A UNIFIED APPROACH TO HINDMAN, RAMSEY AND VAN DER WAERDEN SPACES

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Abstract. For many years, there have been conducting research (e.g. by Bergelson, Furstenberg, Kojman, Kubiś, Shelah, Szeptycki, Weiss) into sequentially compact spaces that are, in a sense, topological counterparts of some combinatorial theorems, for instance Ramsey's theorem for coloring graphs, Hindman's finite sums theorem and van der Waerden's arithmetical progressions theorem. These spaces are defined with the aid of different kinds of convergences: IP-convergence, R-convergence and ordinary convergence.

The first aim of this paper is to present a unified approach to these various types of convergences and spaces. Then, using this unified approach, we prove some general theorems about existence of the considered spaces and show that all results obtained so far in this subject can be derived from our theorems.

The second aim of this paper is to obtain new results about the specific types of these spaces. For instance, we construct a Hausdorff Hindman space that is not an $\mathcal{I}_{1/n}$-space and a Hausdorff differentially compact space that is not Hindman. Moreover, we compare Ramsey spaces with other types of spaces. For instance, we construct a Ramsey space that is not Hindman and a Hindman space that is not Ramsey.

The last aim of this paper is to provide a characterization that shows when there exists a space of one considered type that is not of the other kind. This characterization is expressed in purely combinatorial manner with the aid of the so-called Katětov order that has been extensively examined for many years so far.

This paper may interest the general audience of mathematicians as the results we obtain are on the intersection of topology, combinatorics, set theory and number theory.

Contents

Date: August 18, 2023.

2020 *Mathematics Subject Classification.* Primary: 54A20 05D10, 03E75. Secondary: 03E02, 54D30, 54H05, 26A03, 40A35, 40A05, 03E17, 03E15, 03E05, 03E35.

Key words and phrases. Ramsey's theorem, Hindman's theorem, van der Waerden's theorem, sequentially compact space, compact space, Bolzano–Weierstrass theorem, Ramsey space, Hindman space, differentially compact space, ideal, filter, almost disjoint families, Katětov order.

1. Introduction

For more than twenty years, many mathematicians have been examining sequentially compact spaces that are, in a sense, topological counterparts of some combinatorial theorems, for instance Ramsey's theorem for coloring graphs, Hindman's finite sums theorem and van der Waerden's arithmetical progressions theorem [5, 6, 18, 19, 20, 22, 24, 25, 26, 29, 33, 34, 50, 55, 56, 57, 58, 59, 60, 61, 71]. These spaces are defined with the aid of different kinds of convergences: IP-convergence, R-convergence and ordinary convergence.

We start our brief overview of these spaces with the ones defined using ordinary convergence. A topological space X is called:

- *van der Waerden* [56] if for every sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X there exists a convergent subsequence $\langle x_n\rangle_{n\in A}$ with A being an AP-set (i.e. A contains arithmetic progressions of arbitrary finite length);
- an $\mathcal{I}_{1/n}$*-space* [29] if for every sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X there exists a convergent subsequence $\langle x_n\rangle_{n\in A}$ with A having the property that the series of reciprocals of elements of A diverges.

In fact both mentioned classes of spaces are special cases of a more general notion. A nonempty family $\mathcal{I}\subseteq\mathcal{P}(\mathbb{N})$ of subsets of $\mathbb{N}$ is an *ideal on* $\mathbb{N}$ if it is closed under taking subsets and finite unions of its elements, $\mathbb{N}\notin\mathcal{I}$ and $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$ (it is easy to see that the family $\mathcal{I}_{1/n}=\{A\subseteq\mathbb{N}:\sum_{n\in A}1/n<\infty\}$ is an ideal on $\mathbb{N}$, and it follows from van der Waerden's theorem [73] that the family $\mathcal{W}=\{A\subseteq\mathbb{N}: A \text{ is not an AP-set}\}$ is an ideal on $\mathbb{N}$). If $\mathcal{I}$ is an ideal on $\mathbb{N}$ then a topological space X is called an $\mathcal{I}$*-space* [29] if for every sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X there exists a converging subsequence $\langle x_n\rangle_{n\in A}$ with $A\notin\mathcal{I}$. In particular, van der Waerden spaces coincide with $\mathcal{W}$-spaces.

Now we want to turn our attention to spaces defined with the aid of different kinds of convergence. We start with Hindman spaces. A set $A\subseteq\mathbb{N}$ is an *IP-set* [34] if there exists an infinite set $D\subseteq\mathbb{N}$ such that $\mathrm{FS}(D)\subseteq A$ where $\mathrm{FS}(D)$ denotes the set of all finite sums of distinct elements of D. The family $\mathcal{H}=\{A\subseteq\mathbb{N}: A \text{ is not an IP-set}\}$ is an ideal on $\mathbb{N}$ (it follows from Hindman's theorem [42]).

An *IP-sequence* in X is a sequence indexed by $\mathrm{FS}(D)$ for some infinite $D\subseteq\mathbb{N}$. An IP-sequence $\langle x_n\rangle_{n\in\mathrm{FS}(D)}$ in a topological space X is *IP-convergent* [34] to a point $x \in X$ if for every neighborhood U of x there exists $m \in \mathbb{N}$ so that $x_n \in U$ for every $n \in \mathrm{FS}(D \setminus \{0, 1, \dots, m\})$ (then x is called the *IP-limit* of the sequence).

Since only finite spaces are $\mathcal{H}$-spaces [55], Kojman replaced the ordinary convergence with IP-convergence (introduced by Furstenberg and Weiss [34]) to define a meaningful topological counterpart of Hindman's finite sums theorem. Namely, a topological space X is called *Hindman* [55] if for every sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in X there exists an infinite set $D \subseteq \mathbb{N}$ such that the subsequence $\langle x_n \rangle_{n \in \mathrm{FS}(D)}$ IP-converges to some $x \in X$.

We finish our brief overview of classes of sequentially compact spaces with Ramsey spaces. Let $[A]^2$ denote the set of all pairs of elements of A. A sequence $\langle x_n \rangle_{n \in [D]^2}$ in X (indexed by pairs of natural numbers from some infinite set $D \subseteq \mathbb{N}$) *R-converges* [6, 5] to a point $x \in X$ if for every neighborhood U of x there is a finite set F such that $x_{\{a,b\}} \in U$ for all distinct $a, b \in D \setminus F$. A topological space X is called *Ramsey* [59] if for every sequence $\langle x_n \rangle_{n \in [\mathbb{N}]^2}$ in X there exists an infinite set $D \subseteq \mathbb{N}$ such that the subsequence $\langle x_n \rangle_{n \in [D]^2}$ R-converges to some $x \in X$.

We say that an ideal $\mathcal{I}$ (on $\mathbb{N}$) is below an ideal $\mathcal{J}$ in the *Katětov order* [53] if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $f^{-1}[A] \in \mathcal{J}$ for every $A \in \mathcal{I}$. Note that Katětov order has been extensively examined (even in its own right) for many years so far [2, 3, 8, 12, 10, 39, 40, 43, 44, 46, 47, 48, 65, 66, 68, 70, 76].

There are three objectives of this paper. The first aim is to present a unified approach to these various types of convergences and spaces. This is achieved in sections in Part 1 with the aid of partition regular functions (Definition 3.1), a convergence with respect to partition regular functions (Definition 9.1) and a subclass of sequentially compact spaces defined using this new kind of convergence (see Definition 10.1). Then using this approach, we prove some general theorems about those classes of spaces (Theorem 10.5) and show that all results obtained so far in this subject can be derived from our theorems (see sections in Parts 2 and 3).

The second aim of this paper is to obtain new results concerning specific types of these spaces: Ramsey spaces, Hindman spaces, van der Waerden spaces and $\mathcal{I}_{1/n}$-spaces. For instance, we construct a Hausdorff Hindman space that is not an $\mathcal{I}_{1/n}$-space (Corollary 14.10(2)) – this gives a positive answer to a question posed by Flašková [27] (so far only non-Hausdorff answer to this question was known [22, Theorem 2.5]). We also construct a Hausdorff so-called differentially compact space that is not Hindman (Corollary 14.9(3)) which yields the negative answer to a question posed by Shi [71, Question 4.2.2] and other authors [20, Problem 1],[58, Question 3]. Moreover, we compare Ramsey spaces with other types of spaces (so far Ramsey spaces were only examined in their own right without comparing them with other kinds of spaces [6, 59, 11]). For instance, we construct a Ramsey space that is not Hindman and a Hindman space that is not Ramsey (Corollary 14.9).

The final aim of this paper is to provide a characterization that shows when there exists a space of one considered type that is not of the other type (Theorem 16.1 and other results in Part 4). This characterization is expressed in purely combinatorial manner with the aid of the Katětov order or its counterpart in the realm of partition regular functions (Definition 7.3).

2. Preliminaries

In the paper we are exclusively interested in Hausdorff topological spaces with one exception (Sections 17 and 18) where we were unable to obtain results for Hausdorff spaces but succeeded in constructing a topological space with unique limits of sequences.

Following von Neumann, we identify an ordinal number α with the set of all ordinal numbers less than α. In particular, the smallest infinite ordinal number

$\omega = \{0, 1, \dots\}$ is equal to the set $\mathbb{N}$ of all natural numbers, and each natural number $n = \{0, \dots, n-1\}$ is equal to the set of all natural numbers less than n. Using this identification, we can for instance write $n \in k$ instead of $n < k$ and $n < \omega$ instead of $n \in \omega$ or $A \cap n$ instead of $A \cap \{0, 1, \dots, n-1\}$.

If $A \subseteq \omega$ and $n \in \omega$, we write $A + n = \{a + n : a \in A\}$ and $A - n = \{a - n : a \in A, a > n\}$.

We write $[A]^2$ to denote the set of all unordered pairs of elements of A, $[A]^{<\omega}$ to denote the family of all finite subsets of A, $[A]^\omega$ to denote the family of all infinite countable subsets of A and $\mathcal{P}(A)$ to denote the family of all subsets of A.

We say that a family $\mathcal{A}$ of subsets of a set Λ is an *almost disjoint family on* Λ if

(1) $|A| = |\Lambda|$ for every $A \in \mathcal{A}$ and
(2) $|A \cap B| < |\Lambda|$ for all distinct elements $A, B \in \mathcal{A}$.

By $A \sqcup B$ we denote the *disjoint union* of sets A and B:

$$A \sqcup B = (A \times \{0\}) \cup (B \times \{1\}) = \{(x, 0) : x \in A\} \cup \{(y, 1) : y \in B\}.$$

For families of sets $\mathcal{A} \subseteq \mathcal{P}(\Lambda)$ and $\mathcal{B} \subseteq \mathcal{P}(\Sigma)$, we write $\mathcal{A} \oplus \mathcal{B} = \{A \sqcup B : A \in \mathcal{A}, B \in \mathcal{B}\}$.

A nonempty family $\mathcal{I} \subseteq \mathcal{P}(\Lambda)$ of subsets of Λ is an *ideal on* Λ if it is closed under taking subsets and finite unions of its elements, $\Lambda \notin \mathcal{I}$ and $\mathcal{I}$ contains all finite subsets of Λ. By $\mathrm{Fin}(\Lambda)$ we denote the family of all finite subsets of Λ. For $\Lambda = \omega$, we write Fin instead of $\mathrm{Fin}(\omega)$. For an ideal $\mathcal{I}$ on Λ, we write $\mathcal{I}^+ = \{A \subseteq \Lambda : A \notin \mathcal{I}\}$ and call it the *coideal of* $\mathcal{I}$, and we write $\mathcal{I}^* = \{\Lambda \setminus A : A \in \mathcal{I}\}$ and call it the *filter dual to* $\mathcal{I}$. For an ideal $\mathcal{I}$ on Λ and $A \in \mathcal{I}^+$, it is easy to see that $\mathcal{I} \restriction A = \{A \cap B : B \in \mathcal{I}\}$ is an ideal on A.

In our research the following ideal on ω^2 plays an important role:

$$\mathrm{Fin}^2 = \left\{C \subseteq \omega^2 : \{n \in \omega : \{k \in \omega : (n, k) \in C\} \notin \mathrm{Fin}\} \in \mathrm{Fin}\right\}.$$

We say that a function $f : \Lambda \to \Sigma$ is *$\mathcal{I}$-to-one* if $f^{-1}(\sigma) \in \mathcal{I}$ for every $\sigma \in \Sigma$.

A set $A \subseteq X$ is F_σ (G_δ, $F_{\sigma\delta}$, etc., resp.) in a topological space X if A is a union of a countable family of closed sets (A is an intersection of a countable family of open sets, A is an intersection of a countable family of F_σ sets, etc., resp.).

For a function $f : X \to Y$ and a set $A \subseteq X$, we write $f \restriction A$ to denote the restriction of f to the set A.

Part 1. Partition regular operations

3. Partition regular operations and ideals associated with them

Below we introduce a notion that proved to be a convenient tool allowing to grasp the common feature of different kinds of convergences related to Hindman, Ramsey and van der Waerden spaces.

Definition 3.1. Let Λ and Ω be countable infinite sets. Let $\mathcal{F}$ be a nonempty family of infinite subsets of Ω such that $F \setminus K \in \mathcal{F}$ for every $F \in \mathcal{F}$ and a finite set $K \subseteq \Omega$. We say that a function $\rho : \mathcal{F} \to [\Lambda]^\omega$ is *partition regular* if

(M): $\forall E, F \in \mathcal{F}\ (E \subseteq F \implies \rho(E) \subseteq \rho(F))$,
(R): $\forall F \in \mathcal{F}\, \forall A, B \subseteq \Lambda\ (\rho(F) = A \cup B \implies \exists E \in \mathcal{F}\, (\rho(E) \subseteq A \vee \rho(E) \subseteq B))$.
(S): $\forall F \in \mathcal{F}\, \exists E \in \mathcal{F}\, (E \subseteq F \wedge \forall a \in \rho(E)\, \exists K \in [\Omega]^{<\omega} (a \notin \rho(E \setminus K)))$.

In our considerations, we use the following easy observation concerning condition (S) of Definition 3.1.

Proposition 3.2. *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ (with $\mathcal{F} \subseteq [\Omega]^\omega$) be a partition regular function. Then for every $F \in \mathcal{F}$ there is $E \in \mathcal{F}$ such that $E \subseteq F$ and for every finite set $L \subseteq \Lambda$ there exists a finite set $K \subseteq \Omega$ such that $\rho(E \setminus K) \subseteq \rho(E) \setminus L$.*

Proof. For $F \in \mathcal{F}$, let $E \in \mathcal{F}$ be as in condition (S) of Definition 3.1. Let $L \subseteq \Lambda$ be a finite set. For every $a \in \rho(E)$, we take a finite set K_a such that $a \notin \rho(E \setminus K_a)$. Then $K = \bigcup\{K_a : a \in \rho(E) \cap L\}$ is finite and $\rho(E \setminus K) \subseteq \rho(E) \setminus L$. □

The following easy proposition reveals basic relationships between partition regular functions and ideals.

Proposition 3.3.

(1) If $\rho : \mathcal{F} \to [\Lambda]^\omega$ is partition regular, then

$$\mathcal{I}_\rho = \{A \subseteq \Lambda : \forall F \in \mathcal{F}\, (\rho(F) \not\subseteq A)\}.$$

is an ideal on Λ.

(2) If $\mathcal{I}$ is an ideal on Λ, then the function

$$\rho_{\mathcal{I}} : \mathcal{I}^+ \to [\Lambda]^\omega \quad \text{given by} \quad \rho_{\mathcal{I}}(A) = A$$

is partition regular and $\mathcal{I} = \mathcal{I}_{\rho_{\mathcal{I}}}$.

Remark. If ρ is partition regular and $\tau = \rho_{\mathcal{I}_\rho}$, then $\mathcal{I}_\tau = \mathcal{I}_\rho$ but, as we will see, in general, $\rho \neq \tau$. More important, τ may miss some crucial properties which ρ possesses (e.g. P-like properties – see Proposition 6.5(3)(4)).

Below we present the most important examples of partition regular functions that were our prototypes while we were thinking on a unified approach to Hindman, Ramsey and van der Waerden spaces.

3.1. **Hindman's finite sums theorem.** Let the function $\mathrm{FS} : [\omega]^\omega \to [\omega]^\omega$ be given by

$$\mathrm{FS}(D) = \left\{ \sum_{n \in \alpha} n : \alpha \in [D]^{<\omega} \setminus \{\emptyset\} \right\}$$

i.e. $\mathrm{FS}(D)$ is the set of all finite non-empty sums of distinct elements of D.

A set $D \subseteq \omega$ is *sparse* [55, p. 1598] if for each $n \in \mathrm{FS}(D)$ there exists the unique set $\alpha \subseteq D$ such that $n = \sum_{i \in \alpha} i$. This unique set will be denoted by $\alpha_D(n)$. For instance, the set $E = \{2^i : i \in \omega\}$ is sparse, and in the sequel, we write $\alpha(n)$ instead of $\alpha_E(n)$.

A sparse set $D \subseteq \omega$ is *very sparse* [22, p. 894] if $\alpha_D(x) \cap \alpha_D(y) \neq \emptyset$ implies $x + y \notin \mathrm{FS}(D)$ for every $x, y \in \mathrm{FS}(D)$.

Theorem 3.4 (Hindman)**.** *The function* FS *is partition regular and the family*

$$\mathcal{H} = \mathcal{I}_{\mathrm{FS}} = \{A \subseteq \omega : \forall D \in [\omega]^\omega\, (\mathrm{FS}(D) \not\subseteq A)\}$$

is an ideal on ω. The ideal $\mathcal{H}$ is called the Hindman ideal [29, p. 109]*. It is known that sets from $\mathcal{H}^+$ (that are called* IP-sets*) are examples of so-called Poincaré sequences*[1] *that play an important role in the study of recurrences in topological dynamics* [33, p. 74]*.*

Proof. It is easy to see that condition (M) of Definition 3.1 is satisfied for FS. Condition (R) of Definition 3.1 holds for FS as in this case it is the well known Hindman's finite sums theorem [42, Theorem 3.1],[4, Theorem 3.5]. To see that condition (S) of Definition 3.1 holds for FS, it is enough to notice [55, p. 1598] that every infinite set $F \subseteq \omega$ has an infinite sparse subset $G \subseteq F$ which obviously satisfies condition (S). Finally, Proposition 3.3(1) shows that $\mathcal{H}$ is an ideal on ω. □

The following lemma will be used in some proofs regarding properties of the function FS.

[1] A set $W \subseteq \mathbb{Z}$ is called a *Poincaré sequence* [33, Definition 3.6 at p. 72] if for any measure preserving system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ with $\mu(A) > 0$ we have $\mu(T^{-n}[A] \cap A) > 0$ for some $n \in W$, $n \neq 0$.

Lemma 3.5 ([55, Lemma 7]). *If D is an infinite sparse set, then there exists a set $S = \{s_i : i \in \omega\} \subseteq \mathrm{FS}(D)$ such that for every $i \in \omega$ we have $s_i < s_{i+1}$ and*

$$\max \alpha_D(s_i) < \min \alpha_D(s_{i+1}) \text{ and } \max \alpha(s_i) < \min \alpha(s_{i+1}).$$

3.2. **Ramsey's theorem for coloring graphs.**

Theorem 3.6 (Ramsey). *Let $r : [\omega]^\omega \to \left[[\omega]^2\right]^\omega$ be given by*

$$r(H) = [H]^2 = \{\{x, y\} \subseteq [\omega]^2 : x, y \in H, x \neq y\}$$

i.e. $r(H)$ is the set of all unordered pairs of elements of H. Then r is partition regular and the family

$$\mathcal{R} = \mathcal{I}_r = \{A \subseteq [\omega]^2 : \forall H \in [\omega]^\omega\ ([H]^2 \not\subseteq A)\}$$

is an ideal on $[\omega]^2$. The ideal $\mathcal{R}$ is called the Ramsey ideal [65, 48]. *(If we identify a set $A \subseteq [\omega]^2$ with a graph $G_A = (\omega, A)$, the ideal $\mathcal{R}$ can be seen as an ideal consisting of graphs without infinite complete subgraphs).*

Proof. It is easy to see that condition (M) of Definition 3.1 is satisfied for r. Condition (R) of Definition 3.1 holds for r as in this case it is the well known Ramsey's theorem for coloring graphs [69, Theorem A],[38, Theorem 1.5]. To see that condition (S) of Definition 3.1 holds for r, it is enough to notice that for every $\{a, b\} \in [F]^2$ we have $\{a, b\} \notin [F \setminus \{a, b\}]^2$. Finally, Proposition 3.3(1) shows that $\mathcal{R}$ is an ideal on $[\omega]^2$. $\square$

3.3. **The positive differences and the associated ideal.** Let the function $\Delta : [\omega]^\omega \to [\omega]^\omega$ be given by

$$\Delta(E) = \{a - b : a, b \in E, a > b\}$$

i.e. $\Delta(E)$ is the set of all positive differences of distinct elements of E.

We say that a set $E \subseteq \omega$ is *$\mathcal{D}$-sparse* [20, p. 2009] if for every $a \in \Delta(E)$ there are unique elements $b, c \in E$ such that $a = b - c$.

Proposition 3.7. *The function Δ is partition regular and the family*

$$\mathcal{D} = \mathcal{I}_\Delta = \{A \subseteq \omega : \forall E \in [\omega]^\omega\ (\Delta(E) \not\subseteq A)\}$$

is an ideal on ω such that $\mathcal{D} \subsetneq \mathcal{H}$. It is known that sets from $\mathcal{D}^+$ are examples of so-called Poincaré sequences [33, p. 74].

Proof. It is easy to see that condition (M) of Definition 3.1 is satisfied for Δ. It is known [20, Proposition 4.1] that condition (R) of Definition 3.1 holds for Δ. To see that condition (S) of Definition 3.1 holds for Δ, it is enough to notice [20, Proposition 4.3(2)] that every infinite set $F \subseteq \omega$ has an infinite $\mathcal{D}$-sparse subset $G \subseteq F$ which obviously satisfies condition (S). Finally, Proposition 3.3(1) shows that $\mathcal{D}$ is an ideal on ω and it is known [71, Proposition 4.2.1],[20, Proposition 4.1] that $\mathcal{D} \subsetneq \mathcal{H}$. $\square$

3.4. **The summable ideal.**

Proposition 3.8. *The family*

$$\mathcal{I}_{1/n} = \left\{A \subseteq \omega : \sum_{n \in A} \frac{1}{n+1} < \infty\right\}$$

is an ideal on ω. The ideal $\mathcal{I}_{1/n}$ is called the summable ideal [64, Definition 1.6],[62, Example 3],[75, p. 238],[51, p. 411]. *The function $\rho_{\mathcal{I}_{1/n}} : \mathcal{I}_{1/n}^+ \to [\omega]^\omega$ given by $\rho_{\mathcal{I}_{1/n}}(A) = A$ is partition regular and $\mathcal{I}_{1/n} = \mathcal{I}_{\rho_{\mathcal{I}_{1/n}}}$.*

Proof. It is easy to show that $\mathcal{I}_{1/n}$ is an ideal on ω, whereas Proposition 3.3(2) gives the required properties of $\rho_{\mathcal{I}_{1/n}}$. $\square$

3.5. **van der Waerden's arithmetical progressions theorem.**

Theorem 3.9 (van der Waerden). *A set $A \subseteq \omega$ is called an* AP-set *if it contains an arithmetic progressions of arbitrary finite length. The family*

$$\mathcal{W} = \{A \subseteq \omega : A \text{ is not an AP-set}\}$$

is an ideal on ω. The ideal $\mathcal{W}$ is called the van der Waerden ideal [29, p. 107]. *The function $\rho_\mathcal{W} : \mathcal{W}^+ \to [\omega]^\omega$ given by $\rho_\mathcal{W}(A) = A$ is partition regular and $\mathcal{W} = \mathcal{I}_{\rho_\mathcal{W}}$.*

Proof. It is easy to see that all conditions from the definition of an ideal but additivity are satisfied, whereas additivity is the well known van der Waerden's arithmetical progressions theorem [73],[38, Theorem 2.1]. Finally, Proposition 3.3(2) gives the required properties of $\rho_\mathcal{W}$. □

3.6. **Ideals on directed sets.** Finally, we introduce a class of partition regular functions which are connected with ideals on directed sets [17, 18]. Recall, that $(\Lambda, <)$ is a *directed set* if the relation $<$ is an upward directed strict partial order on Λ.

Let $(\Lambda, <)$ be a *directed set* such that Λ is infinite countable. A set $B \subseteq \Lambda$ is *cofinal* in $(\Lambda, <)$ if for every $\lambda \in \Lambda$ there is $b \in B$ with $\lambda < b$. A family $\mathcal{I}$ of subsets of Λ is an *ideal on* $(\Lambda, <)$ [18, Definition 2.2] if $\mathcal{I}$ is an ideal on Λ and $\mathcal{I}$ contains all sets which are *not* cofinal. A family $\mathcal{B}$ of subsets of Λ is a *coideal basis on* $(\Lambda, <)$ [18, Definition 2.4] if $\mathcal{B} \neq \emptyset$, all sets in $\mathcal{B}$ are cofinal and if $C \cup D \in \mathcal{B}$, then there exists $B \in \mathcal{B}$ such that $B \subseteq C$ or $B \subseteq D$. In particular, for every ideal $\mathcal{I}$ on $(\Lambda, <)$ the family $\mathcal{I}^+$ is a coideal basis on $(\Lambda, <)$. It is known [17, Proposition 2.7] that $\mathcal{I}$ is an ideal on $(\Lambda, <)$ if and only if there exists a coideal basis $\mathcal{B}$ on $(\Lambda, <)$ such that $\mathcal{I} = \{A \subseteq \Lambda : \forall B \in \mathcal{B}\, (B \not\subseteq A)\}$.

