by less than $\mathfrak{p}$ sets is nowhere tall (where $\mathfrak{p}$ is the pseudo-intersection number).

Proposition 2.27. *If $\mathcal{I}$ and $\mathcal{J}$ are nowhere tall ideals, then $\emptyset\otimes\mathcal{J}$, $\mathcal{I}\otimes\emptyset$, $\mathcal{I}\oplus\mathcal{J}$ and $\mathcal{I}\oplus\mathcal{P}(\mathbb{N})$ are nowhere tall as well.*

Proof. $\emptyset\otimes\mathcal{J}$: If $A\notin\emptyset\otimes\mathcal{J}$ then there is $n\in\mathbb{N}$ such that $B=\{k:(n,k)\in A\}\notin\mathcal{J}$. Then we have $C\subseteq B$ such that $\mathcal{J}\restriction C=\mathrm{Fin}$ and for the set $D=\{(n,k):k\in C\}\subseteq A$ we have $(\emptyset\otimes\mathcal{J})\restriction D=\mathrm{Fin}$.

$\mathcal{I}\otimes\emptyset$: If $A\notin\mathcal{I}\otimes\emptyset$ then $B=\{n\in\mathbb{N}:A\cap(\{n\}\times\mathbb{N})\neq\emptyset\}\notin\mathcal{I}$. Then we find $C\subseteq B$ such that $\mathcal{I}\restriction C=\mathrm{Fin}$. If we denote by k_n any number such that $(n,k_n)\in A\cap(\{n\}\times\mathbb{N})$ for each $n\in C$ then for the set $D=\{(n,k_n):n\in C\}\subseteq A$ we have $(\mathcal{I}\otimes\emptyset)\restriction D=\mathrm{Fin}$.

$\mathcal{I}\oplus\mathcal{J}$: If $A\notin\mathcal{I}\oplus\mathcal{J}$ then either $A_0=\{n:(n,0)\in A\}\notin\mathcal{I}$ or $A_1=\{n:(n,1)\in A\}\notin\mathcal{J}$. In either case we find $B\subseteq A_i$ such that $(\mathcal{I}\oplus\mathcal{J})\restriction(B\times\{i\})=\mathrm{Fin}$.

$\mathcal{I}\oplus\mathcal{P}(\mathbb{N})$: If $A\notin\mathcal{I}\oplus\mathcal{P}(\mathbb{N})$ then $A_0=\{n:(n,0)\in A\}\notin\mathcal{I}$ Therefore, we can find $B\subseteq A_0$ such that $(\mathcal{I}\oplus\mathcal{P}(\mathbb{N}))\restriction(B\times\{0\})=\mathrm{Fin}$.

□

2.3.3. *Ideals generated by matrices.*

Definition 2.28 (Freedman-Sember [11], see also Drewnowski-Paúl [7])**.** Let A be a nonnegative regular matrix. For a set $B\subseteq\mathbb{N}$ we define the *upper A-density of B* by

$$\overline{d^A}(B)=\limsup_{i\to\infty}d_i^A(B),$$

where $d_i^A(B)=\sum_{k\in B}a_{i,k}$. Moreover, we define the *A-density of B* by

$$d^A(B)=\lim_{i\to\infty}d_i^A(B)$$

provided that the limit exists. Note that $d_i^A(B)=A_i(\mathbf{1}_B)$ and $d^A(B)=\lim^A\mathbf{1}_B$, where $\mathbf{1}_B$ is the characteristic function of B.

Definition 2.29. For a nonnegative regular matrix A we define the family

$$\mathcal{I}(A)=\{B\subseteq\mathbb{N}:d^A(B)=0\}.$$

It is easy to see that $\mathcal{I}(A)$ is an ideal, and we call it the *matrix ideal generated by A*.

Definition 2.30. For an ideal $\mathcal{I}$ on $\mathbb{N}$ we define

$$\mathcal{M}(\mathcal{I})=\{A\in\mathcal{NRM}:\mathcal{I}\subseteq\mathcal{I}(A)\}.$$

(Families $\mathcal{M}(\mathcal{I})$ will be frequently used in Section 5.)

Example 2.31. For the identity matrix I, $d^I(B)=0$ if B is finite, $d^I(B)=1$ if $\mathbb{N}\setminus B$ is finite and $d^I(B)$ is undefined in other cases. The matrix ideal $\mathcal{I}(I)$ equals Fin.

Example 2.32. For the Cesàro matrix C, the C-density is just the asymptotic density and the matrix ideal $\mathcal{I}(C)$ is $\mathcal{I}_d$.

Lemma 2.33 (Folklore, see e.g. [10]). *If A is a regular matrix, then there is a regular matrix B such that*

1. *B has only finitely many nonzero elements in each row,*
2. *each row of B sums to 1,*
3. $\lim^A x = \lim^B x$ *for every* $x \in l^\infty$,
4. $\mathcal{I}(A) = \mathcal{I}(B)$.

Remark. In general, Lemma 2.33(3) cannot be extended for unbounded sequences x (see [10] for an appropriate example).

Proposition 2.34 ([11, Propositions 3.1 and 3.2], [3, Proposition 13], [7, Proposition 7.2]). *Let $A = (a_{i,k})$ be a nonnegative regular matrix.*

1. *$\mathcal{I}(A)$ is an $F_{\sigma\delta}$ P-ideal.*
2. *$\mathcal{I}(A)$ is a tall ideal if and only if* $\lim_{i,k\to\infty} a_{i,k} = 0$ *(i.e. for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $a_{i,k} < \varepsilon$ for all $i, k > n$).*

2.4. **Ideal convergence.**

Definition 2.35. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. A sequence $x \in \mathbb{R}^{\mathbb{N}}$ is *$\mathcal{I}$-convergent* if there exists $L \in \mathbb{R}$ such that $\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$. The real L is called the *$\mathcal{I}$-limit of x*. We denote by $c^{\mathcal{I}}$ the family of all $\mathcal{I}$-convergent sequences. Finally, the *ideal limit function generated by an ideal $\mathcal{I}$* (in short *$\mathcal{I}$-limit*) is the function $\lim^{\mathcal{I}} : c^{\mathcal{I}} \to \mathbb{R}$ mapping x to the $\mathcal{I}$-limit of x.

Remark. In [5, Definition 1], Connor introduced the notion of *μ-statistical convergence*, where μ is a finitely additive complete measure vanishing on finite sets. It is not difficult to see that μ-statistical convergence is equivalent to $\mathcal{I}$-convergence, where $\mathcal{I} = \{A \subseteq \mathbb{N} : \mu(A) = 0\}$.

Example 2.36. The ideal limit function generated by the ideal $\mathcal{I} = \mathrm{Fin}$ is equal to the ordinary limit function ($\lim^{\mathrm{Fin}} = \lim$) i.e. $c^{\mathrm{Fin}} = c$ and $\lim^{\mathrm{Fin}} x = \lim x$ for every $x \in c$.

Example 2.37. The ideal limit function generated by the ideal $\mathcal{I} = \mathcal{I}_d$ is regular and strictly contains the ordinary limit function i.e. $\lim \subsetneq \lim^{\mathcal{I}_d}$. $\mathcal{I}_d$-convergence is also called *statistical convergence*. In "The Scottish Book" (e.g. [25, p. 55]), Mazur used the name *asymptotic convergence* in this case. (In fact, Mazur defined it in a different manner, but by Theorem 2.39(2) both notions coincide).

Remark. In [4, Definition 7], Connor introduced the notion of *A-statistical convergence*, where A is a nonnegative regular matrix. It is not difficult to see that A-statistical convergence is equivalent to $\mathcal{I}(A)$-convergence, where $\mathcal{I}(A)$ is the matrix ideal generated by A.

2.5. **$\mathcal{I}^*$-convergence.**

Definition 2.38. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. A sequence $x \in \mathbb{R}^{\mathbb{N}}$ is *$\mathcal{I}^*$-convergent* if there exists $L \in \mathbb{R}$ and a set $F \in \mathcal{I}^*$ such that the subsequence $(x_n)_{n\in F}$ is ordinarily convergent (i.e. $\{n \in F : |x_n - L| \geq \varepsilon\}$ is finite for every $\varepsilon > 0$). The real L is called the *$\mathcal{I}^*$-limit of x*. We denote by $c^{\mathcal{I}^*}$ the family of all $\mathcal{I}^*$-convergent sequences. Finally, the *$\mathcal{I}^*$-limit function generated by an ideal $\mathcal{I}$* (in short *$\mathcal{I}^*$-limit*) is the function $\lim^{\mathcal{I}^*} : c^{\mathcal{I}^*} \to \mathbb{R}$ mapping x to the $\mathcal{I}^*$-limit of x.

Remark. In [5, Definition 1], Connor introduced the notion of *convergence in μ-density*, where μ is a finitely additive complete measure vanishing on finite sets. It is not difficult to see that μ-density convergence is equivalent to $\mathcal{I}^*$-convergence, where $\mathcal{I} = \{A \subseteq \mathbb{N} : \mu(A) = 0\}$.

The following theorem shows the basic relationships between $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence.

Theorem 2.39 ([19, Proposition 3.2 and Theorem 3.2], see also [5, Proposition 2 and Theorem 6])**.** *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$.*

1. $\lim^{\mathcal{I}^*} \subseteq \lim^{\mathcal{I}}$.
2. *If $\mathcal{I}$ is a P-ideal, then* $\lim^{\mathcal{I}^*} = \lim^{\mathcal{I}}$.

Remark. In [6, p. 320], Connor introduced the notion of *convergence in A-density*, where A is a nonnegative regular matrix. It is not difficult to see that convergence in A-density is equivalent to $\mathcal{I}(A)^*$-convergence, where $\mathcal{I}(A)$ is the matrix ideal generated by A.

3. Matrix summability versus matrix ideal convergence

The following theorem reveals relationship between matrix summability method generated by a matrix A and the ideal limit function generated by the matrix ideal $\mathcal{I}(A)$.

Theorem 3.1 (Essentially [10, Proposition 4.4])**.** *Let $A = (a_{i,k})$ be a nonnegative and regular matrix.*

1. $\lim^{\mathcal{I}(A)} \restriction \ell^\infty \subseteq \lim^A \restriction \ell^\infty$.
2. *If there exist an infinite set $B = \{b_n : n \in \mathbb{N}\}$ and an increasing sequence (i_n) such that $d^A(B) = 0$ and $a_{i_n,b_n} \neq 0$ for each $n \in \mathbb{N}$, then*
$$\lim^{\mathcal{I}(A)} \not\subseteq \lim^A.$$
3. *If there exists $B \subseteq \mathbb{N}$ such that $d^A(B)$ exists and is not equal to 0 nor 1 then*
$$\lim^{\mathcal{I}(A)} \restriction \ell^\infty \not\supseteq \lim^A \restriction \ell^\infty,$$
and hence, also $\lim^{\mathcal{I}(A)} \not\supseteq \lim^A$.

Proof. (1) and (3) is proved in [10, Proposition 4.4].

(2) Let $x = (x_n)$ be given by $x_{b_n} = 1/a_{i_n,b_n}$ and $x_n = 0$ otherwise. Since $B \in \mathcal{I}(A)$, we get that x is $\mathcal{I}(A)$-convergent to 0. On the other hand, $A_{i_n}(x) = \sum_{k=1}^\infty a_{i_n,k}x_k \geq a_{i_n,b_n}x_{b_n} = 1$, hence x is not A-summable to 0. □

Remark. By Proposition 2.34, $\mathcal{I}(A)$ is a P-ideal, so by Theorem 2.39(2)) we have $\lim^{\mathcal{I}(A)} = \lim^{\mathcal{I}(A)^*}$. Thus, Theorem 3.1 remains true if we replace $\mathcal{I}(A)$-limit by $\mathcal{I}(A)^*$-limit.

Example 3.2. Let I be the identity matrix.

1. $\mathcal{I}(I) = \mathrm{Fin}$.
2. $\lim^{\mathcal{I}(I)} = \lim^I$ and $\lim^{\mathcal{I}(I)} \restriction \ell^\infty = \lim^I \restriction \ell^\infty$.

Example 3.3. Let $C = (c_{i,k})$ be the Cesàro matrix.

1. $\mathcal{I}(C) = \mathcal{I}_d$.

(2) $\lim^{\mathcal{I}(C)} \restriction \ell^\infty \not\supseteq \lim^C \restriction \ell^\infty$, and hence, also $\lim^{\mathcal{I}(C)} \not\supseteq \lim^C$.
(3) $\lim^{\mathcal{I}(C)} \not\subseteq \lim^C$.
(4) $\lim^{\mathcal{I}(C)} \restriction \ell^\infty \subseteq \lim^C \restriction \ell^\infty$.

Proof. These properties follows from Proposition 3.1 (however (4) was already proved by Schoenberg [27, Lemma 4]). Indeed, to show (2) take $B = \{1, 3, 5, \dots\}$ and note that $d^C(B) = 1/2$; to show (3) take $B = \{2^n : n \in \mathbb{N}\}$ and note that $d^C(B) = 0$ and $c_{2^n,2^n} = 1/2^n$. □

4. One matrix summability method

In Proposition 4.1 we provide a necessary condition for an ideal limit function to be equal to some matrix summability method. This condition will be later used to characterize ideals for which an ideal limit function is equal to some matrix summability method (see Theorems 4.2 and 4.3).

Proposition 4.1. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and $A \in \mathcal{NRM}$. If $\lim^{\mathcal{I}^*} \restriction \ell^\infty = \lim^A \restriction \ell^\infty$ or $\lim^{\mathcal{I}} \restriction \ell^\infty = \lim^A \restriction \ell^\infty$, then $\mathcal{I} = \mathcal{I}(A)$.*

Proof. ($\subseteq$) Let $B \in \mathcal{I}$. Then $\lim^{\mathcal{I}^*} \mathbf{1}_B = \lim^{\mathcal{I}} \mathbf{1}_B = 0$, so $\lim^A \mathbf{1}_B = 0$. Thus, $d^A(B) = 0$, and consequently $B \in \mathcal{I}(A)$.

($\supseteq$) Let $B \in \mathcal{I}(A)$. Then $\lim^A \mathbf{1}_B = 0$, hence $\lim^{\mathcal{I}} \mathbf{1}_B = 0$ or $\lim^{\mathcal{I}^*} \mathbf{1}_B = 0$. In the first case we immediately obtain $B \in \mathcal{I}$. In the second case there is $C \in \mathcal{I}^*$ such that $\mathbf{1}_B \restriction C$ is ordinarily convergent to 0. Hence $B \cap C$ is finite and $B \setminus C \in \mathcal{I}$, so $B = (B \cap C) \cup (B \setminus C) \in \mathcal{I}$. □

Theorem 4.2 (Essentially [10, Theorem 3.2])**.** *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following conditions are equivalent.*

(1) $\exists A \in \mathcal{NRM}(\lim^{\mathcal{I}} = \lim^A)$.
(2) $\exists A \in \mathcal{NRM}(\lim^{\mathcal{I}^*} = \lim^A)$.
(3) $\mathcal{I} = \mathrm{Fin}$ *or* $\mathcal{I} \approx \mathrm{Fin} \oplus \mathcal{P}(\mathbb{N})$.

Proof. (2) $\implies$ (1) By Proposition 4.1, $\mathcal{I} = \mathcal{I}(A)$. By Theorem 2.34, $\mathcal{I}$ is a P-ideal, and consequently, by Theorem 2.39(2), $\lim^{\mathcal{I}} = \lim^{\mathcal{I}^*}$. Thus, $\lim^{\mathcal{I}} = \lim^A$ for some $A \in \mathcal{NRM}$.

(1) $\implies$ (3) It was proved in [10, Theorem 3.2].