The following easy proposition reveals basic relationships between partition regular functions and ideals on directed sets.

Proposition 3.10. *Let $(\Lambda, <)$ be a directed set.*

1. *If $\rho : \mathcal{F} \to [\Lambda]^\omega$ is a partition regular function such that $\rho(F)$ is cofinal for every $F \in \mathcal{F}$, then*
$$\mathcal{I}_\rho = \{A \subseteq \Lambda : \forall F \in \mathcal{F}\, (\rho(F) \not\subseteq A)\}.$$
is an ideal on $(\Lambda, <)$.
2. *For a coideal basis $\mathcal{B}$ on $(\Lambda, <)$ (in particular for $\mathcal{B} = \mathcal{I}^+$, where $\mathcal{I}$ is an ideal on $(\Lambda, <)$), we define*
$$\widehat{\mathcal{B}} = \{B \setminus K : B \in \mathcal{B}, K \in [\Lambda]^{<\omega}\}.$$
Then the function $\rho_\mathcal{B} : \widehat{\mathcal{B}} \oplus \mathrm{Fin}(\Lambda)^* \to [\Lambda]^\omega$ given by
$$\rho_\mathcal{B}((B \setminus K) \sqcup C) = (B \setminus K) \cap \{\lambda \in \Lambda : \forall \lambda' \in (\Lambda \setminus C)\, (\lambda' < \lambda)\}$$
is a partition regular function such that $\rho_\mathcal{B}((B \setminus K) \sqcup C)$ is cofinal for every $(B \setminus L) \sqcup C \in \widehat{\mathcal{B}} \oplus \mathrm{Fin}(\Lambda)^*$ and $\mathcal{I}_{\rho_\mathcal{B}} = \{A \subseteq \Lambda : \forall B \in \mathcal{B}\, (B \not\subseteq A)\}$.

4. Restrictions and small accretions

4.1. **Restrictions of partition regular operations.** For $B \notin \mathcal{I}_\rho$, we define a family $\mathcal{F} \restriction B = \{E \in \mathcal{F} : \rho(E) \subseteq B\}$ and a function $\rho \restriction B : \mathcal{F} \restriction B \to [B]^\omega$ by $(\rho \restriction B)(E) = \rho(E)$ (i.e. $\rho \restriction B = \rho \restriction (\mathcal{F} \restriction B)$). The following easy proposition reveals relationships between restriction of a function ρ and restriction of an ideal $\mathcal{I}_\rho$.

Proposition 4.1. *If $\rho : \mathcal{F} \to [\Lambda]^\omega$ is partition regular and $B \notin \mathcal{I}_\rho$, then $\rho \restriction B$ is partition regular and $\mathcal{I}_{\rho \restriction B} = \mathcal{I}_\rho \restriction B$.*

4.2. **Small accretions of partition regular operations.** We will need the following notion in the last part of the paper for characterization that shows when there exists a space of one considered type that is not of the other type (Theorem 16.1).

Definition 4.2. Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ (with $\mathcal{F} \subseteq [\Omega]^\omega$) be a partition regular function.

(1) A set $F \in \mathcal{F}$ has *small accretions* if $\rho(F) \setminus \rho(F \setminus K) \in \mathcal{I}_\rho$ for every finite set K.
(2) ρ has *small accretions* if for every $E \in \mathcal{F}$ there is $F \in \mathcal{F}$ such that $F \subseteq E$ and F has small accretions.

Proposition 4.3. *If $\rho \in \{\mathrm{FS}, r, \Delta\} \cup \{\rho_\mathcal{I} : \mathcal{I}$ is an ideal$\}$, then ρ has small accretions.*

Proof for $\rho = \rho_\mathcal{I}$ where $\mathcal{I}$ is an ideal. The function ρ has small accretions, since for every $A \in \mathcal{I}^+$ and finite $K \subseteq \Lambda$ we have $\rho_\mathcal{I}(A)\setminus\rho_\mathcal{I}(A\setminus K) = A\setminus(A\setminus K) \subseteq K \in \mathcal{I}$. □

Proof for $\rho = \mathrm{FS}$. It is known [22, Lemma 2.2] that every infinite set $E \subseteq \omega$ has an infinite very sparse subset $F \subseteq E$, so if we show that every very sparse set has small accretions, the proof will be finished.

Let $F \subseteq \omega$ be an infinite very sparse set and $K \subseteq \omega$ be a finite set. Assume towards contradiction that $\mathrm{FS}(D) \subseteq \mathrm{FS}(F) \setminus \mathrm{FS}(F \setminus K) = \{x \in \mathrm{FS}(F) : \ \alpha_F(x) \cap K \neq \emptyset\}$ for some $D \in [\omega]^\omega$. Since K is finite, we can find $x, y \in D$, $x \neq y$, such that $\alpha_F(x) \cap \alpha_F(y) \neq \emptyset$. But then $x + y \in \mathrm{FS}(D) \setminus \mathrm{FS}(F)$, a contradiction. □

Proof for $\rho = r$. The function r has small accretions, since for every $A \in [\omega]^\omega$ and finite $K \subseteq \omega$ we have $r(A) \setminus r(A \setminus K) = [A]^2 \setminus [A \setminus K]^2 = \{\{i, j\} : \ i \in A \cap K, j \in A\} \in \mathcal{R}$. □

Proof for $\rho = \Delta$. It is known [20, Proposition 4.3(2)] that every infinite set $E \subseteq \omega$ has an infinite $\mathcal{D}$-sparse subset $F \subseteq E$, so if we show that every $\mathcal{D}$-sparse set has small accretions, the proof will be finished.

Let $F \subseteq \omega$ be an infinite $\mathcal{D}$-sparse set and $K \subseteq \omega$ be a finite set. It is known [20, Proposition 4.3(1)] that then $F - n \in \mathcal{D}$ for every $n < \min F$, and consequently, $\{a - b : a \in F \setminus K, b \in F \cap K\} \cap \omega \in \mathcal{D}$. Thus, $\Delta(F) \setminus \Delta(F \setminus K) = \{a - b : a \in F \cap K, b \in F, a > b\} \cup (\{a - b : a \in F \setminus K, b \in F \cap K\} \cap \omega) \in \mathcal{D}$ as a finite union of sets from $\mathcal{D}$. □

5. Topological complexity of partition regular operations

If Λ is a countable infinite set, then we consider $2^\Lambda = \{0, 1\}^\Lambda$ as a product (with the product topology) of countably many copies of a discrete topological space $\{0, 1\}$. Since 2^Λ is a Polish space [54, p. 13] and $[\Lambda]^\omega$ is a G_δ subset of 2^Λ, we obtain that $[\Lambda]^\omega$ is a Polish space as well [54, Theorem 3.11]. In particular, if Λ and Ω are countable infinite and $\mathcal{F} \subseteq [\Omega]^\omega$, we say that a partition regular function $\rho : \mathcal{F} \to [\Lambda]^\omega$ is *continuous* if ρ is a continuous function from a topological subspace $\mathcal{F}$ into a topological space $[\Lambda]^\omega$.

By identifying subsets of Λ with their characteristic functions, we equip $\mathcal{P}(\Lambda)$ with the topology of the space 2^Λ and therefore we can assign topological notions to ideals on Λ. In particular, an ideal $\mathcal{I}$ is *Borel* (*analytic*, *coanalytic*, resp.) if $\mathcal{I}$ is a Borel (analytic, coanalytic, resp.) subset of 2^Λ. Recall, a set $A \subseteq X$ is *analytic* if there is a Polish space Y and a Borel set $B \subseteq X \times Y$ such that A is a projection of B onto the first coordinate [54, Exercise 14.3], and a set $C \subseteq X$ is coanalytic if $X \setminus C$ is an analytic set.

Proposition 5.1. *If a partition regular function $\rho : \mathcal{F} \to [\Lambda]^\omega$ (with $\mathcal{F} \subseteq [\Omega]^\omega$) is continuous and $\mathcal{F}$ is a closed subset of $[\Omega]^\omega$, then the ideal $\mathcal{I}_\rho$ is coanalytic.*

Proof. We will show that $\mathcal{I}_\rho^+ = \mathcal{P}(\Lambda) \setminus \mathcal{I}_\rho$ is an analytic set. Let $B = \{(A, F) \in \mathcal{P}(\Lambda) \times \mathcal{F} : \rho(F) \subseteq A\}$. Since $B \subseteq \mathcal{P}(\Lambda) \times [\Omega]^\omega$ and $\mathcal{I}_\rho^+$ is a projection of B onto the first coordinate, we only need to show that B is a Borel set. It suffices to show that $C = (\mathcal{P}(\Lambda) \times [\Omega]^\omega) \setminus B$ is an open set, since

$$B = ((\mathcal{P}(\Lambda) \times [\Omega]^\omega) \setminus C) \cap (\mathcal{P}(\Lambda) \times \mathcal{F}).$$

Let $(A, F) \in C$. We have two cases: (1) $F \notin \mathcal{F}$ or (2) $F \in \mathcal{F}$.

Case (1). Since $\mathcal{F}$ is closed, there is an open set $U \subseteq [\Omega]^\omega$ with $F \in U$ and $U \cap \mathcal{F} = \emptyset$. Then $W = \mathcal{P}(\Lambda) \times U$ is open and $(A, F) \in W \subseteq C$.

Case (2). Since $\rho(F) \not\subseteq A$, there is $a \in \rho(F) \setminus A$. Let $V = \{D \in \mathcal{P}(\Lambda) : a \in D\}$. Then V is an open and closed set, $A \notin V$ and $\rho(F) \in V$. Since ρ is continuous at the point F, there is an open set $U \subseteq [\Omega]^\omega$ such that $F \in U$ and $\rho[U] \subseteq V$. Then $W = (\mathcal{P}(\Lambda) \setminus V) \times U$ is open and $(A, F) \in W \subseteq C$. □

Proposition 5.2.

(1) The ideals $\mathcal{I}_{1/n}$ and $\mathcal{W}$ are F_σ.
(2) The functions FS *and r are continuous.*
(3) The function Δ is not continuous. In fact, the function Δ is discontinuous at every point A such that $\Delta(A) \neq \omega$.
(4) If $\mathcal{L} = \{A \in [\omega]^\omega : \forall n \in \omega\, (e_A(n+1) - e_A(n) > e_A(n))\}$ where $e_A : \omega \to A$ is the increasing enumeration of a set $A \subseteq \omega$, then $\mathcal{I}_\Delta = \mathcal{I}_{\Delta \restriction \mathcal{L}}$, $\mathcal{L}$ is closed and $\Delta \restriction \mathcal{L}$ is continuous.
(5) The ideals $\mathcal{H}$, $\mathcal{R}$ and $\mathcal{D}$ are coanalytic.

Proof. (1) It is known that $\mathcal{I}_{1/n}$ and $\mathcal{W}$ are F_σ [64, Example 1.5], [23, Example 4.12].

(2) *Case of* FS. Let $D \in [\omega]^\omega$ and let U be an open basic neighborhood of $\mathrm{FS}(D)$. Then there exists a finite set $G \subseteq \omega$ such that $U = \{B \in [\omega]^\omega : B \cap \{0, 1, \ldots, \max G\} = G\}$. Let $F = D \cap \{0, 1, \ldots, \max G\}$. Then $V = \{A \in [\omega]^\omega : A \cap \{0, 1, \ldots, \max G\} = F\}$ is an open neighborhood of D and $\mathrm{FS}[V] \subseteq U$.

Case of r. Let $D \in [\omega]^\omega$ and let U be an open basic neighborhood of $[D]^2$. There exists a finite set $G \subseteq [\omega]^2$ such that $U = \{B \in \left[[\omega]^2\right]^\omega : B \cap [N]^2 = G\}$, where $N = \max\{\max\{p, q\} : \{p, q\} \in G\}$. Then $V = \{A \in [\omega]^\omega : A \cap N = D\}$ is an open neighborhood of D and $r[V] \subseteq U$.

(3) Let $A \subseteq \omega$ be such that $b \notin \Delta(A)$ for some $b \in \omega$. Then $U = \{B \subseteq \omega : b \notin B\}$ is an open neighborhood of $\Delta(A)$. Let V be an open basic neighborhood of A. There is $N \in \omega$ such that $V = \{C \subseteq \omega : C \cap N = A \cap N\}$. Then $C = (A \cap N) \cup (\omega \setminus N) \in V$ and $\Delta(C) = \omega \notin U$. Hence the function Δ is discontinuous at the point A.

(4) It is obvious that $\mathcal{I}_\Delta = \mathcal{I}_{\Delta \restriction \mathcal{L}}$. To show that $\mathcal{L}$ is closed, notice that $[\omega]^\omega \setminus \mathcal{L}$ is open as for each $A \in [\omega]^\omega \setminus \mathcal{L}$ there is $n \in \omega$ such that $e_A(n+1) - e_A(n) \leq e_A(n)$ and $U = \{C \in [\omega]^\omega : C \cap (e_A(n+1)+1) = A \cap (e_A(n+1)+1)\}$ is an open neighborhood of A disjoint with $\mathcal{L}$.

Below we show that $\Delta \restriction \mathcal{L}$ is continuous. Let $A \in \mathcal{L}$. We are going to show that the function $\Delta \restriction \mathcal{L}$ is continuous at the point A. Let U be a neighborhood of $\Delta(A)$. Without loss of generality, we can assume that there is $N \in \omega$ such that $U = \{B \in [\omega]^\omega : B \cap N = \Delta(A) \cap N\}$. There exists $M \in \omega$ such that $e_A(M) > N$. Then $V = \{C \in [\omega]^\omega : C \cap (e_A(M)+1) = A \cap (e_A(M)+1)\}$ is an open neighborhood of A. Once we show that $\Delta[V \cap \mathcal{L}] \subseteq U$, the proof will be finished. Let $C \in V \cap \mathcal{L}$. Since $A, C \in \mathcal{L}$, we obtain $\Delta(C) \cap (e_A(M)+1) = \Delta(C \cap (e_A(M)+1)) = \Delta(A \cap (e_A(M)+1)) = \Delta(A) \cap (e_A(M)+1)$. But $N < e_A(M)$, hence $\Delta(C) \cap N = \Delta(A) \cap N$ and consequently $\Delta(C) \in U$.

(5) It is known that $\mathcal{H}$ and $\mathcal{R}$ are coanalytic [23, Example 4.11],[65, Lemma 1.6.24] (but it also follows from item (2) and Proposition 5.1). It follows from item (4) and Proposition 5.1 that $\mathcal{D}$ is coanalytic. □

6. P-like properties

6.1. **P-like properties of ideals.** For $A, B \subseteq \Lambda$, we write $A \subseteq^* B$ if there is a finite set $K \subseteq \Lambda$ with $A \setminus K \subseteq B$.

Let us recall definitions of P-like properties of ideals that are considered in the literature [43, p. 2030]. An ideal $\mathcal{I}$ on Λ is

- $P^-(\Lambda)$ if for every $\subseteq$-decreasing sequence $A_n \in \mathcal{I}^+$ with $A_0 = \Lambda$ and $A_n \setminus A_{n+1} \in \mathcal{I}$ for each $n \in \omega$ there exists $B \in \mathcal{I}^+$ such that $B \subseteq^* A_n$ for each $n \in \omega$;
- P^- if for every $\subseteq$-decreasing sequence $A_n \in \mathcal{I}^+$ with $A_n \setminus A_{n+1} \in \mathcal{I}$ for each $n \in \omega$ there exists $B \in \mathcal{I}^+$ such that $B \subseteq^* A_n$ for each $n \in \omega$;
- P^+ if for every $\subseteq$-decreasing sequence $A_n \in \mathcal{I}^+$ there exists $B \in \mathcal{I}^+$ such that $B \subseteq^* A_n$ for each $n \in \omega$.

The following proposition reveals some implications between P-like properties and provides equivalent forms of the properties $P^-(\Lambda)$ and P^- that were considered in the literature [61] under the names *weak P-ideals* and *hereditary weak P-ideals*, where the author used them for in-depth research on $\mathcal{I}$-spaces.

Proposition 6.1. *Let $\mathcal{I}$ be an ideal on an infinite countable set Λ.*

(1) $\mathcal{I}$ is P^+ $\implies$ $\mathcal{I}$ is P^- $\implies$ $\mathcal{I}$ is $P^-(\Lambda)$.
(2) The implications from item (1) cannot be reversed.
(3) The following conditions are equivalent.
  (a) $\mathcal{I}$ is $P^-(\Lambda)$ ($\mathcal{I}$ is P^-, resp.).
  (b) For every partition $\mathcal{A}$ of Λ (of any set $C \in \mathcal{I}^+$, resp.) into sets from $\mathcal{I}$ there exists $B \in \mathcal{I}^+$ such that $B \subseteq \Lambda$ ($B \subseteq C$, resp.) and $B \cap A$ is finite for each $A \in \mathcal{A}$.
  (c) $\mathcal{I}$ is a weak P-ideal *(*hereditary weak P-ideal*, resp.) i.e. for every countable family $\mathcal{A} \subseteq \mathcal{I}$ of subsets of Λ (subsets of any $C \in \mathcal{I}^+$, resp.) there exists $B \in \mathcal{I}^+$ such that $B \subseteq \Lambda$ ($B \subseteq C$, resp.) and $B \cap A$ is finite for each $A \in \mathcal{A}$.*

Proof. (1) Straightforward.

(2) The ideal $\mathrm{Fin} \oplus \mathrm{Fin}^2$ is $P^-(\omega \sqcup \omega^2)$ (the set $B = \omega \sqcup \emptyset$ works for every sequence) but not P^- (as witnessed by the sets $A_n = \emptyset \sqcup ((\omega \setminus n) \times \omega)$).

Below we show an example of a P^- ideal that is not P^+. For a set $A \subseteq \omega$, we define the asymptotic density of A by $\overline{d}(A) = \limsup_{n\to\infty} |A \cap n|/n$. Then the ideal $\mathcal{I}_d = \{A \subseteq \omega : \overline{d}(A) = 0\}$ is P^- (see e.g. [9, Corollary 1.1]). Now we show that $\mathcal{I}_d$ is not P^+. Take a decreasing sequence $B_n \subseteq \omega$ such that $0 < \overline{d}(B_n) < 1/n$ for each $n \in \omega$. If $C \subseteq \omega$ is such that $C \subseteq^* B_n$ for all $n \in \omega$, then $\overline{d}(C) \leq \overline{d}(B_n) \to 0$ as $n \to \infty$. Hence $C \in \mathcal{I}_d$. This shows that $\mathcal{I}_d$ is not P^+.

(3) Straightforward. □

There are known relationships between topological complexity and P-like properties.

Theorem 6.2 ([61, Proposition 4.9],[51, Lemma 1.2],[43, Theorem 3.7])**.**

(1) Each $G_{\delta\sigma\delta}$ (in particular, $F_{\sigma\delta}$) ideal is P^- (hence $P^-(\Lambda)$).
(2) Each F_σ ideal is P^+ (hence P^- and $P^-(\Lambda)$).
(3) If $\mathcal{I}$ is an analytic ideal, then the following conditions are equivalent.
  (a) There exists a P^+ ideal $\mathcal{J}$ with $\mathcal{I} \subseteq \mathcal{J}$.
  (b) There exists an F_σ ideal $\mathcal{K}$ with $\mathcal{I} \subseteq \mathcal{K}$.

6.2. **P-like properties of partition regular operations.**

Definition 6.3. Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular. For sets $F \in \mathcal{F}$ and $B \subseteq \Lambda$, we write $\rho(F) \subseteq^\rho B$ if there is a finite set $K \subseteq \Omega$ with $\rho(F \setminus K) \subseteq B$.

Remark. We want to stress here that the relation "$\rho(F) \subseteq^\rho B$" is in fact a relation between F and B and not between $\rho(F)$ and B because it can happen that $\rho(F) = \rho(G)$ and $\rho(F) \subseteq^\rho B$ but $\rho(G) \not\subseteq^\rho B$. We decided that we write $\rho(F) \subseteq^\rho B$ instead of $F \subseteq^\rho B$ as the former seems more natural for us. The same remark applies to other notions involving "$\rho(F)$" we defined earlier or we define later (e.g. Definitions 6.4 and 9.1).

The following properties will prove useful in the studies of classes of sequentially compact spaces defined with the aid of partition regular functions.

Definition 6.4. Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular. We say that ρ is

(1) $P^-(\Lambda)$ if for every $\subseteq$-decreasing sequence $A_n \in \mathcal{I}_\rho^+$ with $A_0 = \Lambda$ and $A_n \setminus A_{n+1} \in \mathcal{I}_\rho$ for each $n \in \omega$ there exists $F \in \mathcal{F}$ such that $\rho(F) \subseteq^\rho A_n$ for each $n \in \omega$;
(2) P^- if for every $\subseteq$-decreasing sequence $A_n \in \mathcal{I}_\rho^+$ with $A_n \setminus A_{n+1} \in \mathcal{I}_\rho$ for each $n \in \omega$ there exists $F \in \mathcal{F}$ such that $\rho(F) \subseteq^\rho A_n$ for each $n \in \omega$;
(3) P^+ if for every $\subseteq$-decreasing sequence $A_n \in \mathcal{I}_\rho^+$ there exists $F \in \mathcal{F}$ such that $\rho(F) \subseteq^\rho A_n$ for each $n \in \omega$;
(4) *weak* P^+ if for every $E \in \mathcal{F}$ there exists $F \in \mathcal{F}$ such that $\rho(F) \subseteq \rho(E)$ and for every sequence $\{F_n : n \in \omega\} \subseteq \mathcal{F}$ such that $\rho(F) \supseteq \rho(F_n) \supseteq \rho(F_{n+1})$ for each $n \in \omega$ there exists $G \in \mathcal{F}$ such that $\rho(G) \subseteq^\rho \rho(F_n)$ for each $n \in \omega$.

The following result reveals basic properties of the above defined notions and their connections with P-like properties of ideals.

Proposition 6.5. *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$. Let $\mathcal{I}$ be an ideal on Λ.*

(1) ρ is P^+ $\implies$ ρ is weak P^+ $\implies$ ρ is P^- $\implies$ ρ is $P^-(\Lambda)$.
(2) $\mathcal{I}$ is P^+ $\iff$ $\rho_\mathcal{I}$ is P^+, for every ideal $\mathcal{I}$. Similar equivalences hold for P^- and $P^-(\Lambda)$, resp.
(3) The implications from item (1) cannot be reversed.
(4) If $\mathcal{I}_\rho$ is $P^-(\Lambda)$ (P^-, P^+, resp.), then ρ is $P^-(\Lambda)$ (P^-, P^+, resp.).
(5) The implications from item (4) cannot be reversed in case of $P^-(\Lambda)$ and P^- properties.

Proof. (1) Below we only show that if ρ is weak P^+ then it is P^- since other implications are straightforward.

Let $A_n \in \mathcal{I}_\rho^+$ be a $\subseteq$-decreasing sequence with $A_n \setminus A_{n+1} \in \mathcal{I}_\rho$ for each $n \in \omega$. Since $A_0 \in \mathcal{I}_\rho^+$, there is $E \in \mathcal{F}$ such that $\rho(E) \subseteq A_0$. Using the fact that ρ is weak P^+ we can find $F \in \mathcal{F}$ with $\rho(F) \subseteq \rho(E)$ and such as in the definition of weak P^+ property.

We will show that there is a sequence $\{F_n : n \in \omega\} \subseteq \mathcal{F}$ such that $\rho(F_0) \subseteq \rho(F)$ and $\rho(F_{n+1}) \subseteq \rho(F_n) \cap A_{n+1}$ for each $n \in \omega$. Indeed, since $\rho(F) \subseteq A_0$, it suffices to put $F_0 = F$. Suppose now that F_i have been constructed for $i \leq n$. Since $\rho(F_n) \cap A_{n+1} = \rho(F_n) \setminus (A_n \setminus A_{n+1}) \in \mathcal{I}_\rho^+$, there is $F_{n+1} \in \mathcal{F}$ with $\rho(F_{n+1}) \subseteq \rho(F_n) \cap A_{n+1}$.

Since F is as in the definition of weak P^+ property, there exists $G \in \mathcal{F}$ such that $\rho(G) \subseteq^\rho \rho(F_n)$ for each $n \in \omega$. Thus, $\rho(G) \subseteq^\rho A_n$ for each $n \in \omega$.

(2) Straightforward.

(3) The cases of the second and third implications follow from Proposition 6.1(2) and item (2), where the proof of the fact that $\rho_{\mathcal{I}_d}$ is not weak P^+ is just a slight modification of the proof that $\mathcal{I}_d$ is not P^+.

Now we show that the first implication cannot be reversed. Consider the ideal $\mathcal{I} = \{A \subseteq \omega \times \omega : A \cap (\{n\} \times \omega) \text{ is finite for every } n \in \omega\}$. Then $\mathcal{I}$ is not P^+ as witnessed by $A_n = (\omega \setminus n) \times \omega$, so $\rho_\mathcal{I}$ is not P^+ (by item (2)). However, we will show that $\rho_{\mathcal{I}}$ is weak P^+. Let $E \in \mathcal{I}^+$. Then there is $n \in \omega$ such that $F = E \cap (\{n\} \times \omega)$ is infinite. Then $F \in \mathcal{I}^+$ and it is easy to see that if $F_n \in \mathcal{I}^+$ are such that $F \supseteq F_n \supseteq F_{n+1}$ then one can pick $x_n \in F_n$ for each $n \in \omega$ and $G = \{x_n : n \in \omega\} \in \mathcal{I}^+$ is such that $G \setminus \{x_i : i < n\} \subseteq F_n$ for all $n \in \omega$.