(3) $\implies$ (2) It is not difficult to define an appropriate matrix A (in the first case it is just the identity matrix) such that $\lim^{\mathcal{I}^*} = \lim^A$ (for details see [10, Theorem 3.2]). □

Theorem 4.3 (Essentially Khan and Orhan [18, Theorem 2.2])**.** *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following conditions are equivalent.*

(1) $\exists A \in \mathcal{NRM}(\lim^{\mathcal{I}} \restriction \ell^\infty = \lim^A \restriction \ell^\infty)$.
(2) $\exists A \in \mathcal{NRM}(\lim^{\mathcal{I}^*} \restriction \ell^\infty = \lim^A \restriction \ell^\infty)$.
(3) $\mathcal{I} = \mathcal{I}(B)$ *for some* $B \in \mathcal{NRM}$.

Proof. (1) $\implies$ (2) By Proposition 4.1, $\mathcal{I} = \mathcal{I}(A)$. By Theorem 2.34, $\mathcal{I}$ is a P-ideal, and consequently, by Theorem 2.39(2), $\lim^{\mathcal{I}} = \lim^{\mathcal{I}^*}$. Thus, $\lim^{\mathcal{I}^*} \restriction \ell^\infty = \lim^A \restriction \ell^\infty$ for some $A \in \mathcal{NRM}$.

(2) $\implies$ (3) Follows from Proposition 4.1.

(3) $\implies$ (1) It was proved in [18, Theorem 2.2]. □

5. Intersections of matrix summability methods

Fridy and Miller [12, Theorems 1 and 4] proved that the ideal limit function generated by a matrix ideal $\mathcal{I}$ (in particular $\mathcal{I}_d$) is equal to an intersection of some matrix summability methods in the realm of all bounded sequences, more specifically

$$\lim^{\mathcal{I}} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}.$$

Later, Gogola, Mačaj and Visnyai [14, Theorem 4.4] proved that the similar result holds for the ideals $\mathcal{I}_{1/n^\alpha}$ with $\alpha \in (0;1]$, and they asked ([14, Problem 4.6]) whether the same holds for every ideal $\mathcal{I}$. In [10], we showed that the answer is negative (see also Theorem 5.5 below). In this section we study a similar problem in the realm of all sequences and also for $\mathcal{I}^*$-convergence. In Proposition 5.1 we provide a necessary condition for an ideal limit function to be equal to an intersection of some matrix summability methods, and in Proposition 5.2 we add one more necessary condition for $\mathcal{I}^*$-convergence. Next we use these conditions to characterized ideals for which the ideal limit function is equal to an intersection of some matrix summability methods (see Theorems 5.3, 5.4, 5.5 and 5.6).

Proposition 5.1. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and $\mathcal{M} \subseteq \mathcal{NRM}$. If $\lim^{\mathcal{I}} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}\}$ or $\lim^{\mathcal{I}^*} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}\}$, then*

(1) $\mathcal{M} \subseteq \mathcal{M}(\mathcal{I})$,
(2) $\mathcal{I} = \bigcap\{\mathcal{I}(A) : A \in \mathcal{M}\}$.

Proof. (1) Let $A \in \mathcal{M}$. We have to show that $\mathcal{I} \subseteq \mathcal{I}(A)$. Let $B \in \mathcal{I}$. Then $\lim^{\mathcal{I}} \mathbf{1}_B = 0$ and $\lim^{\mathcal{I}^*} \mathbf{1}_B = 0$, so $\lim^A \mathbf{1}_B = 0$. Thus $B \in \mathcal{I}(A)$.

(2,$\subseteq$) Follows from (1) and the definition of $\mathcal{M}(\mathcal{I})$.

(2,$\supseteq$) Let $B \in \bigcap\{\mathcal{I}(A) : A \in \mathcal{M}\}$ and $x = \mathbf{1}_B$. Then x is bounded and A-summable to 0 for every $A \in \mathcal{M}$, so x is $\mathcal{I}$-convergent to 0 or $\mathcal{I}^*$-convergent to 0. In the first case $B = \{n \in \mathbb{N} : |x_n - 0| > 1/2\} \in \mathcal{I}$. In the second case there is $F \in \mathcal{I}^*$ with $C = \{n \in F : |x_n - 0| > 1/2\} \in \text{Fin}$. Then $B \subseteq (\mathbb{N} \setminus F) \cup C \in \mathcal{I}$, so $B \in \mathcal{I}$. □

Proposition 5.2. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and $\mathcal{M} \subseteq \mathcal{NRM}$. If $\lim^{\mathcal{I}^*} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}\}$, then $\mathcal{I}$ is a P-ideal.*

Proof. Let $P_1, P_2, \dots \in \mathcal{I}$ be pairwise disjoint sets such that $\bigcup_{n\in\mathbb{N}} P_n = \mathbb{N}$. If we find $P \in \mathcal{I}$ such that $P_n \setminus P$ is finite for every $n \in \mathbb{N}$, the proof will be finished.

Let us define $x \in \ell^\infty$ by $x_k = 1/n^2 \iff k \in P_n$.

We show that $\lim^A x = 0$ for every $A \in \mathcal{M}$.

Let $A \in \mathcal{M}$. By Lemma 2.33, we can assume that $d^A_i(\mathbb{N}) = 1$ for every $i \in \mathbb{N}$. Let $\varepsilon > 0$. Let $n_0 \in \mathbb{N}$ be such that $\sum_{n>n_0} 1/n^2 < \varepsilon/2$. Since $\mathcal{I} \subseteq \mathcal{I}(A)$, $P_1 \cup \dots \cup P_{n_0} \in \mathcal{I}(A)$, hence $d^A(P_1 \cup \dots \cup P_{n_0}) = 0$. Let $i_0 \in \mathbb{N}$ be such that

$d_i^A(P_1 \cup \dots \cup P_{n_0}) < \varepsilon/2$ for every $i > i_0$. Then

$$\begin{aligned} A_i(x) &= \sum_{k=1}^{\infty} x_k a_{i,k} = \sum_{n=1}^{\infty} \sum_{k \in P_n} x_k a_{i,k} = \sum_{n=1}^{\infty} \sum_{k \in P_n} \frac{1}{n^2} a_{i,k} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2} \sum_{k \in P_n} a_{i,k} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} d_i^A(P_n) = \sum_{n \le n_0} \frac{1}{n^2} d_i^A(P_n) + \sum_{n > n_0} \frac{1}{n^2} d_i^A(P_n) \\ &\le \sum_{n \le n_0} 1 \cdot d_i^A(P_n) + \sum_{n > n_0} \frac{1}{n^2} \cdot 1 = d_i^A(P_1 \cup \dots \cup P_{n_0}) + \sum_{n > n_0} \frac{1}{n^2} \le \varepsilon \end{aligned}$$

for every $i > i_0$. Since ε is arbitrary, $\lim^A x = 0$.

Since $\lim^A x = 0$ for every $A \in \mathcal{M}$ and $\lim^{\mathcal{I}^*} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}\}$, we obtain $\lim^{\mathcal{I}^*} x = 0$. Consequently there is $F \in \mathcal{I}^*$ such that $x \restriction F$ is ordinarily convergent to 0.

If we take $P = \mathbb{N} \setminus F$, then $P \in \mathcal{I}$ and for every $n \in \mathbb{N}$ we have $P_n \setminus P = P_n \cap F \subseteq \{k \in F : |x_k - 0| \ge 1/n^2\}$ and the latter set is finite. $\square$

Theorem 5.3. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following conditions are equivalent.*

1. $\exists \mathcal{M} \subseteq \mathcal{NRM}(\lim^{\mathcal{I}} = \bigcap\{\lim^A : A \in \mathcal{M}\})$.
2. *$\mathcal{I}$ is nowhere tall.*

Proof. (1) $\implies$ (2) Suppose to the contrary that there is $B \notin \mathcal{I}$ such that $\mathcal{I} \restriction B$ is tall. By Proposition 5.1(2), there is $A \in \mathcal{M}$ such that $B \notin \mathcal{I}(A)$. Then it is not difficult to see that for every $n \in \mathbb{N}$ there are $i, k > n$ such that $k \in B$ and $a_{i,k} \neq 0$. Consequently we can construct two increasing sequences i_n, k_n such that $k_n \in B$ and $a_{i_n,k_n} \neq 0$ for every $n \in \mathbb{N}$. Since $\mathcal{I} \restriction B$ is dense, we can assume that $C = \{k_n : n \in \mathbb{N}\} \in \mathcal{I}$.

Now we define a sequence $x \in \mathbb{R}^{\mathbb{N}}$ by $x_k = n/a_{i_n,k_n}$ for $k = k_n$ and $x_k = 0$ otherwise. Then x is $\mathcal{I}$-convergent to 0 (as $C \in \mathcal{I}$). On the other hand, $A_{i_n}(x) \ge x_{k_n} \cdot a_{i_n,k_n} = n$, so x is not A-summable to 0, a contradiction.

(2) $\implies$ (1) For every $B \notin \mathcal{I}$ we take $C_B \in [B]^{\aleph_0}$ such that $\mathcal{I} \restriction C_B = \mathrm{Fin}$ (note that then $C_B \notin \mathcal{I}$) and a matrix $A_B = (a_{i,k})$ such that $a_{i,c_i} = 1$ for every $i \in \mathbb{N}$ and $a_{i,k} = 0$ otherwise, where (c_i) is the increasing enumeration of C_B (note that then $A_B \in \mathcal{NRM}$).

Let $\mathcal{M} = \{A_B : B \notin \mathcal{I}\}$.

We claim that $\lim^{\mathcal{I}} = \bigcap\{\lim^A : A \in \mathcal{M}\}$.

($\subseteq$) Let $x \in \mathbb{R}^{\mathbb{N}}$ be $\mathcal{I}$-convergent to L and $B \notin \mathcal{I}$. Then $x \restriction C_B$ is $\mathcal{I} \restriction C_B$-convergent to L. But $\mathcal{I} \restriction C_B = \mathrm{Fin}$, hence $x \restriction C_B$ is ordinarily convergent to L. Then it is not difficult to see that x is A_B-summable to L.

($\supseteq$) Let $x \in \mathbb{R}^{\mathbb{N}}$ be A_B-summable to L for every $B \notin \mathcal{I}$. Suppose to the contrary that x is not $\mathcal{I}$-convergent to L. Let $\varepsilon > 0$ be such that $B_\varepsilon = \{n \in \mathbb{N} : |x_n - L| > \varepsilon\} \notin \mathcal{I}$.

Since x is A_{B_ε}-summable to L, so $x \restriction C_{B_\varepsilon}$ is ordinarily convergent to L. Then $\{n \in C_{B_\varepsilon} : |x_n - L| > \varepsilon\}$ is finite. On the other hand $C_{B_\varepsilon} \subseteq B_\varepsilon$, a contradiction. $\square$

Theorem 5.4. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following conditions are equivalent.*

1. $\exists \mathcal{M} \subseteq \mathcal{NRM}(\lim^{\mathcal{I}^*} = \bigcap\{\lim^A : A \in \mathcal{M}\})$.
2. *$\mathcal{I}$ is a nowhere tall P-ideal.*

Proof. (1) $\implies$ (2) By Proposition 5.2, $\mathcal{I}$ is a P-ideal. Then, by Theorem 2.39(2), $\lim^{\mathcal{I}^*} = \lim^{\mathcal{I}}$, hence, by Theorem 5.3, $\mathcal{I}$ is nowhere tall.

(2) $\implies$ (1) By Theorem 5.3, there is $\mathcal{M} \subseteq \mathcal{NRM}$ with $\lim^{\mathcal{I}} = \bigcap\{\lim^A : A \in \mathcal{M}\}$, and then, by Theorem 2.39(2), $\lim^{\mathcal{I}^*} = \lim^{\mathcal{I}}$. □

Theorem 5.5 (Essentially [10, Theorem 6.9])**.** *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following conditions are equivalent.*

1. $\exists \mathcal{M} \subseteq \mathcal{NRM}(\lim^{\mathcal{I}} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}\})$.
2. $\mathcal{I} = \bigcap\{\mathcal{I}(A) : A \in \mathcal{M}(\mathcal{I})\}$.
3. $\lim^{\mathcal{I}} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$.

Proof. (1) $\implies$ (2) It follows from Proposition 5.1(2).

(2) $\implies$ (3) The proof is heavily based on Khan-Orhan theorem (Theorem 4.3), and it is included in [10, Theorem 6.9]. We provide the proof for the readers convenience.

($\subseteq$) Let x be a bounded sequence with $\lim^{\mathcal{I}} x = L$. Let $A \in \mathcal{M}(\mathcal{I})$. Then $\mathcal{I} \subseteq \mathcal{I}(A)$, so $\lim^{\mathcal{I}(A)} x = L$. By Theorem 3.1(1) we obtain that $\lim^A x = L$.

($\supseteq$) Let x be a bounded sequence such that $\lim^A x = L$ for every $A \in \mathcal{M}(\mathcal{I})$. Suppose to the contrary that x is not $\mathcal{I}$-convergent. Let $\varepsilon > 0$ be such that $C = \{n \in \mathbb{N} : |x_n - L| > \varepsilon\} \notin \mathcal{I}$. Then there is $A \in \mathcal{M}(\mathcal{I})$ with $C \notin \mathcal{I}(A)$. By Theorem 4.3 there is a nonnegative regular matrix B such that $\mathcal{I}(A) = \mathcal{I}(B)$ and $\lim^{\mathcal{I}(A)} = \lim^B$. Then $B \in \mathcal{M}(\mathcal{I})$, so $\lim^B x = L$. Thus $\lim^{\mathcal{I}(A)} x = L$ as well. But $C \notin \mathcal{I}(A)$, a contradiction.

(3) $\implies$ (1) The family $\mathcal{M} = \mathcal{M}(\mathcal{I})$ works. □

Theorem 5.6. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following conditions are equivalent.*

1. $\exists \mathcal{M} \subseteq \mathcal{NRM}(\lim^{\mathcal{I}^*} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}\})$.
2. *$\mathcal{I}$ is a P-ideal and* $\mathcal{I} = \bigcap\{\mathcal{I}(A) : A \in \mathcal{M}(\mathcal{I})\}$.
3. $\lim^{\mathcal{I}^*} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$.

Proof. (1) $\implies$ (2) By Proposition 5.2, $\mathcal{I}$ is a P-ideal. Then, by Theorem 2.39(2), $\lim^{\mathcal{I}^*} = \lim^{\mathcal{I}}$, hence, by Theorem 5.5, $\mathcal{I} = \bigcap\{\mathcal{I}(A) : A \in \mathcal{M}(\mathcal{I})\}$.

(2) $\implies$ (3) By Theorem 5.5, $\lim^{\mathcal{I}} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$, and by Theorem 2.39(2), $\lim^{\mathcal{I}^*} = \lim^{\mathcal{I}}$.

(3) $\implies$ (1) The family $\mathcal{M} = \mathcal{M}(\mathcal{I})$ works. □

5.1. Ideals defined by submeasures.

In this subsection we focus on representation of ideal limit functions for two classes of ideals defined with the aid of submeasure, namely analytic P-ideals and F_σ ideals (see Definitions 2.19 and 2.21, and Theorems 2.20 and 2.23).

In the case of $\mathcal{I}$-limits in the realm of all sequences, we only have to check, by Theorem 5.3, if $\mathcal{I}$ is nowhere tall ideal. Thus, by Theorem 2.26, we essentially have only 3 analytic P-ideals for which $\mathcal{I}$-limit can be represented as an intersection of matrix summability methods. Unfortunately, we do not know of any simpler characterization of nowhere tall ideals in the realm of F_σ ideals than the definition of nowhere tallness itself.