(4) Proofs in all cases are very similar, so we only present a proof for the property $P^-(\Lambda)$. Let $A_n \in \mathcal{I}_\rho^+$ be a $\subseteq$-decreasing sequence with $A_0 = \Lambda$ and $A_n \setminus A_{n+1} \in \mathcal{I}_\rho$ for each $n \in \omega$. Since $\mathcal{I}_\rho$ is $P^-(\Lambda)$, there is $B \notin \mathcal{I}_\rho$ such that for every $n \in \omega$ one can find a finite set $K_n \subseteq \Omega$ such that $B \setminus K_n \subseteq A_n$. From the fact that $B \notin \mathcal{I}_\rho$, there is $F \in \mathcal{F}$ such that $\rho(F) \subseteq B$. Using Proposition 3.2, we can find $E \in \mathcal{F}$ such that $E \subseteq F$ and for every K_n there exists a finite set $L_n \subseteq \Omega$ such that $\rho(E \setminus L_n) \subseteq \rho(E) \setminus K_n$. Then $\rho(E \setminus L_n) \subseteq \rho(E) \setminus K_n \subseteq B \setminus K_n \subseteq A_n$, so ρ is $P^-(\Lambda)$.

(5) In Proposition 6.7(3)(4) we will show that $\rho = \mathrm{FS}$ is weak P^+, but $\mathcal{I}_\rho = \mathcal{H}$ is not $P^-(\omega)$. $\square$

We will need the following lemma to show that FS, r and Δ are not P^+.

Lemma 6.6. *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ (with $\mathcal{F} \subseteq [\Omega]^\omega$) be a partition regular function such that there exists a function $\tau : [\Omega]^{<\omega} \to \Lambda$ such that*

(1) $\forall F \in \mathcal{F}\, \forall \{a,b\} \in [F]^2\, (\tau(\{a,b\}) \in \rho(F))$,
(2) $\forall F \in \mathcal{F}\, \forall c \in \rho(F)\, \exists S \in [F]^{<\omega}\, (\tau(S) = c)$,
(3) there exists a pairwise disjoint family $\{P_n : n \in \omega\} \subseteq \mathcal{F}$ such that the family $\{\rho(P_n) : n \in \omega\}$ is also pairwise disjoint and the restriction $\tau \restriction [\bigcup\{P_n : n \in \omega\}]^{<\omega}$ is one-to-one.

Then ρ is not P^+.

Proof. Let $\{P_n : n \in \omega\}$ be as in item (3) of the lemma. For each $n \in \omega$, we define $B_n = \bigcup\{\rho(P_i) : i \geq n\}$. Then $B_n \in \mathcal{I}_\rho^+$ and $B \supseteq B_n \supseteq B_{n+1}$ for each $n \in \omega$. If we show that there is no $G \in \mathcal{F}$ such that $\rho(G) \subseteq^\rho B_n$ for every $n \in \omega$, the proof will be finished. Suppose for sake of contradiction that there exists $G \in \mathcal{F}$ such that for every $n \in \omega$ there exists a finite set $K_n \subseteq \Omega$ with $\rho(G \setminus K_n) \subseteq B_n$. We have two cases:

(1) $|G \cap P_{n_0}| = \omega$ for some $n_0 \in \omega$,
(2) $|G \cap P_n| < \omega$ for all $n \in \omega$.

Case (1). We take distinct $a, b \in (G \cap P_{n_0}) \setminus K_{n_0+1}$. Since $a, b \in P_{n_0} \in \mathcal{F}$, we have $\tau(\{a,b\}) \in \rho(P_{n_0})$. On the other hand, $a, b \in G \setminus K_{n_0+1} \in \mathcal{F}$, so $\tau(\{a,b\}) \in \rho(G \setminus K_{n_0+1}) \subseteq B_{n_0+1}$. Hence, there exists $i \geq n_0 + 1$ such that $\tau(\{a,b\}) \in \rho(P_i)$. A contradiction with $\rho(P_i) \cap \rho(P_{n_0}) = \emptyset$.

Case (2). In this case, there exists a strictly increasing sequence $\{k_n : n \in \omega\}$ such that we can choose an element $x_{k_n} \in G \cap P_{k_n}$ for each $n \in \omega$. Since x_{k_n} are pairwise distinct, there is $N \in \omega$ such that $x_{k_n} \in G \setminus K_0$ for every $n \geq N$. In particular, $\tau\left(\{x_{k_N}, x_{k_{N+1}}\}\right) \in \rho(G \setminus K_0) \subseteq B_0$, and consequently there exists $i \in \omega$ such that $\tau\left(\{x_{k_N}, x_{k_{N+1}}\}\right) \in \rho(P_i)$. Therefore there is a finite set $S \subseteq P_i$ such that $\tau(S) = \tau\left(\{x_{k_N}, x_{k_{N+1}}\}\right)$. Since P_n are pairwise disjoint and $x_{k_n} \in P_n$, we obtain that $x_{k_N} \notin P_i$ or $h_{k_N+1} \notin P_i$. Consequently, $\{x_{k_N}, x_{k_{N+1}}\} \neq S$, so $\tau \restriction [\bigcup\{P_n : n \in \omega\}]^{<\omega}$ is not one-to-one, a contradiction. $\square$

Proposition 6.7.

(1) The ideals $\mathcal{W}$ and $\mathcal{I}_{1/n}$ are P^+ (hence, P^- and $P^-(\omega)$) while $\rho_{\mathcal{W}}$ and $\rho_{\mathcal{I}_{1/n}}$ are P^+, weak P^+, P^- and $P^-(\omega)$.
(2) If $\rho \in \{\mathrm{FS}, r, \Delta\}$, then ρ is not P^+,
(3) If $\rho \in \{\mathrm{FS}, r, \Delta\}$, then ρ is weak P^+ (hence P^- and $P^-(\Lambda)$).
(4) If $\mathcal{I} \in \{\mathcal{H}, \mathcal{R}, \mathcal{D}\}$, then $\mathcal{I}$ is not $P^-(\Lambda)$ (hence not P^+ and not P^-).

Proof. (1) It follows from Theorem 6.2(2) and the fact that $\mathcal{W}$ and $\mathcal{I}_{1/n}$ are F_σ ideals (see Proposition 5.2(1)). The "hence" part follows from Proposition 6.5.

(2) Below we show that ρ is not P^+ separately for each ρ.

Case of $\rho = FS$. We define a function $\tau : [\omega]^{<\omega} \to \omega$ by $\tau(S) = \sum_{i\in S} i$. Then we take an infinite sparse set P and a partition $\{P_n : n \in \omega\}$ of P into infinite sets. Lemma 6.6 shows that FS is not P^+.

Case of $\rho = r$. We define a function $\tau : [\omega]^{<\omega} \to [\omega]^2$ by $\tau(\{a, b\}) = \{a, b\}$ for distinct $a > b$ and $\tau(S) = \{0, 1\}$ otherwise. Then we take a partition $\{P_n : n \in \omega\}$ of ω into infinite sets. Lemma 6.6 shows that r is not P^+.

Case of $\rho = \Delta$. We define a function $\tau : [\omega]^{<\omega} \to \omega$ by $\tau(\{a, b\}) = a - b$ for distinct $a > b$ and $\tau(S) = 0$ otherwise. Then we take an infinite $\mathcal{D}$-sparse set P and a partition $\{P_n : n \in \omega\}$ of P into infinite sets. Lemma 6.6 shows that Δ is not P^+.

(3) The "hence" part follows from Proposition 6.5(1). Below we show that ρ is weak P^+ separately for each ρ.

Case of $\rho =$ FS. It is proved in [22, Lemma 2.3] (see also [18, Example 2.9(2)]).

Case of $\rho = r$. For any $E \in [\omega]^\omega$ we take $F = E$. Let $F_n \in [\omega]^\omega$ be such that $[F]^2 \supseteq [F_n]^2 \supseteq [F_{n+1}]^2$ for each $n \in \omega$. We pick $x_n \in F_n \setminus \{x_i : i < n\}$ for each $n \in \omega$. Then $G = \{x_n : n \in \omega\} \in [\omega]^\omega$ and $[G]^2 \subseteq^r [F_n]^2$ for each $n \in \omega$.

Case of $\rho = \Delta$. Fix any $F \in [\omega]^\omega$. Inductively pick a sequence $(x_i)_{i\in\omega} \subseteq \omega$ such that $x_i \in F$, $x_i < x_{i+1}$ and $x_{i+1} - x_i > x_i - x_0$ for all $i \in \omega$. Let $E = \{x_i : i \in \omega\} \in [F]^\omega$.

Define $a_i = x_{i+1} - x_i$ for all $i \in \omega$ and observe that $a_i = x_{i+1} - x_i > x_i - x_0 = \sum_{j<i} a_j$. Put $A = \{a_i : i \in \omega\}$. By [22, proof of Lemma 2.2] the set A is very sparse, i.e. A is sparse and if $\alpha_A(x) \cap \alpha_A(y) \neq \emptyset$ then $x + y \notin \mathrm{FS}(A)$. Note that $\Delta(E) = \{\sum_{i\in I} a_i : I \text{ is a finite interval in } \omega\} \subseteq \mathrm{FS}(A)$.

Observe that if $\Delta(\{y_n : n \in \omega\}) \subseteq \Delta(E)$, where $y_n < y_{n+1}$ for all $n \in \omega$, then there is a partition of ω into finite intervals $(I_n)_{n\in\omega}$ such that $\max I_n < \min I_{n+1}$ and $y_{n+1} - y_n = \sum_{i\in I_n} a_i$. Indeed, as $y_{n+1} - y_n \in \Delta(\{y_n : n \in \omega\}) \subseteq \Delta(E) \subseteq \mathrm{FS}(A)$, for each $n \in \omega$ there is a finite interval I_n such that $y_{n+1} - y_n = \sum_{i\in I_n} a_i$ (because A is sparse, we get $I_n = \alpha_A(y_{n+1} - y_n)$). We need to show that the intervals I_n are pairwise disjoint and cover ω. Suppose first that $\sup I_n + 1 < \inf I_{n+1}$ for some $n \in \omega$. Then $y_{n+2} - y_n = (y_{n+2} - y_{n+1}) + (y_{n+1} - y_n) = \sum_{i\in I_n \cup I_{n+1}} a_i$. On the other hand, $y_{n+2} - y_n \in \Delta(\{y_n : n \in \omega\}) \subseteq \Delta(E)$, so $y_{n+2} - y_n = \sum_{i\in I} a_i$ for some interval I. This contradicts uniqueness of $\alpha_A(y_{n+2} - y_n)$ (because A is sparse). Suppose now that $I_n \cap I_{n+1} \neq \emptyset$. Then $y_{n+2} - y_n = (y_{n+2} - y_{n+1}) + (y_{n+1} - y_n) \notin \mathrm{FS}(A)$ (because A is very sparse), which contradicts $\Delta(\{y_n : n \in \omega\}) \subseteq \Delta(E) \subseteq FS(A)$.

Fix any sequence $(F_k)_{k\in\omega} \subseteq [\omega]^\omega$ such that $\Delta(F_{k+1}) \subseteq \Delta(F_k) \subseteq \Delta(E)$ for all $k \in \omega$. By the previous paragraph, with each $k \in \omega$ we can associate a partition of ω into finite intervals I^k_n, i.e., $\Delta(F_k) = \{\sum_{i\in I} a_i : I = I^k_j \cup I^k_{j+1} \cup \ldots \cup I^k_{j'} \text{ for some } j < j'\}$.

Observe that actually for each $n, k \in \omega$ we have that $I^{k+1}_n = \bigcup_{i\in I} I^k_i$ for some interval I. Indeed, otherwise for some $n, k \in \omega$ we would have $x = \sum_{i\in I^{k+1}_n} a_i \in \Delta(F_{k+1}) \subseteq \Delta(F_k)$, so $x = \sum_{i\in I} a_i$ for some $I = I^k_j \cup I^k_{j+1} \cup \ldots \cup I^k_{j'}$, which contradicts that A is sparse.

Inductively pick a sequence $(n_k)_{k\in\omega} \subseteq \omega$ such that for each $k \in \omega$ we have $n_{k+1} > n_k$ (so also $a_{n_{k+1}} > a_{n_k}$) and $n_k = \min I^k_j$ for some $j \in \omega$. Define $E' = \{x_{n_k} : k \in \omega\}$. Notice that $x_{n_{k+1}} - x_{n_k} = \sum_{i\in[n_k, n_{k+1})} a_i$. Then for each $k \in \omega$ we have $\Delta(E' \setminus [0, x_{n_k})) = \Delta(\{x_{n_i} : i \geq k\}) \subseteq \{\sum_{i\in I} a_i : I = I^k_j \cup I^k_{j+1} \cup \ldots \cup I^k_{j'} \text{ for some } j < j'\} = \Delta(F_k)$.

(4) The "hence" part follows from Proposition 6.1. Below we show that $\mathcal{I}$ is not $P^-(\Lambda)$ separately for each $\mathcal{I}$.

Case of $\mathcal{I} = \mathcal{H}$. Let $A_k = \{2^k(2n+1) : n \in \omega\}$ for each $k \in \omega$. In [22, item (2) in the proof of Proposition 1.1], the authors showed that $A_k \in \mathcal{H}$ for every $k \in \omega$, whereas in [22, item (1) in the proof of Proposition 1.1] it is shown that for every $B \notin \mathcal{H}$ there is $k \in \omega$ such that $B \cap A_k$ is infinite. Thus, the family $\{A_k : k \in \omega\}$ witnesses the fact that $\mathcal{H}$ is not $P^-(\omega)$.

Case of $\mathcal{I} = \mathcal{R}$. Let $A_n = \{\{k, i\} : i > k \geq n\}$ for every $n \in \omega$. Then $A_n \notin \mathcal{R}$, $A_0 = [\omega]^2$ and $A_n \setminus A_{n+1} = \{\{n, i\} : i > n\} \in \mathcal{R}$. Suppose, for sake of contradiction, that there is $B \notin \mathcal{R}$ such that $B \subseteq^* A_n$ for every $n \in \omega$. Let $H = \{h_n : n \in \omega\}$ be an infinite set such that $[H]^2 \subseteq B$ and $h_n < h_{n+1}$ for every $n \in \omega$. Since $[H]^2 \subseteq^* A_{h_1}$, there is a finite set F such that $[H]^2 \setminus F \subseteq A_{h_1}$. Since F is finite, there is $k > 0$ such that $\{h_0, h_n\} \notin F$ for every $n \geq k$. Then $\{\{h_0, h_n\} : n \geq k\} \subseteq [H]^2 \setminus F$ and $\{\{h_0, h_n\} : n \geq k\} \cap A_{h_1} = \emptyset$, a contradiction.

Case of $\mathcal{I} = \mathcal{D}$. Let $A_k = \{2^k(2n+1) : n \in \omega\}$ for each $k \in \omega$. In [58, item (2) in the proof of Theorem 2.1], the author showed that $A_k \in \mathcal{D}$ for every $k \in \omega$, whereas in [58, item (1) in the proof of Theorem 2.1] it is shown that for every $B \notin \mathcal{D}$ there is $k \in \omega$ such that $B \cap A_k$ is infinite. Thus, the family $\{A_k : k \in \omega\}$ witnesses the fact that $\mathcal{D}$ is not $P^-(\omega)$. $\square$

The following easy observation will be useful in our considerations.

Proposition 6.8. *If $\rho : \mathcal{F} \to [\Lambda]^\omega$ is partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$, then the following conditions are equivalent.*

(1) ρ is P^- ($P^-(\Lambda)$, resp.).

(2) For every countable family $\mathcal{B} \subseteq \mathcal{I}_\rho$ with $\bigcup \mathcal{B} \notin \mathcal{I}_\rho$ ($\bigcup \mathcal{B} = \Lambda$, resp.) there exists $F \in \mathcal{F}$ such that $\rho(F) \subseteq \bigcup \mathcal{B}$ and for every finite subfamily $\mathcal{C} \subseteq \mathcal{B}$ there is a finite $K \subseteq \Omega$ such that $\rho(F \setminus K) \cap \bigcup \mathcal{C} = \emptyset$.

Proof. We will assume that ρ is P^-, as the proof in the case of $P^-(\Lambda)$ is similar.

(1) $\implies$ (2). Let $\mathcal{B} = \{B_n : n \in \omega\}$, where $\bigcup \mathcal{B} \notin \mathcal{I}_\rho$ and $B_n \in \mathcal{I}_\rho$ for every $n \in \omega$. For each $n \in \omega$, we define $A_n = \bigcup \mathcal{B} \setminus \bigcup \{B_i : i < n\}$. Since ρ is P^-, there exists $F \in \mathcal{F}$ such that $\rho(F) \subseteq^\rho A_n$ for each $n \in \omega$. Let $\mathcal{C} \subseteq \mathcal{B}$ be a finite subfamily. Let $n \in \omega$ be such that $\mathcal{C} \subseteq \{B_i : i < n\}$. Then $\bigcup \mathcal{C} \subseteq \bigcup \{B_i : i < n\}$. Let $K \subseteq \Omega$ be a finite set such that $\rho(F \setminus K) \subseteq A_n$. Then $\rho(F \setminus K) \cap \bigcup \{B_i : i < n\} = \emptyset$, so $\rho(F \setminus K) \cap \bigcup \mathcal{C} = \emptyset$.

(2) $\implies$ (1). Let $A_n \in \mathcal{I}_\rho^+$ be such that $A_n \supseteq A_{n+1}$ and $A_n \setminus A_{n+1} \in \mathcal{I}_\rho$ for each $n \in \omega$. For each $n \in \omega$ we define $B_n = A_n \setminus A_{n+1}$. Let $\mathcal{B} = \{B_n : n \in \omega\}$. Then there exists $F \in \mathcal{F}$ such that $\rho(F) \subseteq \bigcup \mathcal{B}$ and for every finite subfamily $\mathcal{C} \subseteq \mathcal{B}$ there is a finite $K \subseteq \Omega$ such that $\rho(F \setminus K) \cap \bigcup \mathcal{C} = \emptyset$. Thus for any $n \in \omega$, we find a finite set $K \subseteq \Omega$ such that $\rho(F \setminus K) \cap \bigcup \{B_i : i < n\} = \emptyset$. Hence $\rho(F \setminus K) \subseteq \bigcup \{B_i : i \geq n\} = A_n$, so $\rho(F) \subseteq^\rho A_n$. $\square$

7. Katětov order

7.1. **Katětov order between ideals.** We say that an *ideal $\mathcal{I}_1$ on Λ_1 is above an ideal $\mathcal{I}_2$ on Λ_2 in the Katětov order* (in short: $\mathcal{I}_2 \leq_K \mathcal{I}_1$) [53] if there exists a function $\phi : \Lambda_1 \to \Lambda_2$ such that $\phi[A] \notin \mathcal{I}_2$ for each $A \notin \mathcal{I}_1$. If $\Lambda_1 = \Lambda_2$ and $\mathcal{I}_2 \subseteq \mathcal{I}_1$, then obviously the identity function on Λ_1 witnesses that $\mathcal{I}_2 \leq_K \mathcal{I}_1$.

There are known relationships between Katětov order, P-like properties and topological complexity.

Proposition 7.1 ([43, Theorem 3.8]). *Let $\mathcal{I}$ be an ideal on Λ.*

(1) $\mathcal{I}$ is $P^-(\Lambda)$ $\iff$ $\mathrm{Fin}^2 \not\leq_K \mathcal{I}$.

(2) $\mathcal{I}$ is P^- $\iff$ $\mathrm{Fin}^2 \not\leq_K \mathcal{I} \restriction A$ for every $A \in \mathcal{I}^+$.

Proposition 7.2.

(1) $\mathrm{Fin}^2 \leq_K \mathcal{I}$ *for* $\mathcal{I} \in \{\mathcal{D}, \mathcal{H}, \mathcal{R}\}$.

(2) If $\mathcal{I}$ *is an* $G_{\delta\sigma\delta}$ *ideal, then* $\mathrm{Fin}^2 \not\leq_K \mathcal{I} \restriction A$ *for every* $A \in \mathcal{I}^+$ *In particular,* $\mathrm{Fin}^2 \not\leq_K \mathcal{W}$ *and* $\mathrm{Fin}^2 \not\leq_K \mathcal{I}_{1/n}$.

Proof. (1) Using Proposition 7.1(1), we need to show that $\mathcal{D}$, $\mathcal{H}$ and $\mathcal{R}$ are not $P^-(\Lambda)$ ideals, but this follows from Proposition 6.7(4). (For $\mathcal{I} = \mathcal{R}$, this item was earlier proved by Meza-Alcántara [65, Lemma 1.6.25].)

(2) It follows from Theorem 6.2(1) and Propositions 7.1(2) and 6.7(1). □

7.2. **Katětov order between partition regular operations.** The following notion will be crucial for showing when a class of sequentially compact spaces defined by ρ_1 is contained in a class of sequentially compact spaces defined by ρ_2.

Definition 7.3. Let $\rho_i : \mathcal{F}_i \to [\Lambda_i]^\omega$ be partition regular (with $\mathcal{F}_i \subseteq [\Omega_i]^\omega$) for each $i = 1, 2$. We say that ρ_1 *is above* ρ_2 *in the Katětov order* (in short: $\rho_2 \leq_K \rho_1$) if there is a function $\phi : \Lambda_1 \to \Lambda_2$ such that

$$\forall F_1 \in \mathcal{F}_1 \, \exists F_2 \in \mathcal{F}_2 \, \forall K_1 \in [\Omega_1]^{<\omega} \, \exists K_2 \in [\Omega_2]^{<\omega} \, (\rho_2(F_2 \setminus K_2) \subseteq \phi[\rho_1(F_1 \setminus K_1)])$$

or equivalently:

$$\forall F_1 \in \mathcal{F}_1 \, \exists F_2 \in \mathcal{F}_2 \, \forall K_1 \in [\Omega_1]^{<\omega} \, (\rho_2(F_2) \subseteq^{\rho_2} \phi[\rho_1(F_1 \setminus K_1)]).$$

The following proposition reveals some basic properties of this new order on partition regular functions.

Proposition 7.4.

(1) The relation $\leq_K$ *is a preorder (a.k.a. quasi order) i.e. it is reflexive and transitive.*

(2) The preorder $\leq_K$ *is upward and downward directed.*

(3) Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ *(with* $\mathcal{F} \subseteq [\Omega]^\omega$*) be partition regular.*

- *(a)* $\rho \leq_K \rho \restriction \rho(F)$ *for every* $F \in \mathcal{F}$.
- *(b)* $\rho_{\mathrm{Fin}(\Lambda)} \leq_K \rho$.

Proof. (1) Reflexivity of $\leq_K$ is obvious. To show transitivity, fix $\mathcal{F}_i \subseteq [\Omega_i]^\omega$ and $\rho_i : \mathcal{F}_i \to [\Lambda_i]^\omega$, $i = 1, 2, 3$, and suppose that $\rho_1 \leq_K \rho_2$ is witnessed by f and $\rho_2 \leq_K \rho_3$ is witnessed by g. We claim that $\rho_1 \leq_K \rho_3$ is witnessed by $h : \Lambda_3 \to \Lambda_1$ given by $h(x) = f(g(x))$ for all $x \in \Lambda_3$. Let $F_3 \in \mathcal{F}_3$. Then we can find $F_2 \in \mathcal{F}_2$ such that for every $K \in [\Omega_3]^{<\omega}$ there exists $L_K \in [\Omega_2]^{<\omega}$ such that $\rho_2(F_2 \setminus L_K) \subseteq g\left[\rho_3(F_3 \setminus K)\right]$. Then for F_2 we can find $F_1 \in \mathcal{F}_1$ such that for every $L \in [\Omega_2]^{<\omega}$ there exists $M_L \in [\Omega_1]^{<\omega}$ such that $\rho_1(F_1 \setminus M_L) \subseteq f\left[\rho_2(F_2 \setminus L)\right]$. Now for a given $K \in [\Omega_3]^{<\omega}$ we have $\rho_1(F_1 \setminus M_{L_K}) \subseteq f\left[\rho_2(F_2 \setminus L_K)\right] \subseteq f\left[g\left[\rho_3(F_3 \setminus K)\right]\right] = h\left[\rho_3(F_3 \setminus K)\right]$, so the proof is finished.

(2) Let $\rho_i : \mathcal{F}_i \to [\Lambda_i]^\omega$ with $\mathcal{F}_i \subseteq [\Omega_i]^\omega$ be partition regular for $i = 0, 1$. We define the following partition regular functions $\pi : \{F_0 \times F_1 : F_0 \in \mathcal{F}_0, F_1 \in \mathcal{F}_1\} \to [\Lambda_0 \times \Lambda_1]^\omega$ by $\pi(F_0 \times F_1) = \rho_0(F_0) \times \rho_1(F_1)$ and $\sigma : \mathcal{F}_0 \oplus \mathcal{F}_1 \to [\Lambda_0 \oplus \Lambda_1]^\omega$ by $\sigma((F_0 \times \{0\}) \cup (F_1 \times \{1\})) = (\rho_0(F_0) \times \{0\}) \cup (\rho_1(F_1) \times \{1\})$.