In the case of $\mathcal{I}$-limits in the realm of bounded sequences, we show (see Theorems 5.7 and 5.14) that the ideal limit functions generated by F_σ ideals and analytic P-ideals defined with the aid of nonpathological submeasures are equal to an intersection of some matrix summability methods. These results generalize theorems obtained by Gogola, Mačaj and Visnyai [14, Theorem 4.4] and Fridy and Miller [12, Theorems 1 and 4] to a vast class of ideals as Farah writes in [9, p. 31]: "The class of nonpathological ideals is rather extensive since it includes essentially all analytic P-ideals occurring in the literature. [...] A large class of nonpathological ideals was constructed by Louveau and Veličkoić ([20]) [...]. In §1.13 we will introduce *density ideals*, a class of nonpathological ideals which includes all Erdős-Ulam ideals."

We know that there is a pathological submeasure giving an ideal for which $\mathcal{I}$-limit in the realm of bounded sequences is not representable as an intersection of matrix summability methods (see Example 5.16). However, we do not know if the same holds for every pathological submeasure (see Question 1).

5.1.1. *F_σ ideals.*

Theorem 5.7. *If Φ is a nonpathological lower semicontinuous submeasure and $\mathcal{I} = \mathrm{Fin}(\Phi)$, then $\lim^{\mathcal{I}} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$. Moreover, $\lim^{\mathcal{I}^*} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\} \iff \mathcal{I}$ is a P-ideal.*

Proof. Let $B \notin \mathcal{I}$. Since Φ is lower semicontinuous, $\infty = \Phi(B) = \lim_{n\to\infty} \Phi(B \cap \{1,\dots,n\})$. Then for every $n \in \mathbb{N}$ there is k_n^B such that $\Phi(B \cap \{1,\dots,k_n^B\}) > n$. Since Φ is nonpathological, there is a measure $\mu_n^B \leq \Phi$ such that $\mu_n^B(B \cap \{1,\dots,k_n^B\}) > n$. Now we define a matrix $A^B = (a_{n,k})$ by

$$a_{n,k} = \begin{cases} \dfrac{\mu_n^B(\{k\})}{\mu_n^B(B \cap \{1,\dots,k_n^B\})} & \text{for } k \in B \cap \{1,\dots,k_n^B\} \text{ and } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

If we show that for every $B \notin \mathcal{I}$ we have

(1) A^B is a nonnegative regular matrix,
(2) $A^B \in \mathcal{M}(\mathcal{I})$,
(3) $B \notin \mathcal{I}(A^B)$,

then $\mathcal{I} = \bigcap\{\mathcal{I}(A) : A \in \mathcal{M}(\mathcal{I})\}$, and the proof will be finished by Theorem 5.5.

Below we show properties (1)–(3), but first, observe that for every $C \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} d_n^{A^B}(C) = \sum_{k\in C} a_{n,k} &= \sum_{k \in C\cap B\cap\{1,\dots,k_n^B\}} \frac{\mu_n^B(\{k\})}{\mu_n^B(B \cap \{1,\dots,k_n^B\})} \\ &= \frac{\mu_n^B(C \cap B \cap \{1,\dots,k_n^B\})}{\mu_n^B(B \cap \{1,\dots,k_n^B\})}. \end{aligned}$$

(1) First, it is obvious that $a_{n,k} \geq 0$ for all n, k.
Second,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = \lim_{n\to\infty} d_n^{A^B}(\mathbb{N}) = \lim_{n\to\infty} \frac{\mu_n^B(\mathbb{N} \cap B \cap \{1,\dots,k_n^B\})}{\mu_n^B(B \cap \{1,\dots,k_n^B\})} = 1.$$

Third, for every k we have

$$\begin{aligned} \lim_{n\to\infty} a_{n,k} &= \lim_{n\to\infty} \left(\frac{\mu_n^B(\{k\})}{\mu_n^B(B \cap \{1,\dots,k_n^B\})} \cdot \chi_{B\cap\{1,\dots,k_n^B\}}(k) \right) \\ &\leq \lim_{n\to\infty} \frac{\Phi(\{k\})}{\mu_n^B(B \cap \{1,\dots,k_n^B\})} \leq \lim_{n\to\infty} \frac{\Phi(\{k\})}{n} = 0. \end{aligned}$$

Thus, by Theorem 2.9, A^B is a nonnegative regular matrix.

(2) We have to show that $\mathcal{I} \subseteq \mathcal{I}(A^B)$. Let $C \in \mathcal{I}$ (i.e. $\Phi(C) < \infty$). Since

$$\overline{d_n^{A^B}}(C) = \limsup_{n\to\infty} d_n^{A^B}(C) = \limsup_{n\to\infty} \frac{\mu_n^B(C \cap B \cap \{1,\dots,k_n^B\})}{\mu_n^B(B \cap \{1,\dots,k_n^B\})} \leq \limsup_{n\to\infty} \frac{\Phi(C)}{n} = 0,$$

we obtain $C \in \mathcal{I}(A^B)$.

(3) It follows from

$$d^{A^B}(B) = \lim_{n\to\infty} d_n^{A^B}(B) = \lim_{n\to\infty} \frac{\mu_n^B(B \cap B \cap \{1,\dots,k_n^B\})}{\mu_n^B(B \cap \{1,\dots,k_n^B\})} = 1.$$

Now we show the "moreover" part of the theorem. The implication "$\implies$" follows from Theorem 5.6. The implication "$\impliedby$" follows from the previous part of the theorem and Theorem 2.39(2). □

Corollary 5.8. *If* $\mu : \mathcal{P}(\mathbb{N}) \to [0;\infty]$ *is a measure and* $\mathcal{I} = \mathrm{Fin}(\mu)$*, then* $\lim^{\mathcal{I}} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$.

Proof. First note that every measure is a nonpathological lower semicontinuous submeasure. Then apply Theorem 5.7. □

The following example shows that "measure" cannot be replaced by "finitely additive measure" in Corollary 5.8.

Example 5.9. Let $\mathcal{I}$ be a maximal ideal on $\mathbb{N}$. Define $\mu(A) = 0$ if $A \in \mathcal{I}$ and $\mu(A) = \infty$ if $A \notin \mathcal{I}$. Then μ is a finitely additive measure with $\mathcal{I} = \mathrm{Fin}(\mu)$. On the other hand, $\mathcal{M}(\mathcal{I}) = \emptyset$ (see [10, Proposition 6.5]). Thus $\lim^{\mathcal{I}} \restriction \ell^\infty \neq \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$ as the intersection is undefined in this case.

Definition 5.10. For every $f : \mathbb{N} \to [0,\infty)$ such that $\sum_{n=1}^\infty f(n) = \infty$ we define a *summable ideal generated by a function* f by

$$\mathcal{I}_f = \left\{ B \subseteq \mathbb{N} : \sum_{n\in B} f(n) < \infty \right\}.$$

In particular, if $f(n) = 1/n^\alpha$ with $0 < \alpha \leq 1$ we obtain the ideal

$$\mathcal{I}_{1/n^\alpha} = \left\{ B \subseteq \mathbb{N} : \sum_{n\in B} \frac{1}{n^\alpha} < \infty \right\}.$$

Corollary 5.11. *If* $\mathcal{I} = \mathcal{I}_f$ *is a summable ideal, then* $\lim^{\mathcal{I}} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$.

Proof. First note that $\mu(A) = \sum_{n\in A} f(n)$ is a measure and $\mathcal{I} = \mathrm{Fin}(\mu)$. Then apply Corollary 5.8. □

Corollary 5.12 (Gogola, Mačaj and Visnyai [14, Theorem 4.4])**.** *If* $\alpha \in (0;1]$ *and* $\mathcal{I} = \mathcal{I}_{1/n^\alpha}$*, then* $\lim^{\mathcal{I}} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$.

Proof. Since $\mathcal{I}_{1/n^\alpha}$ is a summable ideal, Corollary 5.11 finishes the proof. □

Remark. An example of a nonpathological submeasure Φ such that the ideal $\mathcal{I} = \mathrm{Fin}(\Phi)$ is not a summable ideal is given in [9, Example 1.11.1].

5.1.2. *Analytic P-ideals.*

Lemma 5.13. *For every lower semicontinuous submeasure* $\Phi : \mathcal{P}(\mathbb{N}) \to [0;\infty]$ *there exists a lower semicontinuous submeasure* $\Psi : \mathcal{P}(\mathbb{N}) \to [0;\infty]$ *such that* $\mathrm{Exh}(\Phi) = \mathrm{Exh}(\Psi)$ *and* $\|\mathbb{N}\|_\Psi < \infty$*. Moreover, if* Φ *is nonpathological then* Ψ *can also be taken nonpathological.*

Proof. It is enough to take $\Psi(A) = \min\{1, \Phi(A)\}$. □

Theorem 5.14. *If* Φ *is a nonpathological lower semicontinuous submeasure and* $\mathcal{I} = \mathrm{Exh}(\Phi)$*, then* $\lim^{\mathcal{I}^*} \restriction \ell^\infty = \lim^{\mathcal{I}} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$.

Proof. By Theorem 2.23, $\mathcal{I}$ is a P-ideal, so, by Theorem 2.39(2), we have $\lim^{\mathcal{I}} = \lim^{\mathcal{I}^*}$. Below we show the second equality.

By Theorem 5.5, it is enough to show that $\mathcal{I} = \bigcap\{\mathcal{I}(A) : A \in \mathcal{M}(\mathcal{I})\}$. By Lemma 5.13, we can assume that $\|\mathbb{N}\|_\Phi < \infty$.

Let $B \notin \mathcal{I}$. Since Φ is lower semicontinuous, for every $n \in \mathbb{N}$ there is $k_n^B \in \mathbb{N}$ such that $|\Phi(B \setminus \{1,\ldots,n\}) - \Phi((B \setminus \{1,\ldots,n\}) \cap \{1,\ldots,k_n^B\})| < 1/n$ and $\Phi((B \setminus \{1,\ldots,n\}) \cap \{1,\ldots,k_n^B\}) > 0$. Since Φ is nonpathological, for every n there is a measure $\mu_n^B \leq \Phi$ such that $|\Phi((B \setminus \{1,\ldots,n\}) \cap \{1,\ldots,k_n^B\}) - \mu_n^B((B \setminus \{1,\ldots,n\}) \cap \{1,\ldots,k_n^B\})| < 1/n$ and $\mu_n^B((B \setminus \{1,\ldots,n\}) \cap \{1,\ldots,k_n^B\}) > 0$.

Now we define a matrix $A^B = (a_{n,k})$ by

$$a_{n,k} = \frac{\mu_n^B(\{k\})}{\mu_n^B((B \setminus \{1,\ldots,n\}) \cap \{1,\ldots,k_n^B\})}$$

for $k \in (B \setminus \{1,\ldots,n\}) \cap \{1,\ldots,k_n^B\}$ and $n \in \mathbb{N}$, and $a_{n,k} = 0$ otherwise.

If we show that for every $B \notin \mathcal{I}$ we have

1. A^B is a nonnegative regular matrix,
2. $A^B \in \mathcal{M}(\mathcal{I})$,
3. $B \notin \mathcal{I}(A^B)$,

then $\mathcal{I} = \bigcap\{\mathcal{I}(A) : A \in \mathcal{M}(\mathcal{I})\}$, and the proof will be finished by Theorem 5.5.

Below we show properties (1)–(3), but first, observe that for every $C \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} d_n^{A^B}(C) = \sum_{k\in C} a_{n,k} &= \sum_{k \in C\cap(B\setminus\{1,\ldots,n\})\cap\{1,\ldots,k_n^B\}} \frac{\mu_n^B(\{k\})}{\mu_n^B((B \setminus \{1,\ldots,n\}) \cap \{1,\ldots,k_n^B\})} \\ &= \frac{\mu_n^B(C \cap (B \setminus \{1,\ldots,n\}) \cap \{1,\ldots,k_n^B\})}{\mu_n^B((B \setminus \{1,\ldots,n\}) \cap \{1,\ldots,k_n^B\})}. \end{aligned}$$

(1) First, it is obvious that $a_{n,k} \geq 0$ for all n, k.

Second,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = \lim_{n\to\infty} d_n^{A^B}(\mathbb{N}) = \lim_{n\to\infty} \frac{\mu_n^B(\mathbb{N} \cap (B \setminus \{1,\ldots,n\}) \cap \{1,\ldots,k_n^B\})}{\mu_n^B((B \setminus \{1,\ldots,n\}) \cap \{1,\ldots,k_n^B\})} = 1.$$

Third, for every k we have $a_{n,k} = 0$ for every $n > k$, so $\lim_{n\to\infty} a_{n,k} = 0$.

Thus, by Theorem 2.9, A^B is a nonnegative regular matrix.

(2) We have to show that $\mathcal{I} \subseteq \mathcal{I}(A^B)$. Let $C \in \mathcal{I}$ (i.e. $\|C\|_\Phi = 0$). Then

$$\begin{aligned}\overline{d^{A^B}}(C) &= \limsup_{n\to\infty} d_n^{A^B}(C) = \limsup_{n\to\infty} \frac{\mu_n^B(C \cap (B \setminus \{1,\dots,n\}) \cap \{1,\dots,k_n^B\})}{\mu_n^B((B \setminus \{1,\dots,n\}) \cap \{1,\dots,k_n^B\})}\\ &\leq \limsup_{n\to\infty} \frac{\Phi(C \cap (B \setminus \{1,\dots,n\}) \cap \{1,\dots,k_n^B\})}{\mu_n^B((B \setminus \{1,\dots,n\}) \cap \{1,\dots,k_n^B\})}\\ &\leq \limsup_{n\to\infty} \frac{\Phi(C \setminus \{1,\dots,n\})}{\mu_n^B((B \setminus \{1,\dots,n\}) \cap \{1,\dots,k_n^B\})} = \frac{\|C\|_\Phi}{\|B\|_\Phi} = 0.\end{aligned}$$

Thus $C \in \mathcal{I}(A^B)$.

(3) It follows from

$$d^{A^B}(B) = \lim_{n\to\infty} d_n^{A^B}(B) = \lim_{n\to\infty} \frac{\mu_n^B(B \cap (B \setminus \{1,\dots,n\}) \cap \{1,\dots,k_n^B\})}{\mu_n^B((B \setminus \{1,\dots,n\}) \cap \{1,\dots,k_n^B\})} = 1.$$

□

Corollary 5.15 (Fridy and Miller [12, Theorems 1 and 4])**.** *If $A \in \mathcal{NRM}$ and $\mathcal{I} = \mathcal{I}(A)$, then $\lim^{\mathcal{I}} \restriction \ell^\infty = \bigcap\{\lim^B \restriction \ell^\infty : B \in \mathcal{M}(\mathcal{I})\}$. In particular, $\lim^{\mathcal{I}_d} \restriction \ell^\infty = \bigcap\{\lim^B \restriction \ell^\infty : B \in \mathcal{M}(\mathcal{I}_d)\}$.*

Proof. By Lemma 2.33, we can assume that A has only finitely many nonzero elements in each row and each row of A sums to 1. Let $\Phi : \mathcal{P}(\mathbb{N}) \to [0;\infty]$ be given by $\Phi(B) = \sup\{d_i^A(B) : i \in \mathbb{N}\}$. Then it is not difficult to show that Φ is a nonpathological lower semicontinuous submeasure and $\mathcal{I}(A) = \mathrm{Exh}(\Phi)$. Thus Theorem 5.14 finishes the proof. □

Example 5.16. In [10, Propositions 6.8 and 6.15] we constructed an example of a pathological lower semicontinuous submeasure Φ such that

$$\lim^{\mathcal{I}} \restriction \ell^\infty \neq \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$$

where $\mathcal{I} = \mathrm{Fin}(\Phi)$ and $\mathcal{I} = \mathrm{Exh}(\Phi)$.

Question 1. Does there exist a pathological lower semicontinuous submeasure Φ such that

$$\lim^{\mathcal{I}} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}$$

where $\mathcal{I} = \mathrm{Fin}(\Phi)$ or $\mathcal{I} = \mathrm{Exh}(\Phi)$?