Then $\sigma \leq_K \rho_i$ $(i = 0, 1)$ is witnessed by a function $\phi_i : \Lambda_i \to \Lambda_0 \oplus \Lambda_1$ given by $\phi_i(x) = (x, i)$, whereas $\rho_i \leq_K \pi$ $(i = 0, 1)$ is witness by a function $\psi_i : \Lambda_0 \times \Lambda_1 \to \Lambda_i$ given by $\psi_i(x_0, x_1) = x_i$.

(3a) Let $F \in \mathcal{F}$. We claim that $\phi : \rho(F) \to \Lambda$ given by $\phi(\lambda) = \lambda$ is a witness for $\rho \leq_K \rho \restriction \rho(F)$. Let $F_1 \in \mathcal{F} \restriction \rho(F)$. Then $F_2 = F_1$ is such that for every finite set $K_1 \subseteq \Omega$ we take $K_2 = K_1$ and see that $\rho(F_2 \setminus K_2) \subseteq \phi(\rho(F_1 \setminus K_1))$.

(3b) We claim that $\phi : \Lambda \to \Lambda$ given by $\phi(\lambda) = \lambda$ is a witness for $\rho_{\mathrm{Fin}(\Lambda)} \leq_K \rho$. Let $F \in \mathcal{F}$. Let $\Omega = \{o_n : n \in \omega\}$. Since $\rho(F \setminus \{o_i : i < n\})$ is infinite for every $n \in \omega$, we can pick a one-to-one sequence $(a_n : n \in \omega)$ such that $a_n \in \rho(F \setminus \{o_i :$

$i < n\}$) for every $n \in \omega$. Then $A = \{a_n : n \in \omega\} \in \mathrm{Fin}(\Lambda)^+$ is an infinite set. For a finite set $K \subseteq \Omega$ there is $n \in \omega$ such that $K \subseteq \{o_i : i < n\}$. Then $L = \{a_i : i < n\}$ is finite subset of Λ and $A \setminus L \subseteq \rho(F \setminus \{o_i : i < n\}) \subseteq \rho(F \setminus K)$. □

Now we compare the relation $\leq_K$ between partition regular operations with the relation $\leq_K$ between ideals.

Proposition 7.5. *Let $\rho_i : \mathcal{F}_i \to [\Lambda_i]^\omega$ for each $i = 1, 2$ and $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular. Let $\mathcal{I}$ be ideal.*

(1) $\rho_2 \leq_K \rho_1 \implies \mathcal{I}_{\rho_2} \leq_K \mathcal{I}_{\rho_1}$ with the same witnessing function.
(2) (a) If ρ_2 is P^+ (in particular, if $\rho_2 = \rho_{\mathcal{I}}$ and $\mathcal{I}$ is P^+), then $\rho_2 \leq_K \rho_1 \iff \mathcal{I}_{\rho_2} \leq_K \mathcal{I}_{\rho_1}$;
(b) $\rho \leq_K \rho_{\mathcal{I}} \iff \mathcal{I}_\rho \leq_K \mathcal{I}$;

Proof. (1) Let $\mathcal{F}_i \subseteq [\Omega_i]^\omega$ for $i = 1, 2$. Let ϕ be a witness for $\rho_2 \leq_K \rho_1$. We claim that ϕ is also a witness for $\mathcal{I}_{\rho_2} \leq_K \mathcal{I}_{\rho_1}$. Let $A \notin \mathcal{I}_{\rho_1}$. Then there is $F_1 \in \mathcal{F}_1$ with $\rho_1(F_1) \subseteq A$. Since $\rho_2 \leq_K \rho_1$, there is $F_2 \in \mathcal{F}_2$ and a finite set $K_2 \subseteq \Omega_2$ such that $\rho_2(F_2 \setminus K_2) \subseteq \phi[\rho_1(F_1 \setminus \emptyset)] = \phi[\rho_1(F_1)]$. Since $F_2 \setminus K_2 \in \mathcal{F}_2$ and $\rho_2(F_2 \setminus K_2) \subseteq \phi[A]$, we obtain that $\phi[A] \notin \mathcal{I}_{\rho_2}$. Thus the proof of this item is finished.

(2a) The "in particular" part follows from Propositions 3.3(2) and 6.5(2).

We only have to show the implication " $\Longleftarrow$ ", because the reversed implication is true by item (1). Let $\phi : \Lambda_1 \to \Lambda_2$ be a witness for $\mathcal{I}_{\rho_2} \leq_K \mathcal{I}_{\rho_1}$. We claim that ϕ is also a witness for $\rho_2 \leq_K \rho_1$. Let $F_1 \in \mathcal{F}_1$, $\Omega_1 = \{o_n : n \in \omega\}$ and $B_n = \phi[\rho_1(F_1 \setminus \{o_i : i < n\})]$ for each $n \in \omega$. Then $B_n \notin \mathcal{I}_{\rho_2}$, $B_n \supseteq B_{n+1}$ for each $n \in \omega$, and since $\mathcal{I}_{\rho_2}$ is P^+, there is $F_2 \in \mathcal{F}_2$ such that for each $n \in \omega$ there is a finite set $L_n \subseteq \Omega_2$ with $\rho_2(F_2 \setminus L_n) \subseteq B_n$. Now, for any finite set $K_1 \subseteq \Omega_1$ there is $n \in \omega$ such that $K_1 \subseteq \{o_i : i < n\}$. Let $K_2 = L_n$. Then $\rho_2(F_2 \setminus K_2) \subseteq B_n \subseteq \phi[\rho_1(F_1 \setminus K_1)]$. Thus the proof of this item is finished.

(2b) The implication " $\implies$ " follows from item (1) and Proposition 3.3(2), so below we show the reverse implication.

Suppose that $\mathcal{I}$ is an ideal on Λ and $\mathcal{F} \subseteq [\Omega]^\omega$. Let $\phi : \Lambda \to \Lambda$ be a witness of $\mathcal{I}_\rho \leq_K \mathcal{I}$. We claim that the same ϕ is also a witness for $\rho \leq_K \rho_{\mathcal{I}}$. Indeed, for $A \notin \mathcal{I}$ we find $E \in \mathcal{F}$ such that $\rho(E) \subseteq \phi[A]$. Using Proposition 3.2, we can find a set $F \in \mathcal{F}$ such that $F \subseteq E$ and for any finite set $K \subseteq \Lambda_1$ there exists a finite set $L \subseteq \Omega$ with $\rho(F \setminus L) \subseteq \rho(F) \setminus \phi[K]$. Consequently, $\rho(F \setminus L) \subseteq \phi[A] \setminus \phi[K] \subseteq \phi[A \setminus K]$. □

The following example shows that in general $\rho_2 \leq_K \rho_1$ and $\mathcal{I}_{\rho_2} \leq_K \mathcal{I}_{\rho_1}$ are not equivalent.

Example 7.6. $\mathrm{Fin}^2 \leq_K \mathcal{H}$, but $\rho_{\mathrm{Fin}^2} \not\leq_K \mathrm{FS}$.

Proof. By Proposition 7.2(1) we know that $\mathcal{I}_{\rho_{\mathrm{Fin}^2}} = \mathrm{Fin}^2 \leq_K \mathcal{H} = \mathcal{I}_{\mathrm{FS}}$. Thus, we only need to show that $\rho_{\mathrm{Fin}^2} \not\leq_K \mathrm{FS}$.

Suppose that $\rho_{\mathrm{Fin}^2} \leq_K \mathrm{FS}$ and let $\phi : \omega \to \omega^2$ be a witness for this. For each $n \in \omega$, we define $A_n = \phi^{-1}[(\omega \setminus n) \times \omega]$. Then $A_0 = \omega$, $A_n \supseteq A_{n+1}$ and $A_n \setminus A_{n+1} \subseteq \phi^{-1}[\{n\} \times \omega] \in \mathcal{H}$ for each $n \in \omega$ by Proposition 7.5(1). Since FS is $P^-(\omega)$ by Proposition 6.7(3), there is $F \in [\omega]^\omega$ such that for every $n \in \omega$ there is a finite set $K_n \subseteq \omega$ with $\mathrm{FS}(F \setminus K_n) \subseteq A_n$. Now, using the fact that $\rho_{\mathrm{Fin}^2} \leq_K \mathrm{FS}$, we find $B \notin \mathrm{Fin}^2$ such that for every $n \in \omega$ there is a finite set $L_n \subseteq \omega^2$ with $B \setminus L_n \subseteq \phi[\mathrm{FS}(F \setminus K_n)] \subseteq \phi[A_n] \subseteq (\omega \setminus n) \times \omega$. In particular, sets $B \cap (\{n\} \times \omega)$ are finite for every n, so $B \in \mathrm{Fin}^2$, a contradiction. □

Remark. The partition regular function ρ_{Fin^2} from Example 7.6 is not P^-. In Example 15.5 we will show that there are partition regular functions ρ_1 and ρ_2 which are P^- and have small accretions such that $\mathcal{I}_{\rho_2} \subseteq \mathcal{I}_{\rho_1}$ (in particular, $\mathcal{I}_{\rho_2} \leq_K \mathcal{I}_{\rho_1}$), but $\rho_2 \not\leq_K \rho_1$.

7.3. Katětov order between $\mathrm{FS}, r, \Delta, \mathcal{W}$ and $\mathcal{I}_{1/n}$.

Theorem 7.7.

(1) $\mathcal{H} \not\leq_K \mathcal{R}$. *In particular,* $\mathrm{FS} \not\leq_K r$.
(2) $\mathcal{R} \not\leq_K \mathcal{H}$. *In particular,* $r \not\leq_K \mathrm{FS}$.
(3) $\Delta \leq_K \mathrm{FS}$ *and* $\mathcal{D} \subseteq \mathcal{H}$. *In particular,* $\mathcal{D} \leq_K \mathcal{H}$.
(4) $\Delta \leq_K r$. *In particular,* $\mathcal{D} \leq_K \mathcal{R}$.
(5) $\mathcal{R} \not\leq_K \mathcal{D}$. *In particular,* $r \not\leq_K \Delta$.
(6) $\mathcal{H} \not\leq_K \mathcal{D}$. *In particular,* $\mathrm{FS} \not\leq_K \Delta$.
(7) $\mathcal{I}_{1/n} \not\leq_K \mathcal{R}$. *In particular,* $\rho_{\mathcal{I}_{1/n}} \not\leq_K r$.
(8) $\mathcal{I}_{1/n} \not\leq_K \mathcal{H}$. *In particular,* $\rho_{\mathcal{I}_{1/n}} \not\leq_K \mathrm{FS}$.
(9) $\mathcal{I}_{1/n} \not\leq_K \mathcal{D}$. *In particular,* $\rho_{\mathcal{I}_{1/n}} \not\leq_K \Delta$.
(10) $\mathcal{I}_{1/n} \not\leq_K \mathcal{W}$. *In particular,* $\rho_{\mathcal{I}_{1/n}} \not\leq_K \rho_{\mathcal{W}}$.
(11) $\mathcal{D} \not\leq_K \mathcal{I}_{1/n}$. *In particular,* $\Delta \not\leq_K \rho_{\mathcal{I}_{1/n}}$.
(12) $\mathcal{H} \not\leq_K \mathcal{I}_{1/n}$. *In particular,* $\mathrm{FS} \not\leq_K \rho_{\mathcal{I}_{1/n}}$.
(13) $\mathcal{R} \not\leq_K \mathcal{I}_{1/n}$. *In particular,* $r \not\leq_K \rho_{\mathcal{I}_{1/n}}$.
(14) $\mathcal{D} \not\leq_K \mathcal{W}$. *In particular,* $\Delta \not\leq_K \rho_{\mathcal{W}}$.
(15) $\mathcal{H} \not\leq_K \mathcal{W}$. *In particular,* $\mathrm{FS} \not\leq_K \rho_{\mathcal{W}}$.
(16) $\mathcal{R} \not\leq_K \mathcal{W}$. *In particular,* $r \not\leq_K \rho_{\mathcal{W}}$.

Proof. The "in particular" parts follow from Proposition 7.5(1).

The proofs of items (1), (2), (7), (8) and (10) can be found in [21].

(3) The inclusion is proved in [71, Proposition 4.2.1] (see also [20, Propositions 4.2]). Below, we show that $\Delta \leq_K \mathrm{FS}$.

We claim that the identity function $\phi : \omega \to \omega$, $\phi(n) = n$ for every $n \in \omega$ is a witness for $\Delta \leq_K \mathrm{FS}$.

For any infinite set $A \subseteq \omega$, we define an infinite set $B = \{\sum_{i \leq n} a_i : n \in \omega\}$, where $\{a_n : n \in \omega\}$ is the increasing enumeration of A. Next, for any finite set K, we define a finite set $L = \{0, 1, \ldots, \sum_{i \leq k} a_i\}$, where $k = \max\{i \in \omega : a_i \in K\}$ (for $K = \emptyset$ we take $k = 0$). Finally, we observe that $\Delta(B \setminus L) \subseteq \mathrm{FS}(A \setminus K) = \phi[\mathrm{FS}(A \setminus K)]$, so the proof is finished.

(4) We claim that $\phi : [\omega]^2 \to \omega$ given by the formula $\phi(\{n, k\}) = n - k$, where $n > k$, is a witness for $\Delta \leq_K r$. For any infinite set $A \subseteq \omega$, we take $B = A$. Then for any finite set $K \subseteq \omega$, we take $L = K$. Next, we notice that $\Delta(B \setminus L) = \Delta(A \setminus K) = \phi[[A \setminus K]^2] = \phi[r(A \setminus K)]$, so the proof is finished.

(5) It follows from items (3) and (2).

(6) It follows from items (4) and (1).

(9) It follows from items (8) and (3).

(11) Suppose otherwise: $\mathcal{D} \leq_K \mathcal{I}_{1/n}$. By Proposition 7.2(1) $\mathrm{Fin}^2 \leq_K \mathcal{D}$, so $\mathrm{Fin}^2 \leq_K \mathcal{I}_{1/n}$. By Proposition 7.1(1), we obtain that $\mathcal{I}_{1/n}$ is not a $P^-(\omega)$ ideal, a contradiction with Proposition 6.7(1).

(12) It follows from items (3) and (11).

(13) It follows from items (4) and (11).

(14) Suppose otherwise: $\mathcal{D} \leq_K \mathcal{W}$. Using Proposition 7.2(1) we get that $\mathrm{Fin}^2 \leq_K \mathcal{D}$, so $\mathrm{Fin}^2 \leq_K \mathcal{W}$. However, since $\mathcal{W}$ is F_σ (see [23, Example 4.12]), $\mathrm{Fin}^2 \not\leq_K \mathcal{W}$ by [13, Theorems 7.5 and 9.1]. A contradiction.

(15) It follows from items (3) and (14).

(16) It follows from items (3) and (14).

$\square$

Question 7.8. Is $\mathcal{W} \leq_K \mathcal{I}$ for $\mathcal{I} \in \{\mathcal{I}_{1/n}, \mathcal{H}, \mathcal{R}, \mathcal{D}\}$?

Remark. The positive answer to Question 7.8 for $\mathcal{I} = \mathcal{I}_{1/n}$ is implied by the inclusion $\mathcal{W} \subseteq \mathcal{I}_{1/n}$ that is known as the Erdős conjecture on arithmetic progressions (a.k.a. the Erdős–Turán conjecture) which can be rephrased in the following manner: if the sum of the reciprocals of the elements of a set $A \subseteq \omega$ diverges, then A contains arbitrarily long finite arithmetic progressions.

8. Tallness and homogeneity

8.1. **Tallness of partition regular functions.** An ideal $\mathcal{I}$ on Λ is *tall* if for every infinite set $A \subseteq \Lambda$ there exists an infinite set $B \subseteq A$ such that $B \in \mathcal{I}$ ([62, p. 210], see also [63, Definition 0.6]). It is not difficult to see that $\mathcal{I}$ is not tall $\iff \mathcal{I} \leq_K \mathcal{J}$ for every ideal $\mathcal{J} \iff \mathcal{I} \leq_K \mathrm{Fin} \iff \mathcal{I} \restriction A = \mathrm{Fin}(A)$ for some $A \in \mathcal{I}^+$.

The following proposition characterizes tallness of the ideal $\mathcal{I}_\rho$ in terms of ρ and serves as a definition of tallness of partition regular functions.

Proposition 8.1. *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular (with $\mathcal{F} \subseteq [\Omega]^\omega$). The following conditions are equivalent.*

(1) $\mathcal{I}_\rho$ is tall.
(2) There exists a partition regular function τ such that $\rho \not\leq_K \tau$.
(3) $\mathcal{I}_\rho \restriction \rho(F) \neq \mathrm{Fin}(\rho(F))$ for every $F \in \mathcal{F}$.
(4) $\rho \not\leq_K \rho_{\mathrm{Fin}(\Lambda)}$.

Proof. (1) $\implies$ (2) If $\mathcal{I}_\rho$ is tall, there is an ideal $\mathcal{J}$ such that $\mathcal{I}_\rho \not\leq_K \mathcal{J}$. Then $\rho \not\leq_K \rho_{\mathcal{J}}$ by Proposition 7.5(1).

(2) $\implies$ (3) Suppose that there is $F \in \mathcal{F}$ such that $\mathcal{I}_\rho \restriction \rho(F) = \mathrm{Fin}(\rho(F))$. We will show that $\rho \leq_K \tau$ for every partition regular function τ .

Take any partition regular function $\tau : \mathcal{G} \to [\Sigma]^\omega$ with $\mathcal{G} \subseteq [\Gamma]^\omega$.

Let $\phi : \Sigma \to \Lambda$ be a one-to-one function such that $\phi[\Sigma] = \rho(F)$. We claim that ϕ is a witness for $\rho \leq_K \tau$.

Let $G \in \mathcal{G}$ and $\Gamma = \{\gamma_n : n \in \omega\}$. Since $\phi[\tau(G \setminus \{\gamma_i : i < n\})]$ is infinite for every $n \in \omega$, we can pick a one-to-one sequence $(b_n : n \in \omega)$ such that $b_n \in \phi[\tau(G \setminus \{\gamma_i : i < n\})]$ for each $n \in \omega$. Define $B = \{b_n : n \in \omega\}$. Since B is infinite, $B \subseteq \rho(F)$ and $\mathcal{I}_\rho \restriction \rho(F) = \mathrm{Fin}(\rho(F))$, there is $H \in \mathcal{F}$ such that $\rho(H) \subseteq B$. Using Proposition 3.2, there is $E \in \mathcal{F}$ with $E \subseteq H$ such that for any $n \in \omega$ there is a finite set $L \subseteq \Omega$ such that $\rho(E \setminus L) \subseteq \rho(E) \setminus \{b_i : i < n\}$. Consequently, for any finite set $K \subseteq \Gamma$ there is $n \in \omega$ such that $K \subseteq \{\gamma_i : i < n\}$, so we can find a finite set $L \subseteq \Omega$ such that $\rho(E \setminus L) \subseteq \rho(E) \setminus \{b_i : i < n\} \subseteq B \setminus \{b_i : i < n\} \subseteq \phi[\tau(G \setminus \{\gamma_i : i < n\})] \subseteq \phi[\tau(G \setminus K)]$.

(3) $\implies$ (4) Let $\phi : \Lambda \to \Lambda$ be a witness for $\rho \leq_K \rho_{\mathrm{Fin}(\Lambda)}$. Since $\phi^{-1}[\{\lambda\}] \in \mathrm{Fin}(\Lambda)$ for every $\lambda \in \Lambda$ and $\phi[\Lambda]$ is infinite, there is an infinite set $A \subseteq \Lambda$, such that $\phi \restriction A$ is one-to-one. Then we can find $F \in \mathcal{F}$ such that $\rho(F) \subseteq \phi[A]$. We claim that $\mathcal{I}_\rho \restriction \rho(F) = \mathrm{Fin}(\rho(F))$. Indeed, let $B \subseteq \rho(F)$ be infinite and observe that $\phi^{-1}[B]$ is infinite, so $B = \phi[\phi^{-1}[B]] \notin \mathcal{I}_\rho$.

(4) $\implies$ (1) If $\rho \not\leq_K \rho_{\mathrm{Fin}(\Lambda)}$ then by Proposition 7.5(2b), $\mathcal{I}_\rho \not\leq_K \mathrm{Fin}(\Lambda)$, and consequently $\mathcal{I}_\rho$ is tall. ☐

Definition 8.2. We say that a partition regular function ρ is *tall* if any item of Proposition 8.1 holds.

Proposition 8.3. *The ideals $\mathcal{H}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{W}$ and $\mathcal{I}_{1/n}$ are tall (hence, FS, r, Δ, $\rho_{\mathcal{W}}$ and $\rho_{\mathcal{I}_{1/n}}$ are tall).*

Proof. For the case of $\mathcal{W}$ and $\mathcal{I}_{1/n}$ see [7, p. 3-4]. For other cases, see [65, under Lemma 1.6.24] and [20, Proposition 4.3 and text above Lemma 3.2]). Tallness of the listed partition regular functions follows then from Proposition 8.1. ☐

8.2. **Homogeneity of partition regular functions.** Let $\mathcal{I}_i$ be an ideal on Λ_i for each $i = 1, 2$. Ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ are *isomorphic* (in short: $\mathcal{I}_1 \approx \mathcal{I}_2$) if there exists a bijection $\phi : \Lambda_1 \to \Lambda_2$ such that $A \in \mathcal{I}_1 \iff \phi[A] \in \mathcal{I}_2$ for each $A \subseteq \Lambda_1$. An ideal

$\mathcal{I}$ on Λ is *homogeneous* if the ideals $\mathcal{I}$ and $\mathcal{I} \restriction A$ are isomorphic for every $A \in \mathcal{I}^+$ ([60, Definition 1.3], see also [32]). We say that $\mathcal{I}$ is *K-homogeneous* if $\mathcal{I} \restriction A \leq_K \mathcal{I}$ for every $A \in \mathcal{I}^+$ (in [44, p. 37], the author uses the name *K-uniform* in this case). Note that we always have $\mathcal{I} \leq_K \mathcal{I} \restriction A$ for every $A \in \mathcal{I}^+$ (see e.g. [44, p. 46]).

Definition 8.4. Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular. We say that ρ is *K-homogeneous* if $\rho \restriction A \leq_K \rho$ for every $A \in \mathcal{I}_\rho^+$ (note that we always have $\rho \leq_K \rho \restriction A$ for every $A \in \mathcal{I}_\rho^+$ by Proposition 7.4(3a)).

Proposition 8.5.

1. *If a partition regular function ρ is K-homogeneous then $\mathcal{I}_\rho$ is K-homogeneous.*
2. *An ideal $\mathcal{I}$ is K-homogeneous $\iff$ $\rho_\mathcal{I}$ is K-homogeneous.*

Proof. (1) It follows from Propositions 7.5(1) and 4.1.

(2) Observe that if $A \in \mathcal{I}^+$ then $\rho_{\mathcal{I}\restriction A} = \rho_\mathcal{I} \restriction A$. Thus, it follows from Propositions 7.5(2b) and 4.1. $\square$

We need the following lemma to show that FS and r are K-homogeneous.

Lemma 8.6. *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular (with $\mathcal{F} \subseteq [\Omega]^\omega$). If $\mathcal{I}_\rho$ is homogeneous and ρ is P^- and has small accretions then ρ is K-homogeneous.*

Proof. Let $A \in \mathcal{I}_\rho^+$. Since $\mathcal{I}_\rho$ is homogeneous, $\mathcal{I}_\rho \restriction A$ and $\mathcal{I}_\rho$ are isomorphic. Let $f : \Lambda \to A$ be a bijection witnessing it. We claim that f witnesses $\rho \restriction A \leq_K \rho$.

Let $F \in \mathcal{F}$. Since ρ has small accretions, there is $G \in \mathcal{F}$ such that $G \subseteq F$ and G has small accretions. Enumerate $\Omega = \{o_n : n \in \omega\}$ and define $K_n = \{o_i : i \leq n\}$ and $A_n = f[\rho(G \setminus K_n)]$ for all $n \in \omega$. Then $A_n \supseteq A_{n+1}$. Since G has small accretions and f is a bijection and witnesses that $\mathcal{I}_\rho \restriction A$ and $\mathcal{I}_\rho$ are isomorphic, $A_n \in (\mathcal{I}_\rho \restriction A)^+$ and $A_n \setminus A_{n+1} \subseteq f[\rho(G\setminus K_n) \setminus \rho(G\setminus K_{n+1})] \subseteq f[\rho(G) \setminus \rho(G\setminus K_{n+1})] \in \mathcal{I}_\rho \restriction A$. Using the fact that ρ is P^-, we can find $H \in \mathcal{F} \restriction A$ such that $\rho(H) \subseteq^\rho A_n = f[\rho(G \setminus K_n)]$ for all $n \in \omega$. Hence, given any finite set $K \subseteq \Omega$ there are $n \in \omega$ and finite $L \subseteq \Omega$ such that $K \subseteq K_n$ and $\rho(H \setminus L) \subseteq A_n = f[\rho(G \setminus K_n)] \subseteq f[\rho(G \setminus K)]$. $\square$

Proposition 8.7.

1. *The ideals $\mathcal{H}$, $\mathcal{R}$ and $\mathcal{W}$ are homogeneous (hence, K-homogeneous).*
2. *The functions* FS *and r are K-homogeneous.*

Proof. (1) See [60, Examples 2.5 and 2.6].

(2) It follows from item (1), Lemma 8.6 and Propositions 6.7(3) and 4.3. $\square$

Question 8.8.

1. Is the function Δ K-homogeneous?
2. Is the ideal $\mathcal{I}_{1/n}$ K-homogeneous?

Part 2. FinBW **spaces**

In this part we define the main object of our studies – classes of sequentially compact spaces defined with the aid of partition regular functions (Definition 10.1). Next, we prove some general results about those classes of spaces (Theorem 10.5).

9. A convergence with respect to partition regular functions

Definition 9.1. Let X be a topological space. Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$.

1. For $F \in \mathcal{F}$, a function $f : \rho(F) \to X$ is called a *ρ-sequence* in X.
2. A ρ-sequence $f : \rho(F) \to X$ is *ρ-convergent* to a point $x \in X$ if for every neighbourhood U of x there is a finite set $K \subseteq \Omega$ such that
$$f[\rho(F \setminus K)] \subseteq U.$$

Remark. Various kinds of convergences considered in the literature can be described in terms of ρ-convergence.

(1) If $\rho = \mathrm{FS}$, then ρ-convergence coincides with *IP-convergence* (see [33], [34] or [55]).
(2) If $\rho = r$, then ρ-convergence coincides with the *$\mathcal{R}$-convergence* (see [5], [6], [59, Definition 2.1]).
(3) If $\rho = \Delta$, then ρ-convergence coincides with the *differential convergence* (see [71, Definition 4.2.4] or [20, p. 2010]).
(4) If $\mathcal{I}$ is an ideal on Λ and $\rho_{\mathcal{I}}$ is defined as in Proposition 3.3(2), then $\rho_{\mathcal{I}}$-convergence coincides with the ordinary convergence.

The following proposition reveals relationships between ρ-convergence and convergence.

Proposition 9.2.

(1) Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$. Let $F \in \mathcal{F}$ and $f : \rho(F) \to X$.
(a) If f is convergent to L, then $f \restriction \rho(E)$ is ρ-convergent to L for some $E \in \mathcal{F} \restriction \rho(F)$.
(b) If f is ρ-convergent to L, then $f \restriction A$ is convergent to L for some infinite set $A \subseteq \rho(F)$.

(2) Let $\mathcal{I}$ be an ideal on Λ and $f : A \to X$ for some $A \in \mathcal{I}^+$. Then f is convergent to $L \iff f$ is $\rho_{\mathcal{I}}$-convergent to L.

Proof. (1a) Let $E \in \mathcal{F}$ with $E \subseteq F$ be as in Proposition 3.2 and let U be a neighborhood of L. Then there exists a finite set K such that $f(n) \in U$ for every $n \in \rho(F) \setminus K$. There is a finite set L such that $\rho(E \setminus L) \subseteq \rho(E) \setminus K \subseteq \rho(F) \setminus K$. Consequently, $f(n) \in U$ for every $n \in \rho(E \setminus L)$.

(1b) Let $\Omega = \{o_n : n \in \omega\}$. For each $n \in \omega$, we pick $\lambda_n \in \rho(F \setminus \{o_i : i < n\}) \setminus \{\lambda_i : i < n\}$. Let $A = \{\lambda_n : n \in \omega\}$. We claim that $f \restriction A$ is convergent to L. Indeed, if U is a neighborhood of L, then there is a finite set $K \subseteq \Omega$ such that $f[\rho(F \setminus K)] \subseteq U$. Let $n \in \omega$ be such that $K \subseteq \{o_i : i < n\}$. Then $f[A \setminus \{\lambda_i : i < n\}] \subseteq f[\rho(F \setminus \{o_i : i < n\})] \subseteq f[\rho(F \setminus K)] \subseteq U$.

(2) It is straightforward. $\square$

10. FinBW spaces

Let $\mathcal{I}$ be an ideal on a countable infinite set Λ. The following classes of topological spaces were extensively examined in the literature (see e.g. [24, 29, 61]):

(1) $\mathrm{FinBW}(\mathcal{I})$ is the class of all topological spaces X such that for every sequence $f : \Lambda \to X$ there exists $A \in \mathcal{I}^+$ such that $f \restriction A$ converges (in [29], spaces from $\mathrm{FinBW}(\mathcal{I})$ are called *$\mathcal{I}$-spaces*);
(2) $\mathrm{hFinBW}(\mathcal{I})$ is the class of all topological spaces X such that for every $B \in \mathcal{I}^+$ and every sequence $f : B \to X$ there exists $A \in \mathcal{I}^+$ such that $A \subseteq B$ and $f \restriction A$ converges.

Remark. The classes $\mathrm{FinBW}(\mathcal{I}_\rho)$ for $\rho \in \{\mathcal{W}, \mathcal{I}_{1/n}\}$ were examined in the literature under other names:

(1) in [56, Definition 3], spaces from $\mathrm{FinBW}(\mathcal{W})$ are called *van der Waerden spaces*;
(2) in [29, Definition 2.1], spaces from $\mathrm{FinBW}(\mathcal{I}_{1/n})$ are called *$\mathcal{I}_{1/n}$-spaces*.

Definition 10.1. Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be a partition regular function.

(1) $\mathrm{FinBW}(\rho)$ is the class of all topological spaces X such that for every sequence $f : \Lambda \to X$ there exists $F \in \mathcal{F}$ such that $f \restriction \rho(F)$ ρ-converges.

(2) $\mathrm{hFinBW}(\rho)$ is the class of all topological spaces X such that for every ρ-sequence $f : \rho(E) \to X$ there exists $F \in \mathcal{F}$ such that $\rho(F) \subseteq \rho(E)$ and $f \restriction \rho(F)$ ρ-converges.

Remark. The classes $\mathrm{FinBW}(\rho)$ for $\rho \in \{\mathrm{FS}, r, \Delta\}$ were examined in the literature under other names:

(1) in [55, Definition 4], spaces from $\mathrm{FinBW(FS)}$ are called *Hindman spaces*;
(2) in [6] (see also [59, Definition 2.1]), spaces from $\mathrm{FinBW}(r)$ are called *spaces with the Ramsey property*, and we will call them *Ramsey spaces* in short;
(3) in [71, Definition 4.2.4] (see also [20, p. 2010]), spaces in $\mathrm{FinBW}(\Delta)$ are called *differentially compact spaces.*

Remark. Recall that if $(\Lambda, <)$ is a directed set, then any function $f : \Lambda \to X$ is called a *net* in X. A net $f : \Lambda \to X$ in a topological space X *converges* to $x \in X$ if for every neighborhood U of x there is $\lambda_0 \in \Lambda$ such that $f(\lambda) \in U$ for every $\lambda > \lambda_0$ (see e.g. [15, p. 49]). In [18, Remark 2.6], the authors notice that if $\mathcal{B}$ is a coideal basis on $(\Lambda, <)$, then $(B, < \cap (B \times B))$ is a directed set and $f \restriction B$ is a subnet of f for every $B \in \mathcal{B}$. Furthermore, they examine topological spaces X having the property that every net $f : \Lambda \to X$ has a convergent subnet $f \restriction B$ with some $B \in \mathcal{B}$ ([18, p. 418]). It is not difficult to see that the class of spaces they examine coincides with the class $\mathrm{FinBW}(\rho_{\mathcal{B}})$ with $\rho_{\mathcal{B}}$ defined as in Proposition 3.10(2).

The following proposition reveals relationships between FinBW-like spaces defined with the aid of partition regular functions and ideals.

Proposition 10.2. *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$. Let $\mathcal{I}$ be an ideal on Λ.*

(1) (a) $\mathrm{hFinBW}(\rho) = \bigcap\{\mathrm{FinBW}(\rho \restriction \rho(F)) : F \in \mathcal{F}\}$.
(b) $\mathrm{hFinBW}(\mathcal{I}) = \bigcap\{\mathrm{FinBW}(\mathcal{I} \restriction A) : A \in \mathcal{I}^+\}$.
(2) (a) $\mathrm{hFinBW}(\rho) \subseteq \mathrm{FinBW}(\rho)$.
(b) $\mathrm{hFinBW}(\mathcal{I}) \subseteq \mathrm{FinBW}(\mathcal{I})$.
(3) $\mathrm{FinBW}(\mathcal{I}_\rho) \subseteq \mathrm{FinBW}(\rho)$ *and* $\mathrm{hFinBW}(\mathcal{I}_\rho) \subseteq \mathrm{hFinBW}(\rho)$.
(4) $\mathrm{FinBW}(\mathcal{I}) = \mathrm{FinBW}(\rho_{\mathcal{I}})$ *and* $\mathrm{hFinBW}(\mathcal{I}) = \mathrm{hFinBW}(\rho_{\mathcal{I}})$.
(5) (a) If ρ is K-homogeneous, then $\mathrm{hFinBW}(\rho) = \mathrm{FinBW}(\rho)$.
(b) If $\mathcal{I}$ is K-homogeneous, then $\mathrm{hFinBW}(\mathcal{I}) = \mathrm{FinBW}(\mathcal{I})$.

Proof. (1) and (2) Straightforward.

(3) It follows from Proposition 9.2(1a) (the other inclusion does not follow from Proposition 9.2(1b) as it gives us only an infinite set A, not necessarily $A \in \mathcal{I}_\rho^+$).

(4) It follows from Proposition 9.2(2).

(5a) We only need to show $\mathrm{FinBW}(\rho) \subseteq \mathrm{hFinBW}(\rho)$. Let $X \in \mathrm{FinBW}(\rho)$ and $f : \rho(E) \to X$ be a ρ-sequence in X for some $E \in \mathcal{F}$. Let $\phi : \Lambda \to \rho(E)$ be a witness for $\rho \restriction (\mathcal{F} \restriction \rho(E)) \leq_K \rho$. Since $f \circ \phi : \Lambda \to X$, there is $F \in \mathcal{F}$ such that $f \circ \phi \restriction \rho(F)$ is ρ-convergent to some $x \in X$. Since $\rho \restriction (\mathcal{F} \restriction \rho(E)) \leq_K \rho$, there is $G \in \mathcal{F} \restriction \rho(E)$ such that for every finite set $K \subseteq \Omega$ there is a finite set $L \subseteq \Omega$ with $\rho(G \setminus L) \subseteq \phi[\rho(F \setminus K)]$. We claim that $f \restriction \rho(G)$ is ρ-convergent to x. Let U be a neighborhood of x. Then there is a finite set $K \subseteq \Omega$ such that $(f \circ \phi)[\rho(F \setminus K)] \subseteq U$. We pick a finite set $L \subseteq \Omega$ such that $\rho(G \setminus L) \subseteq \phi[\rho(F \setminus K)]$. Then $f[\rho(G \setminus L)] \subseteq f[\phi[\rho(F \setminus K)]] \subseteq U$, so the proof is finished.

(5b) It follows from items (5a) and (4) and Proposition 8.5(2). $\square$

Remark. In Theorem 10.2(3), we cannot replace inclusion with equality in general because in [55, Theorems 3 and 10] the author proved that $\mathrm{FinBW}(\mathcal{H})$ contains only finite Hausdorff spaces, whereas $\mathrm{FinBW(FS)}$ contains infinite (even uncountable) ones.

Corollary 10.3.

(1) [56, Proposition 4] $\mathrm{hFinBW}(\mathcal{W}) = \mathrm{FinBW}(\mathcal{W})$*, and consequently the product of two van der Waerden spaces is van der Waerden.*
(2) [55, Lemma 8] $\mathrm{hFinBW}(\mathrm{FS}) = \mathrm{FinBW}(\mathrm{FS})$*, and consequently the product of two Hindman spaces is Hindman.*
(3) [59, Theorem 3.4] $\mathrm{hFinBW}(r) = \mathrm{FinBW}(r)$*, and consequently the product of two Ramsey spaces is Ramsey.*

Proof. It follows from Theorem 10.2(5) and Proposition 8.7. □

Question 10.4 ([28, 30] and [71, Question 4.2.3])**.**

(1) (a) Does $\mathrm{FinBW}(\mathcal{I}_{1/n}) = \mathrm{hFinBW}(\mathcal{I}_{1/n})$?
(b) Is the product of two $\mathcal{I}_{1/n}$-spaces an $\mathcal{I}_{1/n}$-space?
(2) (a) Does $\mathrm{FinBW}(\Delta) = \mathrm{hFinBW}(\Delta)$?
(b) Is the product of two differentially compact spaces a differentially compact space?

Note that the positive answer to the question in item (1a) gives the positive answer to the question in item (1b), and similarly for the questions in the second item. Moreover, the positive answer to the Question 8.8(1) (Question 8.8(2), resp.) gives the positive answer to Question 10.4(2a) (Question 10.4(1a), resp.).

Let us now turn to one of the main result of this paper.

Theorem 10.5. *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$.*

(1) $\mathrm{FinBW}(\rho)$ *contains all finite spaces and is a subclass of the class of all sequentially compact spaces.*
(2) ρ is not tall $\iff$ $\mathrm{FinBW}(\rho)$ *coincides with the class of all sequentially compact spaces.*
(3) The following conditions are equivalent.
(a) ρ is $P^-(\Lambda)$.
(b) There are Hausdorff compact spaces of arbitrary cardinality that belong to $\mathrm{FinBW}(\rho)$*.*
(c) There exists an infinite Hausdorff topological space $X \in \mathrm{FinBW}(\rho)$.
(4) The following conditions are equivalent.
(a) ρ is P^-.
(b) There are Hausdorff compact spaces of arbitrary cardinality that belong to $\mathrm{hFinBW}(\rho)$*.*
(c) There exists an infinite Hausdorff topological space $X \in \mathrm{hFinBW}(\rho)$.
(5) If ρ is P^-, then
(a) the uncountable non-compact Hausdorff space $X = \omega_1$ with the order topology belongs to $\mathrm{FinBW}(\rho)$*,*
(b) assuming Continuum Hypothesis (CH) there are Hausdorff compact and separable spaces of cardinality $\mathfrak{c}$ that belong to $\mathrm{FinBW}(\rho)$*.*
(6) If ρ is weak P^+, then every compact metric space is in $\mathrm{hFinBW}(\rho)$*.*
(7) If ρ is P^+, then every Hausdorff topological space with the property $(*)$ *belongs to* $\mathrm{hFinBW}(\rho)$*.*
(A topological space X has the property (*) *if for every countable set $D \subseteq X$ the closure $\mathrm{cl}_X(D)$ is compact and first-countable – see* [56]*.)*

Proof. (1) It follows from Proposition 9.2(1b).

(2) The implication " $\Longleftarrow$ " will follow from Theorem 13.2(1). To prove the implication " $\Longrightarrow$ " we only need to show that every sequentially compact space belongs to $\mathrm{FinBW}(\rho)$. Fix a sequentially compact space X and $f : \Lambda \to X$. Let $\phi : \Lambda \to \Lambda$ be a witness for $\rho \leq_K \rho_{\mathrm{Fin}(\Lambda)}$. Since $f \circ \phi : \Lambda \to X$, there is an infinite set $A \subseteq \Lambda$ such that $(f \circ \phi) \restriction A$ is convergent to some $x \in X$. Then

there is $F \in \mathcal{F}$ with $\rho(F) \subseteq \phi[A]$. We claim that $f \restriction \rho(F)$ is ρ-convergent to x. Let U be any neighbourhood of x. Then there is a finite set $L \subseteq \Lambda$ such that $f[\phi[A \setminus L]] \subseteq U$. Now, we can find a finite set $K \subseteq \Omega$ such that $\rho(F \setminus K) \subseteq \phi[A \setminus L]$. Thus, $f[\rho(F \setminus K)] \subseteq U$.

(3) (a) $\Longrightarrow$ (b) Let κ be an infinite cardinal number. Let $X = \kappa \cup \{\infty\}$ be the Alexandroff one-point compactification of the discrete space κ. Then X is Hausdorff, compact and has cardinality κ. Moreover, open neighborhoods of ∞ are of the form $X \setminus S$ where S is a compact (hence finite) subset of κ. We show that X is in $\mathrm{FinBW}(\rho)$. Let $f : \Lambda \to X$. If there is $x \in X$ with $f^{-1}[\{x\}] \notin \mathcal{I}_\rho$, then we take $F \in \mathcal{F}$ such that $\rho(F) \subseteq f^{-1}[\{x\}]$ and see that $f \restriction \rho(F)$ is ρ-convergent to x. Now, we assume that $f^{-1}[\{x\}] \in \mathcal{I}_\rho$ for every $x \in X$. By Proposition 6.8, there is $F \in \mathcal{F}$ such that for every finite set $S \subseteq X$ there is a finite set K_S such that $\rho(F \setminus K_S) \cap f^{-1}[S] = \emptyset$. We claim that $f \restriction \rho(F)$ is ρ-convergent to ∞. Let U be an open neighborhood of ∞. Let $S \subseteq \kappa$ be a finite set with $U = X \setminus S$. Then $f[\rho(F \setminus K_S)] \subseteq X \setminus S = U$.

(b) $\Longrightarrow$ (c) Obvious.

(c) $\Longrightarrow$ (a) Suppose that ρ is not $P^-(\Lambda)$ and let $A_n \in \mathcal{I}_\rho^+$ be the witnessing sequence, i.e., $A_0 = \Lambda$, $A_{n+1} \subseteq A_n$, $A_n \setminus A_{n+1} \in \mathcal{I}_\rho$ and for each $F \in \mathcal{F}$ there is $n \in \omega$ such that $\rho(F) \not\subseteq^\rho A_n$. Note that $\bigcap_{n\in\omega} A_n \in \mathcal{I}_\rho$.

Let X be an infinite Hausdorff topological space. We will show that $X \notin \mathrm{FinBW}(\rho)$. If X is not sequentially compact, then $X \notin \mathrm{FinBW}(\rho)$ by item (1). If X is sequentially compact, then find any one-to-one sequence $\{x_n : n \in \omega\}$ in X converging to some $x \in X$. Without loss of generality we may assume that $x \neq x_n$ for all $n \in \omega$. Define $f : \Lambda \to X$ by $f \restriction \bigcap_{n\in\omega} A_n = x_0$ and $f(\lambda) = x_{n+1}$, where n is such that $\lambda \in A_n \setminus A_{n+1}$. Suppose for the sake of contradiction that there are $L \in X$ and $F \in \mathcal{F}$ such that $f \restriction \rho(F)$ ρ-converges to L. By Proposition 9.2(1b) we get that either $L = x_n$ for some $n \in \omega$ or $L = x$.

If $L = x_n$ for some $n \in \omega$, find open U and V such that $x_n \in U$, $x \in V$ and $U \cap V = \emptyset$. Since $x_m \in V$ (so $x_m \notin U$) for almost all $m \in \omega$ and $f^{-1}[\{x_m\}] \in \mathcal{I}_\rho$ for all $m \in \omega$, $f^{-1}[U] \in \mathcal{I}_\rho$. Hence, $f \restriction \rho(F)$ cannot ρ-converge to x_n.

If $L = x$, we can find $n \in \omega$ such that $\rho(F) \not\subseteq^\rho A_n$. Since X is Hausdorff, there is an open neighbourhood U of L such that $x_{i+1} \notin U$ for all $i < n$. Since $f \restriction \rho(F)$ ρ-converges to L, there should be a finite $K \subseteq \Omega$ such that $f[\rho(F \setminus K)] \subseteq U$, however $\rho(F \setminus K) \setminus A_n \neq \emptyset$ (by $\rho(F) \not\subseteq^\rho A_n$), so $f[\rho(F \setminus K)] \cap \{x_{i+1} : i < n\} \neq \emptyset$, which contradicts $x_{i+1} \notin U$ for all $i < n$.

(4) (a) $\Longrightarrow$ (b) Notice that if X is the space defined in the proof of the implication $(3a) \implies (3b)$ then $X \in \mathrm{FinBW}(\rho)$ for every ρ that is $P^-(\Lambda)$ (the definition of X did not depend on ρ). Thus, if ρ is P^- then $\rho \restriction \rho(F)$ is $P^-(\rho(F))$ for every $F \in \mathcal{F}$ and consequently $X \in \bigcap_{F\in\mathcal{F}} \mathrm{FinBW}(\rho \restriction \rho(F)) = \mathrm{hFinBW}(\rho)$ (by Proposition 10.2(1a)).

(b) $\Longrightarrow$ (c) Obvious.

(c) $\Longrightarrow$ (a) If ρ is not P^- then $\rho \restriction \rho(F)$ is not $P^-(\rho(F))$ for some $F \in \mathcal{F}$. Hence, by item (3), $\mathrm{FinBW}(\rho \restriction \rho(F))$ contains only finite Hausdorff spaces. Since $\mathrm{hFinBW}(\rho) \subseteq \mathrm{FinBW}(\rho \restriction \rho(F))$ by Proposition 10.2(1a), $\mathrm{hFinBW}(\rho)$ also contains only finite Hausdorff spaces.

(5a) Let $f : \Lambda \to \omega_1$. If there is $\alpha < \omega_1$ with $f^{-1}[\{\alpha\}] \notin \mathcal{I}_\rho$, then we take $F \in \mathcal{F}$ such that $\rho(F) \subseteq f^{-1}[\{\alpha\}]$ and see that $f \restriction \rho(F)$ is ρ-convergent to α. Now, we assume that $f^{-1}[\{\alpha\}] \in \mathcal{I}_\rho$ for every $\alpha < \omega_1$. Since Λ is countable and the cofinality of ω_1 is uncountable, there is $\alpha < \omega_1$ with $f^{-1}[\alpha] \notin \mathcal{I}_\rho$. Let α_0 be the smallest α such that $f^{-1}[\alpha] \notin \mathcal{I}_\rho$. Note that α_0 is a limit ordinal. Indeed, if $\alpha_0 = \alpha + 1$, then $\alpha < \alpha_0$ and $f^{-1}[\alpha] = f^{-1}[\alpha_0] \setminus f^{-1}[\{\alpha\}] \notin \mathcal{I}_\rho$, a contradiction. Since α_0 is a countable limit ordinal, there is an increasing sequence $\{\beta_n : n \in \omega\}$ such that

$\sup\{\beta_n : n \in \omega\} = \alpha_0$. By Proposition 6.8, there is $F \in \mathcal{F}$ such that $\rho(F) \subseteq f^{-1}[\alpha_0]$ and for each $n \in \omega$ there is a finite set K_n such that $\rho(F \setminus K_n) \cap f^{-1}[\beta_n] = \emptyset$. We claim that $f \restriction \rho(F)$ is ρ-convergent to α_0. Indeed, let U be a neighborhood of α_0. Without loss of generality, we can assume that $U = (\alpha_0 + 1) \setminus \beta_n$ for some $n \in \omega$. Then $f[\rho(F \setminus K_n)] \subseteq \alpha_0 \setminus \beta_n \subseteq U$.

(5b) Spaces with these properties are constructed in Theorem 14.5.

(6) Let $f : \rho(E) \to X$ be a ρ-sequence in a metric compact space X.

Since ρ is weak P^+, there exists $F \in \mathcal{F}$ such that $\rho(F) \subseteq \rho(E)$ and for every sequence $\{F_n : n \in \omega\} \subseteq \mathcal{F}$ such that $\rho(F) \supseteq \rho(F_n) \supseteq \rho(F_{n+1})$ for each $n \in \omega$ there exists $G \in \mathcal{F}$ such that $\rho(G) \subseteq \rho(F)$ and $\rho(G) \subseteq^\rho \rho(F_n)$ for each $n \in \omega$.

For $x \in X$ and $r > 0$ we write $B(x, r)$ and $\overline{B}(x, r)$ to denote an open and closed ball of radius r centered at a point x, respectively.

Since X is compact metric, there are finitely many $x^0_i \in X$, $i < n_0$ such that $X = \bigcup\{B(x^0_i, 1) : i < n_0\}$. Then there exists $i_0 < n_0$ such that $\rho(F) \cap f^{-1}[B(x^0_{i_0}, 1)] \notin \mathcal{I}_\rho$, and consequently there is $F_0 \in \mathcal{F}$ such that $\rho(F_0) \subseteq \rho(F) \cap f^{-1}[B(x^0_{i_0}, 1)]$.

Since $\overline{B}(x^0_{i_0}, 1)$ is compact metric, there are finitely many $x^1_i \in X$, $i < n_1$ such that $\overline{B}(x^0_{i_0}, 1) \subseteq \bigcup\{B(x^1_i, \frac{1}{2}) : i < n_1\}$. Then there exists $i_1 < n_1$ such that $\rho(F_0) \cap f^{-1}[B(x^1_{i_1}, \frac{1}{2})] \notin \mathcal{I}_\rho$, and consequently there is $F_1 \in \mathcal{F}$ such that $\rho(F_1) \subseteq \rho(F_0) \cap f^{-1}[B(x^1_{i_1}, \frac{1}{2})]$.

If we continue the above procedure, we obtain $F_n \in \mathcal{F}$ and $x^n_{i_n} \in X$ such that $\rho(F_n) \subseteq \rho(F_{n-1}) \cap f^{-1}[B(x^n_{i_n}, \frac{1}{n+1})]$ for each $n \in \omega$ (assuming that $F_{-1} = F$).

Let $x \in \bigcap\{\overline{B}(x^n_{i_n}, \frac{1}{n+1}) : n \in \omega\}$.

Since ρ is weak P^+, we have $G \in \mathcal{F}$ such that $\rho(G) \subseteq \rho(F)$ and $\rho(G) \subseteq^\rho \rho(F_n)$ for each $n \in \omega$.