5.2. **The ideals of sets of logarithmic and exponential density zero.**

Beside the asymptotic density (see Definition 2.16), there are other kinds of densities (see e.g. [15] where various densities are surveyed). For instance we have the logarithmic density δ and the exponential density ε and we have the corresponding ideals $\mathcal{I}_\delta$ and $\mathcal{I}_\varepsilon$ (see Definitions 5.17 and 5.18). Visnyai asked [33] whether $\mathcal{I}_\delta$-limit and $\mathcal{I}_\varepsilon$-limit functions can be represented as an intersections of matrix summability methods in the realm of bounded sequences in the manner similar to Corollary 5.12. Below we answer the question in the positive (see Theorems 5.19 and 5.20).

Definition 5.17. For a set $A \subseteq \mathbb{N}$ we define

(1) the *logarithmic density of A* by

$$\delta(A) = \lim_{n\to\infty} \left(\sum_{i\in A, i\leq n} \frac{1}{i}\right) \Big/ \left(\sum_{i\leq n} \frac{1}{i}\right),$$

provided that the limit exists;

(2) the *exponential density of A* by

$$\varepsilon(A) = \lim_{n\to\infty} \frac{\log|A \cap \{1,\ldots,n\}|}{\log n},$$

provided that the limit exists. (For a generalization of the exponential density see [13].)

Definition 5.18.

(1) The family $\mathcal{I}_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$ is an ideal called the *ideal of sets of logarithmic density zero.*

(2) The family $\mathcal{I}_\varepsilon = \{A \subseteq \mathbb{N} : \varepsilon(A) = 0\}$ is an ideal called the *ideal of sets of exponential density zero.*

Theorem 5.19.

(1) *$\mathcal{I}_\delta$ is a matrix ideal.*

(2) $\lim^{\mathcal{I}_\delta} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I}_\delta)\}$.

Proof. First notice that (2) follows from (1) and Corollary 5.15. Below we prove (1).

It is not difficult to see that $\mathcal{I}_\delta$ is a matrix ideal generated by the nonnegative, regular matrix $A = (a_{i,k})$, where $a_{i,k} = \frac{1/i}{\sum_{j\leq k} 1/j}$ for $i \leq k$ and $a_{i,k} = 0$ otherwise. □

Theorem 5.20.

(1) *$\mathcal{I}_\varepsilon$ is a matrix ideal.*

(2) $\lim^{\mathcal{I}_\varepsilon} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I}_\varepsilon)\}$.

Proof. First notice that (2) follows from (1) and Corollary 5.15. Below we prove (1).

We will construct a matrix $A = (a_{i,k})$ such that $\mathcal{I}_\varepsilon = \mathcal{I}(A)$. Put $l_0 = 0$, $l_n = \sum_{j\leq n} j!$ for $n \in \mathbb{N}$ and by $(\alpha_{l_{n-1}+1}, \ldots, \alpha_{l_n})$ enumerate in any way the set of all permutations of $[1,n] \cap \mathbb{N}$.

For a given $l_{n-1} < i \leq l_n$ put

$$a_{i,k} = \frac{1/j}{\sum_{m\leq n} 1/m}$$

when $k = \alpha_i(j)$ for some $j \leq n$ and $a_{i,k} = 0$ for $k > n$.

Clearly, A is nonnegative and $d_i^A(\mathbb{N}) = 1$ for every $i \in \mathbb{N}$. It is also not difficult to see that $\lim_{i\to\infty} a_{i,k} = 0$ for every $k \in \mathbb{N}$, because $\lim_{n\to\infty} \sum_{m\leq n} 1/m = \infty$. Therefore, A is regular. We will now show that $\mathcal{I}_\varepsilon = \mathcal{I}(A)$.

$(\subseteq)$: Let $B \in \mathcal{I}_\varepsilon$. Clearly, $\lim_{n\to\infty} \frac{\log|B\cap\{1,\ldots,n\}|}{\log n} = 0$, which means that

$$\lim_{n\to\infty} \frac{\sum_{m\leq|B\cap\{1,\ldots,n\}|} 1/m}{\sum_{m\leq n} 1/m} = 0.$$

Observe that for $l_{n-1} < i \leq l_n$ we have

$$0 \leq d_i^A(B) \leq \frac{\sum_{m\leq|B\cap\{1,\ldots,n\}|} 1/m}{\sum_{m\leq n} 1/m} \to_{n\to\infty} 0.$$

Thus, $B \in \mathcal{I}(A)$.

$(\supseteq)$: Let $B \notin \mathcal{I}_\varepsilon$. Then

$$\limsup_{n\to\infty} \frac{\log|B\cap\{1,\dots,n\}|}{\log n} > 0,$$

so can we find an increasing sequence (n_i) such that $\lim_{i\to\infty} \frac{\log|B\cap\{1,\dots,n_i\}|}{\log n_i} > 0$. Now, for each of those n_i we can find a permutation α_j, with $l_{n_i-1} < j \leq l_{n_i}$, such that

$$d^A_j(B) = \frac{\sum_{m\leq|B\cap\{1,\dots,n_i\}|} 1/m}{\sum_{m\leq n_i} 1/m}.$$

Since

$$\lim_{i\to\infty} \frac{\sum_{m\leq|B\cap\{1,\dots,n_i\}|} 1/m}{\sum_{m\leq n_i} 1/m} = \lim_{i\to\infty} \frac{\log|B\cap\{1,\dots,n_i\}|}{\log n_i} > 0,$$

we get $\limsup_{i\to\infty} d^A_i(B) > 0$, thus $B \notin \mathcal{I}(A)$. □

5.3. **The ideal of uniform density zero sets.**

Definition 5.21. For a set $A \subseteq \mathbb{N}$ we define the *upper uniform density* of A by

$$\overline{u}(A) = \lim_{h\to\infty} \max_{n\in\mathbb{N}} \frac{|A\cap[n,n+h)|}{h}.$$

In [16] the authors show that the limit in the above definition exists for every set A, and they also show that the upper uniform density is equal to the *upper Banach density.*

Definition 5.22. The family $\mathcal{I}_u = \{A \subseteq \mathbb{N} : \overline{u}(A) = 0\}$ is an ideal called the *ideal of sets of uniform density zero.*

Theorem 5.23. $\lim^{\mathcal{I}_u} \restriction \ell^\infty = \bigcap\{\lim^A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I}_u)\}$.

Proof. In this case we cannot use Corollary 5.15 to prove the theorem because $\mathcal{I}_u$ is not a P-ideal (see [11, p. 299]), hence, by Proposition 2.34, it is not a matrix ideal.

Let $\alpha \in \mathbb{N}^\mathbb{N}$ be an increasing sequence of natural numbers. Consider the sets

$$\begin{aligned} B^\alpha_n =& \{C \subseteq [1,\alpha(n)] : C \text{ is an interval of length } n\} \\ =& \{[1,n]\cap\mathbb{N}, [2,n+1]\cap\mathbb{N}, \dots, [\alpha(n)-n+1,\alpha(n)]\cap\mathbb{N}\} \end{aligned}$$

and take $l_n = \sum_{i\leq n} |B^\alpha_i|$. Enumerate the elements of every B^α_n in any way by $(C_{l_{n-1}+1}, \dots, C_{l_n})$.

We construct the matrix $A_\alpha = (a_{i,k})$ in the following way. If $l_{n-1} < i \leq l_n$ for some $n \in \mathbb{N}$ then put $a_{i,k} = 1/n$ for $k \in C_i$ and $a_{i,k} = 0$ otherwise.

By Theorem 5.5, we will finish the proof by showing that $\mathcal{I}_u = \bigcap\{\mathcal{I}(A_\alpha) : \alpha \in \mathbb{N}^\mathbb{N} \wedge \alpha \text{ is increasing}\}$.

$(\subseteq)$ Take $B \in \mathcal{I}_u$ and an increasing sequence of natural numbers α. Then for every $\varepsilon > 0$ there is $H \in \mathbb{N}$ such that for all $h > H$ and $n \in \mathbb{N}$ we have $|B\cap[n,n+h)|/h < \varepsilon$. It follows that for every $i > l_H$ we get $d^{A_\alpha}_i(B) < \varepsilon$. Thus, $B \in \mathcal{I}(A_\alpha)$.

$(\supseteq)$ Take $B \notin \mathcal{I}_u$. Then there are $\varepsilon > 0$, infinitely many $h \in \mathbb{N}$ and appropriate $n_h \in \mathbb{N}$ such that $|B\cap[n_h,n_h+h)|/h > \varepsilon$. We take an increasing sequence α with $\alpha(h) > n_h+h$ for all h mentioned above. Then for every of those h we can find some $l_{h-1} < i \leq l_h$ for which $C_i = [n_h,n_h+h)$, hence $d^{A_\alpha}_i(B) = |B\cap[n_h,n_h+h)|/h > \varepsilon$. Therefore, $B \notin \mathcal{I}(A_\alpha)$ for this chosen α. □

6. Unions of matrix summability methods

Fridy and Miller [12, Theorem 3] proved that the $\mathcal{I}_d$-limit function is equal to a union of some matrix summability methods in the realm of all sequences, more specifically there exists $\mathcal{W} \subseteq \mathcal{NRM}$ with

$$\lim^{\mathcal{I}_d} = \bigcup\{\lim^A : A \in \mathcal{W}\}.$$

In this section we study a similar problem in the realm of bounded sequences and also for $\mathcal{I}^*$-convergence. Namely we characterize ideals for which the ideal limit function is equal to a union of some matrix summability methods (see Theorem 6.2). In the case of $\mathcal{I}^*$-convergence we show that the ideal limit function is equal to a union of some matrix summability methods for every ideal (Theorem 6.1).

Theorem 6.1. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$.*

1. $\exists \mathcal{W} \subseteq \mathcal{NRM}(\lim^{\mathcal{I}^*} = \bigcup\{\lim^A : A \in \mathcal{W}\})$.
2. $\exists \mathcal{W} \subseteq \mathcal{NRM}(\lim^{\mathcal{I}^*} \restriction \ell^\infty = \bigcup\{\lim^A \restriction \ell^\infty : A \in \mathcal{W}\})$.

Proof. Since (2) follows from (1) it is enough to show (1).

For every $x \in c^{\mathcal{I}^*}$ there is $F_x \in \mathcal{I}^*$ such that $\lim^{\mathcal{I}^*} x = \lim(x \restriction F_x)$. Let $f^x_1 < f^x_2 < \dots$ be the increasing enumeration of F_x. Then we define a matrix $A_x = (a^x_{i,k})$ by $a^x_{i,k} = 1$ if $k = f^x_i$ and $a^x_{i,k} = 0$ otherwise. Then $A_x \in \mathcal{NRM}$ and $\lim^{A_x} x = \lim(x \restriction F_x) = \lim^{\mathcal{I}^*} x$ for every $x \in c^{\mathcal{I}^*}$. Let $\mathcal{W} = \{A_x : x \in c^{\mathcal{I}}\}$. We claim that $\lim^{\mathcal{I}^*} = \bigcup\{\lim^A : A \in \mathcal{W}\}$.

($\subseteq$) If $\lim^{\mathcal{I}^*} x = L$, then $x \in c^{\mathcal{I}^*}$, so $\lim^{A_x} x = L$ and $A_x \in \mathcal{W}$.

($\supseteq$) If $\lim^A y = L$ for some $A \in \mathcal{W}$, then there is $x \in c^{\mathcal{I}^*}$ with $A = A_x$. Since $L = \lim^{A_x} y = \lim(y \restriction F_x)$ and $F_x \in \mathcal{I}^*$, we obtain that y is $\mathcal{I}^*$-convergent to L. □

Theorem 6.2. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following conditions are equivalent.*

1. $\exists \mathcal{W} \subseteq \mathcal{NRM}(\lim^{\mathcal{I}} = \bigcup\{\lim^A : A \in \mathcal{W}\})$.
2. $\exists \mathcal{W} \subseteq \mathcal{NRM}(\lim^{\mathcal{I}} \restriction \ell^\infty = \bigcup\{\lim^A \restriction \ell^\infty : A \in \mathcal{W}\})$.
3. *$\mathcal{I}$ is a P-ideal.*

Proof. (1) $\implies$ (2) Obvious.

(2) $\implies$ (3) Let $B_1, B_2, \dots \in \mathcal{I}$ be pairwise disjoint sets such that $\bigcup_{n\in\mathbb{N}} B_n = \mathbb{N}$. If we find a set $B \in \mathcal{I}$ such that $B_n \setminus B$ is finite for every $n \in \mathbb{N}$, the proof will be finished.

Let $x \in \mathbb{R}^{\mathbb{N}}$ be defined by $x_k = 1/n \iff k \in B_n$. Since $x \in \ell^\infty$ and $\lim^{\mathcal{I}} x = 0$, there is $A \in \mathcal{W}$ with $\lim^A x = 0$.

Note that $B_n \in \mathcal{I}(A)$ for every $n \in \mathbb{N}$. Indeed, if there was n with $B_n \notin \mathcal{I}(A)$, then

$$\lim^A x = \lim_{i\to\infty} A_i(x) \geq \lim_{i\to\infty} \sum_{k\in B_n} x_k a_{i,k} = \frac{1}{n} \lim_{i\to\infty} \sum_{k\in B_n} a_{i,k} = \frac{1}{n} \cdot \overline{d^A}(B_n) > 0,$$

a contradiction.

Now we inductively construct two increasing sequences i_n and k_n satisfying the following conditions:

1. $d^A_i(B_1 \cup \dots \cup B_n) < \frac{1}{n}$ for every $i \geq i_n$, and
2. $d^A_i(\mathbb{N} \setminus \{1, \dots, k_n\}) < \frac{1}{n}$ for every $i \leq i_{n+1}$.

The construction is possible because $B_1 \cup \dots \cup B_n \in \mathcal{I}(A)$ and $\lim_{k\to\infty} d_i^A(\mathbb{N} \setminus \{1,\dots,k\}) = 0$.

We define $y \in \ell^\infty$ by

$$y_k = \begin{cases} 0 & \text{if } k \in B_n \land k \leq k_n \land n \in \mathbb{N}, \\ 1 & \text{if } k \in B_n \land k > k_n \land n \in \mathbb{N}. \end{cases}$$

First we show that $\lim^A y = 0$. Indeed, let $i \in \mathbb{N}$ and $n \in \mathbb{N}$ be such that $i_n \leq i < i_{n+1}$. Since

$$A_i(y) = \sum_{k \in B_1 \cup \dots \cup B_n} y_k a_{i,k} + \sum_{\substack{k \notin B_1 \cup \dots \cup B_n \\ k \leq k_n}} y_k a_{i,k} + \sum_{\substack{k \notin B_1 \cup \dots \cup B_n \\ k > k_n}} y_k a_{i,k} \leq$$

$$\sum_{k \in B_1 \cup \dots \cup B_n} 1 \cdot a_{i,k} + \sum_{\substack{k \notin B_1 \cup \dots \cup B_n \\ k \leq k_n}} 0 \cdot a_{i,k} + \sum_{\substack{k \notin B_1 \cup \dots \cup B_n \\ k > k_n}} 1 \cdot a_{i,k} \leq$$

$$d_i^A(B_1 \cup \dots \cup B_n) + 0 + d_i^A(\mathbb{N} \setminus \{1,\dots,k_n\}) \leq \frac{1}{n} + \frac{1}{n},$$

we get $\lim^A y = \lim_{i\to\infty} A_i(y) = 0$.

Since $y \in \ell^\infty$ and $A \in \mathcal{W}$, we have $\lim^{\mathcal{I}} y = 0$. Let $B = \{k \in \mathbb{N} : |y_k - 0| > 1/2\}$. Then $B \in \mathcal{I}$ and $B_n \setminus B$ is finite for every $n \in \mathbb{N}$.

(3) $\implies$ (1) By Theorem 2.39(2), $\lim^{\mathcal{I}} = \lim^{\mathcal{I}^*}$, so it is enough to apply Theorem 6.1(1). ☐

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