We claim that $f \restriction \rho(G)$ is ρ-convergent to x. Let U be a neighborhood of x. Since the sequence $(x^n_{i_n})_{n\in\omega}$ is convergent to x, there is $n_0 \in \omega$ such that $B(x^n_{i_n}, \frac{1}{n+1}) \subseteq U$ for every $n \geq n_0$. Consequently, there is $n \in \omega$ with $B(x^n_{i_n}, \frac{1}{n+1}) \subseteq U$. Let $K \subseteq \Omega$ be a finite set such that $\rho(G \setminus K) \subseteq \rho(F_n)$. Then $f[\rho(G \setminus K)] \subseteq f[\rho(F_n)] \subseteq B(x^n_{i_n}, \frac{1}{n+1}) \subseteq U$, so the proof is finished.

(7) Let $E \in \mathcal{F}$ and $f : \rho(E) \to X$ be a sequence in a Hausdorff topological space X having the property $(*)$. Since the set $D = \{f(\lambda) : \lambda \in \rho(E)\}$ is countable, the closure $\mathrm{cl}_X(D)$ is compact and first-countable. We claim that there exists $L \in \mathrm{cl}_X(D)$ such that $f^{-1}[U] \in \mathcal{I}^+_\rho$ for every neighborhood U of L.

Suppose, for sake of contradiction, that for every $x \in \mathrm{cl}_X(D)$ there is a neighborhood U_x of x such that $f^{-1}[U_x] \in \mathcal{I}_\rho$. Since $\mathrm{cl}_X(D)$ is compact, there are finitely many $x_i \in \mathrm{cl}_X(D)$ for $i < n$ with $\mathrm{cl}_X(D) \subseteq \bigcup\{U_{x_i} : i < n\}$. Then $\rho(E) = \bigcup\{f^{-1}[U_{x_i}] : i < n\} \in \mathcal{I}_\rho$, a contradiction, so the claim is proved.

Let $\{U_n : n \in \omega\}$ be a base at L. Without loss of generality, we can assume that $U_n \supseteq U_{n+1}$ for each $n \in \omega$. For each $n \in \omega$, we define $A_n = \{\lambda \in \rho(E) : f(\lambda) \in U_n\}$. Since $A_n \in \mathcal{I}^+_\rho$ and $A_n \supseteq A_{n+1}$ for each $n \in \omega$, using the fact that ρ is P^+, there exists $F \in \mathcal{F}$ such that $\rho(F) \subseteq \rho(E)$ and $\rho(F) \subseteq^\rho A_n$ for each $n \in \omega$. We claim that $f \restriction \rho(F)$ is ρ-convergent to L.

Take any neighborhood U of L. Then there exists $n_0 \in \omega$ with $U_{n_0} \subseteq U$. Since $\rho(F) \subseteq^\rho A_{n_0}$, there exists a finite set $K \subseteq \Omega$ such that $\rho(F \setminus K) \subseteq A_{n_0}$. Thus $f[\rho(F \setminus K)] \subseteq U_{n_0} \subseteq U$, so the proof is finished. □

The following series of corollaries shows that many known earlier results can be easily derived from Theorem 10.5.

Corollary 10.6 ([26, Proposition 2.4])**.** *If an ideal $\mathcal{I}$ is not tall, then* $\mathrm{FinBW}(\mathcal{I})$ *coincides with the class of all sequentially compact spaces.*

Proof. It follows from Theorem 10.5(2) and Propositions 10.2(4) and 8.1. □

Corollary 10.7 ([61, Theorem 6.5])**.** $\mathrm{Fin}^2 \leq_K \mathcal{I} \iff \mathrm{FinBW}(\mathcal{I})$ *coincides with the class of all finite spaces in the realm of Hausdorff spaces.*

Proof. It follows from Theorem 10.5(3) and Propositions 7.1(1),6.5(2),10.2(4). □

Corollary 10.8. *Every metric compact space belongs to* $\mathrm{hFinBW}(\rho)$ *in case when*

(1) $\rho = \rho_{\mathcal{I}}$ *and* $\mathcal{I}$ *is* P^+ *ideal,*
(2) [29, Theorem 2.3] $\rho = \rho_{\mathcal{I}}$ *and* $\mathcal{I}$ *is an* F_σ *ideal,*
(3) [56, Theorem 10] $\rho = \rho_{\mathcal{I}}$ *and* $\mathcal{I} = \mathcal{W}$*,*
(4) [29, Theorem 2.3] $\rho = \rho_{\mathcal{I}}$ *and* $\mathcal{I} = \mathcal{I}_{1/n}$*,*
(5) [34, Theorem 2.5] $\rho = \mathrm{FS}$*,*
(6) [6, Theorem 1] *(see also* [5, Theorem 1.16]*)* $\rho = r$*,*
(7) [20, Corollary 4.8] $\rho = \Delta$*.*

Proof. It follows from Theorem 10.5(6), Propositions 6.7 and 6.5. □

Corollary 10.9. *Every Hausdorff space with the property* $(*)$ *belongs to* $\mathrm{hFinBW}(\rho)$ *in case when*

(1) $\rho = \rho_{\mathcal{I}}$ *and* $\mathcal{I}$ *is* P^+ *ideal,*
(2) [29, Theorem 2.3] $\rho = \rho_{\mathcal{I}}$ *and* $\mathcal{I}$ *is an* F_σ *ideal,*
(3) [56, Theorem 10] $\rho = \rho_{\mathcal{I}}$ *and* $\mathcal{I} = \mathcal{W}$*,*
(4) [29, Theorem 2.3] $\rho = \rho_{\mathcal{I}}$ *and* $\mathcal{I} = \mathcal{I}_{1/n}$*.*

Proof. It follows from Theorem 10.5(7), Propositions 6.7 and 6.5. □

In [55, Theorem 11] ([59, Corollary 3.2], resp.), the authors proved that every Hausdorff space with the property $(*)$ belongs to $\mathrm{hFinBW}(\mathrm{FS})$ ($\mathrm{hFinBW}(r)$, resp.). However, their proofs use properties very specific to FS and r. For instance, the proof for FS uses idempotent ultrafilters whereas the proof for r uses the bounding number $\mathfrak{b}$.

Problem 10.10. Find a property W of partition regular functions such that both FS and r have the property W and if ρ has the property W then every Hausdorff space with the property $(*)$ belongs to $\mathrm{hFinBW}(\rho)$.

In [20, Corollary 4.8], the author proved that every Hausdorff space with the property $(*)$ belongs to $\mathrm{FinBW}(\Delta)$.

Question 10.11. Does every Hausdorff space with the property $(*)$ belong to $\mathrm{hFinBW}(\Delta)$?

Note that the positive answer to Question 10.4(2a) gives the positive answer to Question 10.11.

11. Inclusions between FinBW classes

Theorem 11.1. *Let* $\rho_i : \mathcal{F}_i \to [\Lambda_i]^\omega$ *be partition regular with* $\mathcal{F}_i \subseteq [\Omega_i]^\omega$ *for each* $i = 1, 2$*. Let* $\mathcal{I}$ *be an ideal on* Λ*.*

(1) $\rho_2 \leq_K \rho_1 \implies \mathrm{FinBW}(\rho_1) \subseteq \mathrm{FinBW}(\rho_2)$.
(2) *(a) If* ρ_2 *is* P^+*, then*

$$\mathcal{I}_{\rho_2} \leq_K \mathcal{I}_{\rho_1} \implies \mathrm{FinBW}(\rho_1) \subseteq \mathrm{FinBW}(\rho_2).$$

(b) $\mathcal{I}_{\rho_2} \leq_K \mathcal{I} \implies \mathrm{FinBW}(\mathcal{I}) \subseteq \mathrm{FinBW}(\rho_2)$.

Proof. (1) Let $\phi : \Lambda_1 \to \Lambda_2$ be a witness for $\rho_2 \leq_K \rho_1$. Let $X \in \mathrm{FinBW}(\rho_1)$. If $f : \Lambda_2 \to X$, then $f \circ \phi : \Lambda_1 \to X$, so there is $F_1 \in \mathcal{F}_1$ such that $\rho_1(F_1) \subseteq \Lambda_1$ and $(f \circ \phi) \restriction \rho_1(F_1)$ is ρ_1-convergent to some $x \in X$.

Let $F_2 \in \mathcal{F}_2$ be such that for every finite $K_1 \subseteq \Omega_1$ there is a finite $K_2 \subseteq \Omega_2$ with $\rho_2(F_2 \setminus K_2) \subseteq \phi[\rho_1(F_1 \setminus K_1)]$.

We claim that $f \restriction \rho(F_2)$ is ρ_2-convergent to x. Let U be a neighborhood of x. Since $(f \circ \phi) \restriction \rho_1(F_1)$ is ρ_1-convergent to x, then there is a finite set $K_1 \subseteq \Omega_1$ such that $(f \circ \phi)[\rho_1(F_1 \setminus K_1)] \subseteq U$. Hence, we can find a finite $K_2 \subseteq \Omega_2$ with $\rho_2(F_2 \setminus K_2) \subseteq \phi[\rho_1(F_1 \setminus K_1)]$. Then $f[\rho_2(F_2 \setminus K_2)] \subseteq f[\phi[\rho_1(F_1 \setminus K_1)]] \subseteq U$. That finishes the proof.

(2a) It follows from Proposition 7.5(2a) and item (1).

(2b) It follows from item (1) and Propositions 7.5(2b) and 10.2(4). □

The following series of corollaries shows that many known earlier results as well as some new one can be easily derived from Theorem 11.1.

Corollary 11.2.

(1) [71, p. 39] *Every Hindman space is differentially compact.*

(2) *Every Ramsey space is differentially compact.*

(3) *Let $\mathcal{I}$ be a P^+ ideal.*

- *(a)* [22, Proposition 2.6] *If $\mathcal{I} \leq_K \mathcal{H}$ then every Hindman space is in* $\mathrm{FinBW}(\mathcal{I})$.
- *(b)* *If $\mathcal{I} \leq_K \mathcal{R}$ then every Ramsey space is in* $\mathrm{FinBW}(\mathcal{I})$.
- *(c)* *If $\mathcal{I} \leq_K \mathcal{D}$ then every differentially compact space is in* $\mathrm{FinBW}(\mathcal{I})$.

(4) [61, Corollary 10.2(a)] *If $\mathcal{I}_i$ are ideals for $i = 1, 2$ and $\mathcal{I}_2 \leq_K \mathcal{I}_1$, then* $\mathrm{FinBW}(\mathcal{I}_1) \subseteq \mathrm{FinBW}(\mathcal{I}_2)$.

Proof. (1) It follows from Theorems 11.1(1) and 7.7(3).

(2) It follows from Theorems 11.1(1) and 7.7(4).

(3) It follows from Theorem 11.1(2a) and Propositions 6.5(2) and 10.2(4).

(4) It follows from Theorem 11.1(2b) and Proposition 10.2(4). □

Corollary 11.3.

(1) *Let $\rho : \mathcal{F}_i \to [\Lambda]^\omega$ be a partition regular function.*

- *(a)* *If $\rho \leq_K \rho'$ for some weak P^+ partition regular function ρ', then every compact metric space belongs to* $\mathrm{FinBW}(\rho)$.
- *(b)* *If $\rho \leq_K \rho'$ for some P^+ partition regular function ρ', then every Hausdorff topological space with the property $(*)$ belongs to* $\mathrm{FinBW}(\rho_2)$.

(2) *Let $\mathcal{I}$ be an ideal. If an ideal $\mathcal{I}$ can be extended to a P^+ ideal, then:*

- *(a)* *every compact metric space belongs to* $\mathrm{FinBW}(\mathcal{I})$*;*
- *(b)* [25, Corollary 5.6] *every Hausdorff topological space with the property $(*)$ belongs to* $\mathrm{FinBW}(\mathcal{I})$.

Proof. (1) It follows from Theorems 11.1(1), 10.5(6)(7) and Proposition 10.2(2).

(2) It follows from item (1), Propositions 6.5(2), 7.5(2b), 10.2(4). □

The following proposition shows that when comparing classes $\mathrm{FinBW}(\rho_1)$ and $\mathrm{FinBW}(\rho_2)$ for distinct functions ρ_1 and ρ_2 we can in fact assume that both ρ_1 and ρ_2 "live" on the same sets Ω and Λ.

Proposition 11.4. *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$. Suppose that Γ and Σ are countable infinite sets and $\phi : \Omega \to \Gamma$ and $\psi : \Lambda \to \Sigma$ are bijections. Let $\mathcal{G} = \{\phi[F] : F \in \mathcal{F}\}$ and $\tau : \mathcal{G} \to [\Sigma]^\omega$ be given by $\tau(G) = \psi[\rho(\phi^{-1}[G])]$. Then*

(1) *τ is partition regular,*

(2) *ρ and τ are Katětov equivalent: $\rho \leq_K \tau$ and $\tau \leq_K \rho$,*

(3) *ρ and τ are hereditary Katětov equivalent:*

- *(a)* *$\forall F \in \mathcal{F}\, \exists G \in \mathcal{G}\, (\rho \restriction \rho(F) \leq_K \tau \restriction \tau(G))$,*
- *(b)* *$\forall G \in \mathcal{G}\, \exists F \in \mathcal{F}\, (\tau \restriction \tau(G) \leq_K \rho \restriction \rho(F))$,*

(4) $\mathrm{FinBW}(\tau) = \mathrm{FinBW}(\rho)$ *and* $\mathrm{hFinBW}(\tau) = \mathrm{hFinBW}(\rho)$.

Proof. Items (1)–(3) are straightforward, whereas item (4) follows from previous items and Proposition 11.1. □

Part 3. **Distinguishing between** FinBW **classes**

In this part we are interested in finding spaces that are in $\mathrm{FinBW}(\rho_1)$, but are not in $\mathrm{FinBW}(\rho_2)$. Similar investigations concerning the classes $\mathrm{FinBW}(\mathcal{I})$ were conducted in [61]. In that paper all the examples (showing that under some set-theoretic assumption $\mathrm{FinBW}(\mathcal{I}) \setminus \mathrm{FinBW}(\mathcal{J}) \neq \emptyset$ for some ideals $\mathcal{I}$ and $\mathcal{J}$) were inspired by [57] and are of one specific type – they are defined using *maximal* almost disjoint families. It turns out (see Theorem 13.2) that, in general, we cannot use *maximal* almost disjoint families to distinguish between $\mathrm{FinBW}(\rho)$ classes with the aid of spaces defined as in [61]. Fortunately, we managed to use *not necessary maximal* almost disjoint families to prove two main results of this part (Theorems 14.3 and 15.2), which give us $\mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2) \neq \emptyset$ for certain ρ_1 and ρ_2. Our methods were inspired by [59].

12. Mrówka spaces and their compactifications

For an *infinite* almost disjoint family $\mathcal{A}$ on a countable set Λ, we define a set

$$\Psi(\mathcal{A}) = \Lambda \cup \mathcal{A}$$

and introduce a topology on $\Psi(\mathcal{A})$ as follows: the points of Λ are isolated and a basic neighborhood of $A \in \mathcal{A}$ has the form $\{A\} \cup (A \setminus F)$ with F finite.

Topological spaces of the form $\Psi(\mathcal{A})$ were introduced by Alexandroff and Urysohn in [1, chapter V, paragraph 1.3] (as noted in [15, p. 182], [14, p. 1380] and [45, p. 605]) and its topology is known as the *rational sequence topology* (see [72, Example 65]; the same topology was later described by Katětov in [52, p. 74]). Spaces $\Psi(\mathcal{A})$ with *maximal* (with respect to the inclusion) almost disjoint families $\mathcal{A}$ were first examined by Mrówka (see [67]) and Isbell (as noted in [37, p. 269]). It seems that the notation Ψ for these kind of spaces was used for the first time in [37, Problem 5I, p. 79].

Spaces of the form $\Psi(\mathcal{A})$ are known under many names, including Ψ*-spaces*, *Isbell-Mrówka spaces* and *Mrówka spaces*. Recent surveys on these spaces and their numerous applications can be found in [45, 41].

It is known that $\Psi(\mathcal{A})$ is Hausdorff, regular, locally compact, first countable and separable, but it is not compact nor sequentially compact (see [67] or [74, Section 11]). It is not difficult to see that $A \cup \{A\}$ is compact in $\Psi(\mathcal{A})$ for every $A \in \mathcal{A}$ and for every compact set $K \subseteq \Psi(\mathcal{A})$ both sets $K \cap \mathcal{A}$ and $(K \cap \Lambda) \setminus \bigcup\{A : A \in K \cap \mathcal{A}\}$ are finite. In particular, for every compact set $K \subseteq \Psi(\mathcal{A})$ there are finitely many sets $A_i \in \mathcal{A}$ and a finite set F such that $K \subseteq \{A_i : i < n\} \cup \bigcup\{A_i \cup F : i < n\}$.

Let

$$\Phi(\mathcal{A}) = \Psi(\mathcal{A}) \cup \{\infty\} = \Lambda \cup \mathcal{A} \cup \{\infty\}$$

be the Alexandroff one-point compactification of $\Psi(\mathcal{A})$ (recall that open neighborhoods of ∞ are of the form $\Phi(\mathcal{A}) \setminus K$ for compact sets $K \subseteq \Psi(\mathcal{A})$). It is not difficult to see that $\Phi(\mathcal{A})$ is Hausdorff, compact, sequentially compact, separable and first countable at every point of $\Phi(\mathcal{A}) \setminus \{\infty\}$.

Topological spaces of the form $\Phi(\mathcal{A})$ with *maximal* almost disjoint families $\mathcal{A}$ were first used by Franklin [31, Example 7.1] where the author used the notation Ψ^* instead of Φ. Later, these spaces were considered in [36] where the authors use the notation $\mathcal{F}(\mathcal{A})$ and call them the *Franklin compact spaces* associated to $\mathcal{A}$, whereas in [35] the authors use the notation $Fr(\mathcal{A})$ and call them the *Franklin spaces of* $\mathcal{A}$. The notation $\Phi(\mathcal{A})$ for these spaces is used in the following papers [20, 22, 58, 61] Recently, spaces of the form $\Phi(\mathcal{A})$ were also considered for *non* maximal almost disjoint families [59, 11].

It also makes sense to define $\Phi(\mathcal{A})$ for infinite families $\mathcal{A}$ that are *not* almost disjoint, but then $\Phi(\mathcal{A})$ is no longer Hausdorff (almost disjointness of $\mathcal{A}$ is a necessary and sufficient condition for a space $\Phi(\mathcal{A})$ to be Hausdorff).

The following lemma (which will be used repeatedly in the sequel) shows that a sequence in a space $\Phi(\mathcal{A})$ may fail to have a ρ-convergent ρ-subsequence only in one specific case. Hence, checking whether $\Phi(\mathcal{A}) \in \mathrm{FinBW}(\rho)$ will be reduced to considering only sequences of this one specific kind.

Lemma 12.1. *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$. Let $\mathcal{A}$ be an infinite almost disjoint family on Λ. For every sequence $f : \Lambda \to \Phi(\mathcal{A})$, the following five cases can only occur:*

(1) $f^{-1}(\infty) \notin \mathcal{I}_\rho$,
(2) $f^{-1}[\mathcal{A}] \notin \mathcal{I}_\rho$,
(3) $f^{-1}(\infty) \in \mathcal{I}_\rho$, $f^{-1}[\mathcal{A}] \in \mathcal{I}_\rho$, $f^{-1}[\Lambda] \in \mathcal{I}_\rho^*$ and
  (a) $f^{-1}(\lambda) \notin \mathcal{I}_\rho$ for some $\lambda \in \Lambda$,
  (b) $f^{-1}(\lambda) \in \mathcal{I}_\rho$ for every $\lambda \in \Lambda$ and $f^{-1}[A] \notin \mathcal{I}_\rho$ for some $A \in \mathcal{A}$,
  (c) $f^{-1}(\lambda) \in \mathcal{I}_\rho$ for every $\lambda \in \Lambda$ and $f^{-1}[A] \in \mathcal{I}_\rho$ for every $A \in \mathcal{A}$.

If ρ is P^-, then in cases (1), (2), (3a) and (3b) there is $F \in \mathcal{F}$ such that $f \restriction \rho(F)$ is ρ-convergent.

Proof. Case (1). There is $F \in \mathcal{F}$ such that $f \restriction \rho(F)$ is constant (with the value ∞), hence it is ρ-convergent.

Case (2). We find $F \in \mathcal{F}$ with $\rho(F) \subseteq f^{-1}[\mathcal{A}]$. Then we enumerate $f[\rho(F)] = \{A_n : n \in \omega\}$ and define $E_n = f^{-1}[\{A_n\}]$ for each $n \in \omega$.

If there is $n_0 \in \omega$ such that $E_{n_0} \notin \mathcal{I}_\rho$, then we find $F' \in \mathcal{F}$ with $\rho(F') \subseteq E_{n_0}$, and we see that $f \restriction \rho(F')$ is constant, so it is ρ-convergent.

Now assume that $E_n \in \mathcal{I}_\rho$ for each $n \in \omega$. Since $\rho(F) \subseteq \bigcup\{E_n : n \in \omega\}$, we can use Proposition 6.8 to find $E \in \mathcal{F}$ such that for each $n \in \omega$ there is a finite set $K \subseteq \Omega$ with $\rho(E \setminus K) \subseteq \rho(F) \cap \bigcup\{E_i : i \geq n\}$. We claim that $f \restriction \rho(E)$ is ρ-convergent to ∞. Let U be a neighborhood of ∞. Without loss of generality, we can assume that $U = \Phi(\mathcal{A}) \setminus (\{A_i : i < n\} \cup \bigcup_{i<n} A_i)$ for some $n \in \omega$. Let $K \subseteq \Omega$ be a finite set such that $\rho(E \setminus K) \subseteq \rho(F) \cap \bigcup\{E_i : i \geq n\}$. Then $f[\rho(E \setminus K)] \cap \{A_i : i < n\} = \emptyset$, and consequently $f[\rho(E \setminus K)] \subseteq U$.

Case (3a). There is $F \in \mathcal{F}$ such that $f \restriction \rho_1(F)$ is constant, hence it is ρ-convergent.

Case (3b). Let $A \in \mathcal{A}$ be such that $f^{-1}[A] \notin \mathcal{I}_\rho$. Using Proposition 6.8, we find $F \in \mathcal{F}$ such that $\rho(F) \subseteq f^{-1}[A]$ and for every finite $S \subseteq A$ there is a finite set $K \subseteq \Omega$ with $\rho(F \setminus K) \subseteq f^{-1}[A] \setminus f^{-1}[S] = f^{-1}[A \setminus S]$. We claim that $f \restriction \rho(F)$ is ρ-convergent to A. Indeed, let U be a neighborhood of A. Without loss of generality, we can assume that $U = \{A\} \cup (A \setminus S)$ where S is a finite subset of A. Then we take a finite $K \subseteq \Omega$ such that $\rho(F \setminus K) \subseteq f^{-1}[A \setminus S]$, so $f[\rho(F \setminus K)] \subseteq A \setminus S \subseteq U$. □

Proposition 12.2. *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$. Let $\mathcal{A}$ be an infinite almost disjoint family of infinite subsets of Λ. If ρ is P^- and $\mathcal{A}$ is* countable, *then $\Phi(\mathcal{A}) \in \mathrm{FinBW}(\rho)$.*

Proof. Let $f : \Lambda \to \Phi(\mathcal{A})$. By Lemma 12.1, we can assume that $f^{-1}(\infty) \in \mathcal{I}_\rho$, $f^{-1}[\mathcal{A}] \in \mathcal{I}_\rho$, $f^{-1}[\Lambda] \in \mathcal{I}_\rho^*$, $f^{-1}(\lambda) \in \mathcal{I}_\rho$ for every $\lambda \in \Lambda$ and $f^{-1}[A] \in \mathcal{I}_\rho$ for every $A \in \mathcal{A}$.

Since $f^{-1}[A] \in \mathcal{I}_\rho$ for every $A \in \mathcal{A}$, we can use Proposition 6.8 to find $F \in \mathcal{F}$ such that $\rho(F) \subseteq f^{-1}[\Lambda]$ and for any finite set $\mathcal{A}' \subseteq \mathcal{A}$ there is a finite set $K \subseteq \Omega$ with

$$\rho(F \setminus K) \cap f^{-1}\left[\bigcup \mathcal{A}'\right] = \rho(F \setminus K) \cap \left(\bigcup_{A \in \mathcal{A}'} f^{-1}[A]\right) = \emptyset.$$

We claim that $f \restriction \rho(F)$ is ρ-convergent to ∞. Indeed, let U be a neighborhood of ∞. Without loss of generality, we can assume that there is a finite set $\mathcal{A}' \subseteq \mathcal{A}$ such that $U = \Phi(\mathcal{A}) \setminus (\mathcal{A}' \cup \bigcup \mathcal{A}')$. Then we have a finite set $K \subseteq \Omega$ such that $\rho(F \setminus K) \cap f^{-1}[\bigcup \mathcal{A}'] = \emptyset$, and consequently $f[\rho(F \setminus K)] \subseteq U$, so the proof is finished. $\square$

13. Mrówka for maximal almost disjoint families

In [61] the author extensively studied $\mathrm{FinBW}(\mathcal{I})$ spaces. In particular, for a large class of ideals, assuming the continuum hypothesis, he characterized in terms of Katětov order when there is a space in $\mathrm{FinBW}(\mathcal{I})$ that is not in $\mathrm{FinBW}(\mathcal{J})$. In his proofs the right space is always of the form $\Phi(\mathcal{A})$ for some *maximal* almost disjoint family. In our paper we want to generalize results of [61] so that they will apply also for Hindman spaces, Ramsey spaces and differentially compact spaces. As we will see at the end of this section, our generalization requires going beyond *maximal* almost disjoint families (as always $\Phi(\mathcal{A}) \notin \mathrm{FinBW}(\rho)$ for maximal $\mathcal{A}$ and $\rho \in \{FS, r, \Delta\}$ – see Corollary 13.3) and working with almost disjoint families that are not necessarily maximal.

Lemma 13.1. *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$. Let $\mathcal{A}$ be an almost disjoint family on Λ.*

(1) If $\mathcal{A} \subseteq \mathcal{I}_\rho$ and
$$\forall F \in \mathcal{F}\, \exists A \in \mathcal{A}\, \forall K \in [\Omega]^{<\omega}\, (A \cap \rho(F \setminus K) \neq \emptyset),$$
then $\Phi(\mathcal{A}) \notin \mathrm{FinBW}(\rho)$.

(2) If $\mathcal{A} \subseteq \mathcal{I}_\rho$ and $\mathcal{A}$ is a maximal *almost disjoint family, then $\Phi(\mathcal{A}) \notin \mathrm{FinBW}(\rho)$.*

Proof. (1) Let $f : \Lambda \to \Phi(\mathcal{A})$ be given by $f(\lambda) = \lambda$. We claim that there is no $F \in \mathcal{F}$ such that $f \restriction \rho(F)$ is ρ-convergent. Assume, for sake of contradiction, that there is $F \in \mathcal{F}$ such that $f \restriction \rho(F)$ is ρ-convergent to some $L \in \Phi(\mathcal{A})$. We have three cases: (1) $L \in \Lambda$, (2) $L \in \mathcal{A}$, (3) $L = \infty$.

Case (1). The set $U = \{L\}$ is a neighborhood of L. But for any finite set $K \subseteq \Omega$ we have $f[\rho(F \setminus K)] = \rho(F \setminus K) \not\subseteq \{L\} = U$. Thus, $f \restriction \rho(F)$ is not ρ-convergent to L, a contradiction.

Case (2). The set $U = L \cup \{L\}$ is a neighborhood of L. Since $f \restriction \rho(F)$ is ρ-convergent to L, there is a finite set $K \subseteq \Omega$ such that $\rho(F \setminus K) = f[\rho(F \setminus K)] \subseteq U$. Thus, $\rho(F \setminus K) \subseteq L$, so $L \notin \mathcal{I}_\rho$, but $\mathcal{A} \subseteq \mathcal{I}_\rho$, a contradiction.

Case (3). Let $A \in \mathcal{A}$ be such that $A \cap \rho(F \setminus K) \neq \emptyset$ for every finite set $K \subseteq \Omega$. The set $U = \Phi(\mathcal{A}) \setminus (A \cup \{A\})$ is a neighborhood of ∞. Since $f \restriction \rho(F)$ is ρ-convergent to L, there is a finite set $K \subseteq \Omega$ such that $\rho(F \setminus K) = f[\rho(F \setminus K)] \subseteq U$. Hence, $A \cap \rho(F \setminus K) = \emptyset$, a contradiction.

(2) Let $F \in \mathcal{F}$ and enumerate $\Omega = \{o_n : n \in \omega\}$. We pick inductively a point $b_n \in \rho(F \setminus \{o_j : j < n\}) \setminus \{b_j : j < n\}$ for each $n \in \omega$. Then using maximality of $\mathcal{A}$ we can find $A \in \mathcal{A}$ such that $A \cap \{b_n : n \in \omega\}$ is infinite. Thus, the condition form item (1) is satisfied, so the proof is finished. $\square$

Theorem 13.2. *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$.*

(1) If ρ is tall (equivalently, $\mathcal{I}_\rho$ is a tall ideal), then there exists an infinite (even of cardinality $\mathfrak{c}$) maximal *almost disjoint family $\mathcal{A}$ on Λ such that $\Phi(\mathcal{A}) \notin \mathrm{FinBW}(\rho)$.*

(2) If $\mathcal{I}_\rho$ is not *$P^-(\Lambda)$ (equivalently, $\mathrm{Fin}^2 \leq_K \mathcal{I}_\rho$), then $\Phi(\mathcal{A}) \notin \mathrm{FinBW}(\rho)$ for every infinite* maximal *almost disjoint family $\mathcal{A}$ on Λ.*

Proof. (1) It follows from Lemma 13.1(2), because in [26, Proposition 2.2], the authors proved that if an ideal $\mathcal{I}$ is tall, then there exists an infinite maximal almost disjoint family $\mathcal{A}$ of infinite subsets of Λ such that $\mathcal{A} \subseteq \mathcal{I}$. If necessary, we can make $\mathcal{A}$ to be of cardinality $\mathfrak{c}$ (just take one set $A \in \mathcal{A}$, construct your favourite almost disjoint family $\mathcal{B}$ of cardinality $\mathfrak{c}$ on A, then any maximal almost disjoint family extending $\mathcal{A} \cup \mathcal{B}$ is the required family). The equivalence of ρ being tall and $\mathcal{I}_\rho$ being a tall ideal follows from Proposition 8.1.

(2) The equivalence of $\mathcal{I}_\rho$ not being $P^-(\Lambda)$ and $\mathrm{Fin}^2 \leq_K \mathcal{I}_\rho$ follows from Proposition 7.1(1).

Let $\phi : \Lambda \to \omega^2$ be a witness for $\mathrm{Fin}^2 \leq_K \mathcal{I}_\rho$. In [2], the authors proved that we can assume that ϕ is a bijection. For each $n \in \omega$, we define $P_n = \phi^{-1}[\{n\} \times \omega]$. Then $\{P_n : n \in \omega\}$ is a partition of Λ and $P_n \in \mathcal{I}_\rho \cap [\Lambda]^\omega$ for each $n \in \omega$. Let $\mathcal{A} = \{A_\alpha : \alpha < |\mathcal{A}|\}$. Since $\mathcal{A}$ is infinite, $|\mathcal{A}| \geq \omega$. Let $f : \Lambda \to \Phi(\mathcal{A})$ be a bijection such that $f[P_n] = A_n \setminus \bigcup\{A_i : i < n\}$ for each $n \in \omega$. We claim that f does not have a ρ-convergent subsequence. Assume, for the sake of contradiction, that $f \restriction \rho(F)$ is ρ-convergent to some $L \in \Phi(\mathcal{A})$ for some $F \in \mathcal{F}$. We have three cases: (1) $L \in \Lambda$, (2) $L \in \mathcal{A}$, (3) $L = \infty$.

Case (1). The set $U = \{L\}$ is a neighborhood of L. But for any finite set $K \subseteq \Omega$ we have $f[\rho(F \setminus K)] \not\subseteq \{L\} = U$. Thus, $f \restriction \rho(F)$ is not ρ-convergent to L, a contradiction.

Case (2). We have two subcases: (2a) $\exists n \in \omega\, (L = A_n)$, (2b) $\exists \alpha \in |\mathcal{A}| \setminus \omega\, (L = A_\alpha)$.

Case (2a). The set $U = \{A_n\} \cup A_n$ is a neighborhood of L, so there is a finite set $K \subseteq \Omega$ such that $f[\rho(F \setminus K)] \subseteq U$. Then $f[\rho(F \setminus K)] \subseteq A_n$, so $\rho(F \setminus K) \subseteq f^{-1}[A_n] \subseteq \bigcup_{i \leq n} P_i \in \mathcal{I}_\rho$, a contradiction.

Case (2b). The set $U = A_\alpha \cup \{A_\alpha\}$ is a neighborhood of L, so there is a finite set $K \subseteq \Omega$ such that $f[\rho(F \setminus K)] \subseteq U$. Then $f[\rho(F \setminus K)] \subseteq A_\alpha$, so $\rho(F \setminus K) \subseteq f^{-1}[A_\alpha]$. Thus $f^{-1}[A_\alpha] \notin \mathcal{I}_\rho$, and consequently $\phi[f^{-1}[A_\alpha]] \notin \mathrm{Fin}^2$. On the other hand, $A_\alpha \cap A_n$ is finite for each $n \in \omega$, so $f^{-1}[A_\alpha \cap A_n]$ is finite, and consequently $f^{-1}[A_\alpha] \cap P_n$ is finite for every $n \in \omega$. Thus, $\phi[f^{-1}[A_\alpha]] \in \mathrm{Fin}^2$, a contradiction.

Case (3). Using Proposition 9.2(1b) we find an infinite set $B \subseteq \Lambda$ such that $f \restriction B$ is convergent to ∞. Since f is a bijection, $f[B]$ is infinite. Thus, using maximality of $\mathcal{A}$, we find α such that $A_\alpha \cap f[B]$ is infinite. Since $U = \Phi(\mathcal{A}) \setminus (\{A_\alpha\} \cup A_\alpha)$ is a neighborhood of ∞, there is a finite set $K \subseteq \Lambda$ such that $f[B \setminus K] \subseteq U$. Then $A_\alpha \cap f[B \setminus K] = \emptyset$, a contradiction. $\square$

Corollary 13.3. *Let $\mathcal{A}$ be an infinite* maximal *almost disjoint family.*

- *(1) Hindman spaces.*
  - *(a)* [55, Theorem 10] *If $\mathcal{A} \subseteq \mathcal{H}$, then $\Phi(\mathcal{A})$ is not a Hindman space.*
  - *(b)* [22, Proposition 1.1] $\Phi(\mathcal{A})$ *is not a Hindman space.*
- *(2) Ramsey spaces.*
  - *(a)* [59, Example 4.1] *If $\{\{n, k\} : k \in \omega \setminus \{n\}\} \in \mathcal{A}$ for every $n \in \omega$, then $\Phi(\mathcal{A})$ is not a Ramsey space.*
  - *(b)* $\Phi(\mathcal{A})$ *is not a Ramsey space.*
- *(3) Differentially compact spaces.*
  - *(a)* [71, 4.2.2] *or* [20, Theorem 4.9] *If $\mathcal{A} \subseteq \mathcal{D}$, then $\Phi(\mathcal{A})$ is not a differentially compact space.*
  - *(b)* [58, Theorem 2.1] $\Phi(\mathcal{A})$ *is not a differentially compact space.*
- *(4) van der Waerden spaces.*
  - *(a)* [56, Theorem 6] *If $\mathcal{A} \subseteq \mathcal{W}$, then $\Phi(\mathcal{A})$ is not a van der Waerden space.*
- *(5) $\mathcal{I}_{1/n}$-spaces.*
  - *(a)* [29, Proposition 2.2] *If $\mathcal{A} \subseteq \mathcal{I}_{1/n}$, then $\Phi(\mathcal{A})$ is not a $\mathcal{I}_{1/n}$-space.*

(6) $\mathrm{FinBW}(\mathcal{I})$.
   (a) [29, Proposition 2.2] *If $\mathcal{I}$ is a tall F_σ-ideal on Λ and $\mathcal{A} \subseteq \mathcal{I}$, then $\Phi(\mathcal{A}) \notin \mathrm{FinBW}(\mathcal{I})$.*
   (b) [26, Proposition 2.3] *If $\mathcal{I}$ is a tall ideal and $\mathcal{A} \subseteq \mathcal{I}$, then $\Phi(\mathcal{A}) \notin \mathrm{FinBW}(\mathcal{I})$.*

Proof. (1)–(6) In cases when we assume that $\mathcal{A} \subseteq \mathcal{I}$, it follows from Lemma 13.1(2) along with Proposition 10.2(4) in some cases. In other cases, it follows from Theorem 13.2(2) and Proposition 7.2. □

14. Distinguishing between FinBW classes via Katětov order on ideals

In this section we prove first of the two main results of this part and show its various particular cases and consequences. We will need the following two lemmas.

Lemma 14.1. *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$. Let $\mathcal{A}$ be an infinite almost disjoint family on Λ such that for every $\mathcal{I}_\rho$-to-one function $f : \Lambda \to \Lambda$ there is $E \in \mathcal{F}$ such that the family*

$$\{A \in \mathcal{A} : \forall K \in [\Omega]^{<\omega}\, (|A \cap f[\rho(E \setminus K)]| = \omega)\}$$

is at most countable. If ρ is P^-, then $\Phi(\mathcal{A}) \in \mathrm{FinBW}(\rho)$.

Proof. Let $f : \Lambda \to \Phi(\mathcal{A})$. By Lemma 12.1, we can assume that $f^{-1}(\infty) \in \mathcal{I}_\rho$, $f^{-1}[\mathcal{A}] \in \mathcal{I}_\rho$, $f^{-1}[\Lambda] \in \mathcal{I}_\rho^*$, $f^{-1}(\lambda) \in \mathcal{I}_\rho$ for every $\lambda \in \Lambda$ and $f^{-1}[A] \in \mathcal{I}_\rho$ for every $A \in \mathcal{A}$. Then $B = f^{-1}[\Lambda] \in \mathcal{I}_\rho^*$ and $f \restriction B : B \to \Lambda$ is $\mathcal{I}_\rho$-to-one.

We fix an element $\lambda_0 \in \Lambda$ and define $g : \Lambda \to \Lambda$ by $g(\lambda) = f(\lambda)$ for $\lambda \in B$ and $g(\lambda) = \lambda_0$ otherwise. Then g is $\mathcal{I}_\rho$-to-one, so there is $E \in \mathcal{F}$ such that the family

$$\mathcal{C} = \{A \in \mathcal{A} : \forall K \in [\Omega]^{<\omega}\, (|A \cap g[\rho(E \setminus K)]| = \omega)\}$$

is at most countable.

Since $\rho(E) \cap B \notin \mathcal{I}_\rho$ and ρ is P^-, we can use Proposition 6.8 to find $F \in \mathcal{F}$ such that $\rho(F) \subseteq \rho(E) \cap B$ and for any finite sets $S \subseteq \Lambda$ and $\mathcal{T} \subseteq \mathcal{C}$ there is a finite set $L \subseteq \Omega$ with

$$\rho(F \setminus L) \cap \left(f^{-1}[S] \cup \bigcup \{f^{-1}[A] : A \in \mathcal{T}\}\right) = \emptyset.$$

We claim that $f \restriction \rho(F)$ is ρ-convergent to ∞. Indeed, let U be a neighborhood of ∞. Without loss of generality, we can assume that there is a finite set $\Gamma \subseteq \mathcal{A}$ such that $U = \Phi(\mathcal{A}) \setminus (\{A : A \in \Gamma\} \cup \bigcup\{A : A \in \Gamma\})$.

For each $A \in \Gamma \setminus \mathcal{C}$, there is a finite set $K_A \subseteq \Omega$ such that $A \cap f[\rho(F \setminus K_A)] = A \cap g[\rho(F \setminus K_A)]$ is finite. Then $K = \bigcup\{K_A : A \in \Gamma \setminus \mathcal{C}\} \subseteq \Omega$ is a finite set such that $A \cap f[\rho(F \setminus K)]$ is finite for every $A \in \Gamma \setminus \mathcal{C}$.

Then both $S = f[\rho(F \setminus K)] \cap \bigcup\{A : A \in \Gamma \setminus \mathcal{C}\}$ and $\mathcal{T} = \Gamma \cap \mathcal{C}$ are finite, so we can find a finite set $L \subseteq \Omega$ such that

$$\rho(F \setminus L) \cap \left(f^{-1}[S] \cup \bigcup \{f^{-1}[A] : A \in \mathcal{T}\}\right) = \emptyset,$$

and consequently we obtain a finite set $K \cup L$ such that

$$f[\rho(F \setminus (K \cup L))] \subseteq \Lambda \setminus \bigcup\{A : A \in \Gamma\} \subseteq U.$$

That finishes the proof. □

Recall that $\mathfrak{p}$ is the smallest cardinality of a family $\mathcal{F}$ of infinite subsets of ω with the strong finite intersection property (i.e. intersection of finitely many sets from $\mathcal{F}$ is infinite) that does not have a pseudointersection (i.e. there is no infinite set $A \subseteq \omega$ such that $A \setminus F$ is finite for each $F \in \mathcal{F}$; see e.g. [74]).

Lemma 14.2 (Assume $\mathfrak{p} = \mathfrak{c}$)**.** *Let $\rho_i : \mathcal{F}_i \to [\Lambda_i]^\omega$ be partition regular with $\mathcal{F}_i \subseteq [\Omega_i]^\omega$ for each $i = 1, 2$. Let $\{f_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all functions $f : \Lambda_1 \to \Lambda_2$ and $\mathcal{F}_2 = \{F_\alpha : \alpha < \mathfrak{c}\}$.*

If $\mathcal{I}_{\rho_2} \not\leq_K \mathcal{I}_{\rho_1}$, then there exist families $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $\mathcal{C} = \{C_\alpha : \alpha < \mathfrak{c}\}$ such that for every $\alpha < \mathfrak{c}$:

(1) $C_\alpha \in \mathcal{F}_1$,
(2) $f_\alpha[\rho_1(C_\alpha)] \in \mathcal{I}_{\rho_2}$,
(3) $A_\alpha \in \mathcal{I}_{\rho_2} \cap [\Lambda_2]^\omega$,
(4) $\forall \beta < \alpha\, (|A_\alpha \cap A_\beta| < \omega)$,
(5) $\forall \gamma > \alpha\, (|A_\gamma \cap f_\alpha[\rho_1(C_\alpha)]| < \omega)$,
(6) $\forall L \in [\Omega_2]^{<\omega}\, (A_\alpha \subseteq^* \rho_2(F_\alpha \setminus L))$.

Proof. Suppose that A_β and C_β have been constructed for $\beta < \alpha$ and satisfy items (1)-(6).

First, we construct the set C_α. Since $\mathcal{I}_{\rho_2} \not\leq_K \mathcal{I}_{\rho_1}$, there is a set $C_\alpha \in \mathcal{F}_1$ such that $f_\alpha[\rho_1(C_\alpha)] \in \mathcal{I}_{\rho_2}$.

Now, we turn to the construction of the set A_α. Let

$$\mathcal{D} = \{\Lambda_2 \setminus f_\beta[\rho_1(C_\beta)] : \beta < \alpha\} \cup \{\Lambda_2 \setminus A_\beta : \beta < \alpha\} \cup \{\rho_2(F_\alpha \setminus L) : L \in [\Omega_2]^{<\omega}\}.$$

Since $\bigcap\{\rho_2(F_\alpha \setminus L_i) : i < n\} \in \mathcal{I}^+_{\rho_2}$ for every $n \in \omega$ and finite sets $L_i \subseteq \Omega$, and $\Lambda_2 \setminus f_\beta[\rho_1(C_\beta)] \in \mathcal{I}^*_{\rho_2}$ and $\Lambda_2 \setminus A_\beta \in \mathcal{I}^*_{\rho_2}$ for every $\beta < \alpha$, we obtain that the intersection of finitely many sets from $\mathcal{D}$ is in $\mathcal{I}^+_{\rho_2}$. In particular, this intersection is infinite, so $\mathcal{D}$ has the strong finite intersection property. Since $|\mathcal{D}| < \mathfrak{c} = \mathfrak{p}$, there exists an infinite set $A \subseteq \Lambda_2$ such that $A \subseteq^* D$ for every $D \in \mathcal{D}$. Since $\mathcal{I}_{\rho_2} \not\leq_K \mathcal{I}_{\rho_1}$, we obtain that the ideal $\mathcal{I}_{\rho_2}$ is tall, and consequently there is an infinite set $A_\alpha \subseteq A$ such that $A_\alpha \in \mathcal{I}_{\rho_2}$.

It is not difficult to see, that the sets A_α and C_α satisfy all the required conditions, so the proof of the lemma is finished. □

We are ready for the main result of this section.

Theorem 14.3 (Assume CH)**.** *Let $\rho_i : \mathcal{F}_i \to [\Lambda_i]^\omega$ be partition regular for each $i = 1, 2$. If ρ_1 is P^- and $\mathcal{I}_{\rho_2} \not\leq_K \mathcal{I}_{\rho_1}$, then there exists an almost disjoint family $\mathcal{A}$ such that $|\mathcal{A}| = \mathfrak{c}$ and $\Phi(\mathcal{A}) \in \mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2)$. In particular, there is a Hausdorff compact and separable space of cardinality $\mathfrak{c}$ in $\mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2)$.*

Proof. Using Proposition 11.4 we can assume that $\Lambda_1 = \Lambda_2 = \Lambda$.

Let $\{f_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all functions $f : \Lambda \to \Lambda$ and $\mathcal{F}_2 = \{F_\alpha : \alpha < \mathfrak{c}\}$. By Lemma 14.2, there exist families $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $\mathcal{C} = \{C_\alpha : \alpha < \mathfrak{c}\}$ such that for every $\alpha < \mathfrak{c}$ all conditions of Lemma 14.2 are satisfied. We claim that $\mathcal{A}$ is the required family.

First, we see that $\mathcal{A}$ is an almost disjoint family on Λ by item (4) of Lemma 14.2, $|\mathcal{A}| = \mathfrak{c}$ and $\mathcal{A} \subseteq \mathcal{I}_{\rho_2}$. Second, CH together with item (5) of Lemma 14.2 ensures that

$$|\{\beta < \mathfrak{c} : \forall K \in [\Omega]^{<\omega}\, (|A_\beta \cap f_\alpha[\rho_1(C_\alpha \setminus K)]| = \omega)\}| \leq |\alpha + 1| \leq \omega$$

for each $\alpha < \mathfrak{c}$, so knowing that ρ_1 is P^- we can use Lemma 14.1 to see that $\Phi(\mathcal{A}) \in \mathrm{FinBW}(\rho_1)$. Third, we use item (6) of Lemma 14.2 and Lemma 13.1(1) to see that $\Phi(\mathcal{A}) \notin \mathrm{FinBW}(\rho_2)$. □

Now we want to show various applications of Theorem 14.3. Those applications can be divided into three parts. The first part concerns existence of a Hausdorff compact and separable space in $\mathrm{FinBW}(\rho)$. Before applying Theorem 14.3, we need to prove one more result.

Proposition 14.4. *For every ideal $\mathcal{I}$ there is an ideal $\mathcal{J}$ such that $\mathcal{J} \not\leq_K \mathcal{I}$.*

Proof. Suppose for the sake of contradiction that there is an ideal $\mathcal{I}$ on Λ such that $\mathcal{J} \leq_K \mathcal{I}$ for every ideal $\mathcal{J}$. Then for every maximal (with respect to inclusion) ideal $\mathcal{J}$ on ω there exists a function $f_{\mathcal{J}} : \Lambda \to \omega$ such that $f_{\mathcal{J}}^{-1}[A] \in \mathcal{I}$ for every $A \in \mathcal{J}$. Let $\mathcal{K}(f_{\mathcal{J}}) = \{A \subseteq \omega : f_{\mathcal{J}}^{-1}[A] \in \mathcal{I}\}$. Then $\mathcal{K}(f_{\mathcal{J}})$ is an ideal and $\mathcal{J} \subseteq \mathcal{K}(f_{\mathcal{J}})$. Since $\mathcal{J}$ is maximal, $\mathcal{K}(f_{\mathcal{J}}) = \mathcal{J}$. There are $2^{\mathfrak{c}}$ pairwise distinct ultrafilters on ω (see e.g. [49, Theorem 7.6]), so there are $2^{\mathfrak{c}}$ pairwise distinct maximal ideals on ω (given an ultrafilter $\mathcal{U}$ on ω, the family $\{A \subseteq \omega : A \notin \mathcal{U}\}$ is a maximal ideal). However, there are only $\mathfrak{c}$ many function from ω into ω, a contradiction. □

Theorem 14.5 (Assume CH). *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$. If ρ is P^-, then there exists an almost disjoint family $\mathcal{A}$ such that $|\mathcal{A}| = \mathfrak{c}$ and* $\Phi(\mathcal{A}) \in \mathrm{FinBW}(\rho)$. *In particular, there is a Hausdorff compact and separable space of cardinality $\mathfrak{c}$ in* $\mathrm{FinBW}(\rho)$.

Proof. By Proposition 14.4 there is an ideal $\mathcal{J}$ such that $\mathcal{J} \not\leq_K \mathcal{I}_\rho$, so Theorem 14.3 gives us an almost disjoint family $\mathcal{A}$ such that $|\mathcal{A}| = \mathfrak{c}$ and $\Phi(\mathcal{A}) \in \mathrm{FinBW}(\rho) \setminus \mathrm{FinBW}(\rho_{\mathcal{J}})$. □

Corollary 14.6 (Assume CH). *There exists (for each item distinct) an almost disjoint family $\mathcal{A}$ for which $\Phi(\mathcal{A})$ is a Hausdorff compact and separable space of cardinality $\mathfrak{c}$ such that:*

(1) $\Phi(\mathcal{A})$ *is a Hindman space,*
(2) [59, Theorem 4.7] $\Phi(\mathcal{A})$ *is a Ramsey space,*
(3) $\Phi(\mathcal{A})$ *is a differentially compact space,*
(4) [61, Theorem 5.3] $\Phi(\mathcal{A}) \in \mathrm{FinBW}(\mathcal{I})$*, where $\mathcal{I}$ is a P^- ideal (in particular, if $\mathcal{I}$ is a $G_{\delta\sigma\delta}$ ideal).*

Proof. Items (1), (2) and (3) follow from Theorem 14.5 and Proposition 6.7(3). Item (4) follows from Theorem 14.5 and Propositions 6.5(2) and 10.2(4), and the "in particular" part follows from Proposition 6.2(1). □

The second part of applications of Theorem 14.3 concerns a special case when $\mathcal{I}_{\rho_1}$ is $P^-(\Lambda_1)$ while $\mathcal{I}_{\rho_2}$ is *not* $P^-(\Lambda_2)$.

Theorem 14.7 (Assume CH). *Let $\rho_i : \mathcal{F}_i \to [\Lambda_i]^\omega$ be partition regular functions for each $i = 1, 2$. If ρ_1 is P^-, $\mathcal{I}_{\rho_1}$ is $P^-(\Lambda_1)$ (equivalently,* $\mathrm{Fin}^2 \not\leq_K \mathcal{I}_{\rho_1}$*) and $\mathcal{I}_{\rho_2}$ is* not $P^-(\Lambda_2)$ *(equivalently,* $\mathrm{Fin}^2 \leq_K \mathcal{I}_{\rho_2}$*), then there exists an infinite almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{c}$ such that* $\Phi(\mathcal{A}) \in \mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2)$. *In particular, there is a Hausdorff compact and separable space of cardinality $\mathfrak{c}$ in* $\mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2)$.

Proof. The equivalence of $\mathcal{I}_{\rho_1}$ being $P^-(\Lambda_1)$ and $\mathrm{Fin}^2 \not\leq_K \mathcal{I}_{\rho_1}$ follows from Proposition 7.1(1).

Since $\mathrm{Fin}^2 \leq_K \mathcal{I}_{\rho_2}$ and $\mathrm{Fin}^2 \not\leq_K \mathcal{I}_{\rho_1}$, we know that $\mathcal{I}_{\rho_2} \not\leq_K \mathcal{I}_{\rho_1}$, so Theorem 14.3 finishes the proof. □

Corollary 14.8 (Assume CH). *There exists (for each item distinct) an almost disjoint family $\mathcal{A}$ for which $\Phi(\mathcal{A})$ is a Hausdorff compact and separable space of cardinality $\mathfrak{c}$ and the following holds.*

(1) $\Phi(\mathcal{A})$ *is in* $\mathrm{FinBW}(\mathcal{I})$*, where $\mathcal{I}$ is a P^- ideal (in particular, if $\mathcal{I}$ is a $G_{\delta\sigma\delta}$ ideal), but $\Phi(\mathcal{A})$ is not a:*
   (a) [61, Corollary 11.5] *Hindman space,*
   (b) Ramsey space,
   (c) differentially compact space.
(2) *(a)* [57, Theorem 3] $\Phi(\mathcal{A})$ *is a van der Waerden space that is not a Hindman space.*

  - *(b)* [29, Theorem 4.4] *$\Phi(\mathcal{A})$ is an $\mathcal{I}_{1/n}$-space that is not a Hindman space.*
- *(3)*
  - *(a)* [71, Theorem 4.2.2] *$\Phi(\mathcal{A})$ is a van der Waerden space that is not a differentially compace space.*
  - *(b)* [58, Theorem 3.5] *$\Phi(\mathcal{A})$ is in* $\mathrm{FinBW}(\mathcal{I})$ *but it is not a differentially compact space for any P^+ ideal $\mathcal{I}$ (in particular, for any F_σ ideal). For instance, $\Phi(\mathcal{A})$ is an $\mathcal{I}_{1/n}$-space that is not a differentially compact space.*

Proof. Item (1) follows from Theorem 14.7 and Propositions 7.2(2), 10.2(4) and 6.5(2), 6.1(1). Other items follow from item (1), Theorem 6.2(2) and Propositions 6.7(1), 6.1(1). □

Now we deal with the third part of applications of Theorems 14.3, in which we need to use its full strength.

Corollary 14.9 (Assume CH)**.** *There exists (for each item distinct) an almost disjoint family $\mathcal{A}$ for which $\Phi(\mathcal{A})$ is a Hausdorff compact and separable space of cardinality $\mathfrak{c}$ such that*

- *(1) $\Phi(\mathcal{A})$ is a Ramsey space that is not a Hindman space;*
- *(2) $\Phi(\mathcal{A})$ is a Hindman space that is not a Ramsey space;*
- *(3) $\Phi(\mathcal{A})$ is a differentially compact space that is not a Hindman space;*
- *(4) $\Phi(\mathcal{A})$ is a differentially compact space that is not a Ramsey space.*

Proof. It follows from Theorems 14.3 and 7.7 and Proposition 6.7(3). □

Remark. The space from Corollary 14.9(3) yields the negative answer to [71, Question 4.2.2] (see also [20, Problem 1] and [58, Question 3]).

Corollary 14.10 (Assume CH)**.**

- *(1) If $\mathcal{I}$ is an ideal such that $\mathcal{I} \not\leq_K \mathcal{H}$ ($\mathcal{I} \not\leq_K \mathcal{R}$, $\mathcal{I} \not\leq_K \mathcal{D}$, resp.), then there exists an almost disjoint family $\mathcal{A}$ such that the Hausdorff compact and separable space $\Phi(\mathcal{A})$ of cardinality $\mathfrak{c}$ is a Hindman (Ramsey, differentially compact, resp.) space that is not in* $\mathrm{FinBW}(\mathcal{I})$.
- *(2) There exists an almost disjoint family $\mathcal{A}$ such that the Hausdorff compact and separable space $\Phi(\mathcal{A})$ of cardinality $\mathfrak{c}$ is a Hindman (Ramsey, differentially compact, resp.) space that is not an $\mathcal{I}_{1/n}$-space.*

Proof. (1) It follows from Theorem 14.3 and Propositions 6.7(3) and 10.2(4).

(2) It follows from item (1) and Theorem 7.7. □

Remark. In [22, Theorem 2.5], the authors constructed, assuming CH and $\mathcal{I} \not\leq_K \mathcal{H}$, a *non*-Hausdorff Hindman space that is not in $\mathrm{FinBW}(\mathcal{I})$. Corollary 14.10(1) strengthens this result to the case of Hausdorff spaces. Taking $\mathcal{I} = \mathcal{I}_{1/n}$, they obtained a positive answer to the question posed in [27], namely they constructed a (non-Hausdorff) Hindman space which is not $\mathcal{I}_{1/n}$-space. In Corollary 14.10(2), we obtained a *Hausdorff* answer to the above mentioned question.

Corollary 14.11 (Assume CH)**.**

- *(1)* [61, Theorem 9.3] *If $\mathcal{I}_1$ and $\mathcal{I}_2$ are ideals such that $\mathcal{I}_1$ is P^- (in particular, if $\mathcal{I}_1$ is a $G_{\delta\sigma\delta}$ ideal) and $\mathcal{I}_2 \not\leq_K \mathcal{I}_1$, then there exists an almost disjoint family $\mathcal{A}$ such that the Hausdorff compact and separable space $\Phi(\mathcal{A})$ of cardinality $\mathfrak{c}$ belongs to* $\mathrm{FinBW}(\mathcal{I}_1) \setminus \mathrm{FinBW}(\mathcal{I}_2)$.
- *(2)* [29, Theorem 3.3] *There exists an almost disjoint family $\mathcal{A}$ such that the Hausdorff compact and separable space $\Phi(\mathcal{A})$ of cardinality $\mathfrak{c}$ is a van der Waerden space that is not an $\mathcal{I}_{1/n}$-space.*

Proof. (1) It follows from Theorem 14.3 and Propositions 6.5(2), 10.2(4).

(2) It follows from item (1), Theorem 7.7(10) and Proposition 6.7(1). □

15. Distinguishing between FinBW classes via Katětov order on partition regular functions

In this section we prove the second of the main results of Part 3. Then we compare it with Theorem 14.3 and show that none of them can be derived from the other one. We start with a technical lemma.

Lemma 15.1 (Assume CH)**.** *Let $\rho_i : \mathcal{F}_i \to [\Lambda_i]^\omega$ be partition regular with $\mathcal{F}_i \subseteq [\Omega_i]^\omega$ for each $i = 1, 2$. Let $\{f_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all functions $f : \Lambda_1 \to \Lambda_2$ and $\{F_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all sets $F \in \mathcal{F}_2$ having small accretions.*

If ρ_2 is P^- and $\rho_2 \not\leq_K \rho_1$, then there exist families $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $\mathcal{C} = \{C_\alpha : \alpha < \mathfrak{c}\}$ such that for every $\alpha < \mathfrak{c}$:

(1) $A_\alpha = \emptyset \vee A_\alpha \in \mathcal{I}_{\rho_2} \cap [\Lambda_2]^\omega$,
(2) $\forall \beta < \alpha\, (|A_\alpha \cap A_\beta| < \omega)$,
(3) $C_\alpha \in \mathcal{F}_1$,
(4) $\forall F \in \mathcal{F}_2\, \exists K \in [\Omega_1]^{<\omega}\, \forall L \in [\Omega_2]^{<\omega}\, (\rho_2(F \setminus L) \not\subseteq f_\alpha[\rho_1(C_\alpha \setminus K)])$,
(5) $\forall \gamma > \alpha\, \exists K \in [\Omega_1]^{<\omega}\, (|A_\gamma \cap f_\alpha[\rho_1(C_\alpha \setminus K)]| < \omega)$,
(6) $\exists \beta \leq \alpha\, \forall L \in [\Omega_2]^{<\omega}\, (A_\beta \cap \rho_2(F_\alpha \setminus L) \neq \emptyset)$.

Proof. Suppose that A_β and C_β have been constructed for $\beta < \alpha$ and satisfy all the required conditions.

First, we construct a set C_α. Since $\rho_2 \not\leq_K \rho_1$, there is a set $C_\alpha \in \mathcal{F}_1$ such that

$$\forall F \in \mathcal{F}_2\, \exists K \in [\Omega_1]^{<\omega}\, \forall L \in [\Omega_2]^{<\omega}\, (\rho_2(F \setminus L) \not\subseteq f_\alpha[\rho_1(C_\alpha \setminus K)]).$$

Now, we turn to the construction of a set A_α. We have two cases:

(1) $\exists \beta < \alpha\, \forall L \in [\Omega_2]^{<\omega}\, (A_\beta \cap \rho_2(F_\alpha \setminus L) \neq \emptyset)$.
(2) $\forall \beta < \alpha\, \exists L_\beta \in [\Omega_2]^{<\omega}\, (A_\beta \cap \rho_2(F_\alpha \setminus L_\beta) = \emptyset)$.

Case (1). We put $A_\alpha = \emptyset$. Then the sets A_α and C_α satisfy all the required conditions, so the proof of the lemma is finished in this case.

Case (2). Let $\alpha = \{\beta_n : n \in \omega\}$. Let $\{L_n : n \in \omega\}$ be an increasing sequence of finite subsets of Ω_2 such that $\bigcup\{L_{\beta_i} : i < n\} \subseteq L_n$ and $\bigcup\{L_n : n \in \omega\} = \Omega_2$. Notice that $\rho_2(F_\alpha \setminus L_n) \cap \bigcup\{A_{\beta_i} : i < n\} = \emptyset$ for every $n \in \omega$.

We define inductively sequences $\{E_n : n \in \omega\} \subseteq \mathcal{F}_2$, $\{K_n : n \in \omega\} \subseteq [\Omega_1]^{<\omega}$ and $\{a_n : n \in \omega\} \subseteq \Lambda_2$ such that for every $n \in \omega$ the following conditions hold:

(i) $\rho_2(E_{n+1}) \subseteq \rho_2(E_n) \subseteq \rho_2(F_\alpha)$,
(ii) $\rho_2(E_n) \subseteq \rho_2(F_\alpha \setminus L_n) \setminus f_{\beta_n}[\rho_1(C_{\beta_n} \setminus K_n)]$,
(iii) $a_n \in \rho_2(E_n) \setminus \{a_i : i < n\}$.

Suppose that E_i, K_i and a_i have been constructed for $i < n$ and satisfy all the required conditions.

Since F_α has small accretions, we obtain $\rho_2(F_\alpha \setminus L_{n-1}) \setminus \rho_2(F_\alpha \setminus L_n) \in \mathcal{I}_{\rho_2}$, and consequently $\rho_2(E_{n-1}) \cap \rho_2(F_\alpha \setminus L_n) \notin \mathcal{I}_{\rho_2}$ (in the case of $n = 0$ we put $L_{-1} = \emptyset$ and $E_{-1} = F_\alpha$). Let $E \in \mathcal{F}_2$ be such that $\rho_2(E) \subseteq \rho_2(E_{n-1}) \cap \rho_2(F_\alpha \setminus L_n)$. We have 2 subcases:

(2a) $\exists K_n \in [\Omega_1]^{<\omega}\, (\rho_2(E) \setminus f_{\beta_n}[\rho_1(C_{\beta_n} \setminus K_n)] \notin \mathcal{I}_{\rho_2})$,
(2b) $\forall K \in [\Omega_1]^{<\omega}\, (\rho_2(E) \setminus f_{\beta_n}[\rho_1(C_{\beta_n} \setminus K)] \in \mathcal{I}_{\rho_2})$.

Case (2a). We take $E_n \in \mathcal{F}_2$ such that $\rho_2(E_n) \subseteq \rho_2(E) \setminus f_{\beta_n}[\rho_1(C_{\beta_n} \setminus K_n)]$ and pick any $a_n \in \rho_2(E_n) \setminus \{a_i : i < n\}$. Then E_n, K_n and a_n satisfy all the required conditions.

Case (2b). It will turn out, that this subcase is impossible. Let $\{M_i : i \in \omega\}$ be an increasing sequence of finite subsets of Ω_1 such that $\bigcup\{M_i : i \in \omega\} = \Omega_1$.

We have 2 further subcases:

(2b-1) $\bigcup\{\rho_2(E) \setminus f_{\beta_n}[\rho_1(C_{\beta_n} \setminus M_i)] : i \in \omega\} \notin \mathcal{I}_{\rho_2}$, (2b-2) $\bigcup\{\rho_2(E) \setminus f_{\beta_n}[\rho_1(C_{\beta_n} \setminus M_i)] : i \in \omega\} \in \mathcal{I}_{\rho_2}$.

Case (2b-1). Since ρ_2 is P^-, there is $G \in \mathcal{F}_2$ such that $\rho_2(G) \subseteq \bigcup\{\rho_2(E) \setminus f_{\beta_n}[\rho_1(C_{\beta_n} \setminus M_i)] : i \in \omega\}$ and for every $i \in \omega$ there is a finite set $L \subseteq \Omega_2$ such that $\rho_2(G \setminus L) \subseteq f_{\beta_n}[\rho_1(C_{\beta_n} \setminus M_i)]$. On the other hand, from the inductive assumptions (more precisely: since C_{β_n} satisfies item 4), we know that there is a finite set K such that $\rho_2(G \setminus L) \not\subseteq f_{\beta_n}[\rho_1(C_{\beta_n} \setminus K)]$ for any finite set L. Let $i \in \omega$ be such that $K \subseteq M_i$. Then there is a finite set $L \subseteq \Omega_2$ such that $\rho_2(G\setminus L) \subseteq f_{\beta_n}[\rho_1(C_{\beta_n}\setminus M_i)] \subseteq f_{\beta_n}[\rho_1(C_{\beta_n} \setminus K)]$, a contradiction.

Case (2b-2). In this case, there is $G \in \mathcal{F}_2$ such that $\rho_2(G) \subseteq \rho_2(E) \setminus \bigcup\{\rho_2(E) \setminus f_{\beta_n}[\rho_1(C_{\beta_n} \setminus M_i)] : i \in \omega\} = \rho_2(E) \cap \bigcap\{f_{\beta_n}[\rho_1(C_{\beta_n} \setminus M_i)] : i \in \omega\}$. From the inductive assumptions, we know that there is a finite set K such that $\rho_2(G \setminus L) \not\subseteq f_{\beta_n}[\rho_1(C_{\beta_n} \setminus K)]$ for any finite set L. Let $i \in \omega$ be such that $K \subseteq M_i$. Then $\rho_2(G) \subseteq f_{\beta_n}[\rho_1(C_{\beta_n} \setminus M_i)] \subseteq f_{\beta_n}[\rho_1(C_{\beta_n} \setminus K)]$, a contradiction.

The construction of E_n, K_n and a_n is finished.

We define $A = \{a_n : n \in \omega\}$. Since $\rho_2 \not\leq_K \rho_1$, we obtain that ρ_2 is tall. Thus $\mathcal{I}_{\rho_2}$ is a tall ideal (by Proposition 8.1). Since A is infinite, there is an infinite set $A_\alpha \subseteq A$ such that $A_\alpha \in \mathcal{I}_{\rho_2}$.

It is not difficult to see, that the sets A_α and C_α satisfy all the required conditions, so the proof of the lemma is finished. □

The main result of this section is as follows.

Theorem 15.2 (Assume CH)**.** *Let $\rho_i : \mathcal{F}_i \to [\Lambda_i]^\omega$ be partition regular for each $i = 1, 2$. If ρ_1 and ρ_2 are P^-, ρ_2 has small accretions and $\rho_2 \not\leq_K \rho_1$, then there exists an almost disjoint family $\mathcal{A}$ such that $|\mathcal{A}| = \mathfrak{c}$, $\mathcal{A} \subseteq \mathcal{I}_{\rho_2}$ and $\Phi(\mathcal{A}) \in \mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2)$. In particular, there is a Hausdorff compact and separable space of cardinality $\mathfrak{c}$ in $\mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2)$.*

Proof. Using Proposition 11.4 we can assume that $\Lambda_1 = \Lambda_2 = \Lambda$. Let $\{f_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all functions $f : \Lambda_1 \to \Lambda_2$ and $\{F_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all sets $F \in \mathcal{F}_2$ having small accretions. By Lemma 15.1, there exist families $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $\mathcal{C} = \{C_\alpha : \alpha < \mathfrak{c}\}$ such that for every $\alpha < \mathfrak{c}$ all the required conditions of Lemma 15.1 are satisfied. We claim that $\mathcal{A}\setminus\{\emptyset\}$ is the required family.

First, we see that $\mathcal{A} \setminus \{\emptyset\}$ is an almost disjoint family on Λ_2 (by item (2) of Lemma 15.1) and $\mathcal{A} \setminus \{\emptyset\} \subseteq \mathcal{I}_{\rho_2}$.

Second, let CH together with item (5) of Lemma 15.1 ensures that

$$|\{\beta < \mathfrak{c} : \forall K \in [\Omega]^{<\omega}\, (|A_\beta \cap f_\alpha[\rho_1(C_\alpha \setminus K)]| = \omega)\}| \leq |\alpha + 1| \leq \omega$$

for each $\alpha < \mathfrak{c}$, so knowing that ρ_1 is P^- we can use Lemma 14.1 to see that $\Phi(\mathcal{A}) \in \mathrm{FinBW}(\rho_1)$.

Third, we use Lemma 13.1(1) along with item (6) of Lemma 15.1 and the fact that ρ_2 has small accretions to see that $\Phi(\mathcal{A}) \notin \mathrm{FinBW}(\rho_2)$.

Finally, using Proposition 12.2 we know that $\mathcal{A}$ cannot be countable, so $|\mathcal{A} \setminus \{\emptyset\}| = \mathfrak{c}$. □

Now we want to compare Theorem 15.2 with Theorem 14.3. Next two examples show that there are partition regular functions ρ_1 and ρ_2 satisfying the assumptions of Theorem 14.3 (so it gives us, under CH, a space in $\mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2)$), but not satisfying assumptions of Theorem 15.2 (i.e. we cannot apply it).

Example 15.3. There exist partition regular functions ρ_1 and ρ_2 such that ρ_1 is P^-, ρ_2 is not P^- (so we cannot apply Theorem 15.2) and $\mathcal{I}_{\rho_2} \not\leq_K \mathcal{I}_{\rho_1}$.

Proof. Let $\rho_1 = \rho_{\mathcal{I}_{1/n}}$ and $\rho_2 = \rho_{\mathcal{H}}$. By Theorem 6.7, $\mathcal{I}_{1/n}$ is P^+ (hence, P^-) and $\mathcal{H}$ is not $P^-(\omega)$ (hence, not P^-). Applying Proposition 6.5(2), we see that ρ_1 is P^- and ρ_2 is not P^-. By Theorem 7.7(12), $\mathcal{H} \not\leq_K \mathcal{I}_{1/n}$. □

The above example may not be satisfactory as all Hausdorff spaces from the class $\mathrm{FinBW}(\rho_2)$ are finite (by Theorem 10.5(3) and Proposition 10.2(4)), so one could just use Theorem 10.5(3) instead of Theorem 14.3. The next example is more sophisticated.

Example 15.4. There exist partition regular functions ρ_1 and ρ_2 such that ρ_1 is P^-, ρ_2 is not P^- (so we cannot apply Theorem 15.2), $\mathcal{I}_{\rho_2} \not\leq_K \mathcal{I}_{\rho_1}$ and under CH there is a Hausdorff compact separable space of cardinality $\mathfrak{c}$ in $\mathrm{FinBW}(\rho_2)$.

Proof. Let $\rho_1 = \rho_{\mathcal{I}_{1/n}}$ and $\rho_2 = \rho_{\mathrm{conv}}$, where conv is an ideal on $\mathbb{Q} \cap [0,1]$ consisting of those subsets of $\mathbb{Q} \cap [0,1]$ that have only finitely many cluster points in $[0,1]$. Then, $\mathrm{FinBW}(\rho_2) = \mathrm{FinBW}(\mathrm{conv})$ (Proposition 10.2(4)). Applying [61, Definition 4.3, Proposition 4.6 and Theorem 6.6], assuming CH, there is a Hausdorff compact separable space of cardinality $\mathfrak{c}$ in $\mathrm{FinBW}(\rho_2)$. Moreover, ρ_1 is P^- and ρ_2 is not P^- (by Proposition 6.5(2), Theorem 6.7 and [61, proof of Proposition 4.10(b)]). Finally, $\mathcal{I}_{\rho_2} \not\leq_K \mathcal{I}_{\rho_1}$ ([46, Section 2]). □

Next example shows that there are partition regular functions ρ_1 and ρ_2 satisfying the assumptions of Theorem 15.2 (so it gives us, under CH, a space in $\mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2)$), but not satisfying assumptions of Theorem 14.3 (i.e. we cannot apply it).

Example 15.5. There exist partition regular functions ρ_1 and ρ_2 with small accretions which are P^- and such that $\mathcal{I}_{\rho_2} \subseteq \mathcal{I}_{\rho_1}$ (in particular, $\mathcal{I}_{\rho_2} \leq_K \mathcal{I}_{\rho_1}$, so we cannot apply Theorem 14.3), but $\rho_2 \not\leq_K \rho_1$.

Proof. Consider the ideal $\mathrm{nwd} = \{A \subseteq \mathbb{Q} \cap [0,1] : \overline{A} \text{ is meager}\}$. Let $\rho_2 = \rho_{\mathrm{nwd}}$.

Fix an almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{c}$, enumerate it as $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and denote $\mathcal{A}' = \{A \setminus K : A \in \mathcal{A},\ K \in [\omega]^{<\omega}\}$. Let $I_n = [\frac{1}{n+2}, \frac{1}{n+1})$ for all $n \in \omega$. Enumerate also the set $\mathcal{B} = \{B \subseteq \mathbb{Q} \cap [0,1] : B \cap I_n \notin \mathrm{nwd}$ for infinitely many $n \in \omega\}$ as $\{B_\alpha : \alpha < \mathfrak{c}\}$. Let $\rho_1 : \mathcal{A}' \to [\mathbb{Q} \cap [0,1]]^\omega$ be given by $\rho_1(A_\alpha \setminus K) = B_\alpha \setminus \bigcup_{n \in K} I_n$.

Observe that $\mathcal{I}_{\rho_1} = \{A \subseteq \mathbb{Q} \cap [0,1] : \exists_{K \in \mathrm{Fin}}\ \overline{A} \setminus \bigcup_{n \in K} I_n \text{ is meager}\}$. Thus, $\mathrm{nwd} \subseteq \mathcal{I}_{\rho_1}$ and $\mathcal{I}_{\rho_2} \leq_K \mathcal{I}_{\rho_1}$. Moreover, it is easy to see that ρ_1 and ρ_2 both have small accretions (in the case of ρ_2 just apply Proposition 4.3). Since nwd is $F_{\sigma\delta}$ (see [16, Theorem 3]), it is P^- (by Proposition 6.2(1)) and consequently ρ_2 is P^- (by Proposition 6.5(2)).

Now we show that ρ_1 is P^-. Suppose that $\{C_n : n \in \omega\} \subseteq \mathcal{I}^+_{\rho_1}$ is decreasing and such that $C_n \setminus C_{n+1} \in \mathcal{I}_{\rho_1}$ for all $n \in \omega$. For each $n \in \omega$ let $T_n = \{i \in \omega : C_n \cap I_i \notin \mathrm{nwd}\}$.

Assume first that $T = \bigcap_{n \in \omega} T_n$ is infinite. Since nwd is P^-, for each $i \in T$ we can find $D_i \notin \mathrm{nwd}$, $D_i \subseteq I_i$ with $D_i \subseteq^* C_n$ for all $n \in \omega$. Then for $E = \bigcup_{i \in T} D_i \cap C_i$ we have $E \in \mathcal{B}$ (as $D_i \notin \mathrm{nwd}$ and $D_i \setminus C_i$ is finite for all $i \in T$). Hence, $E = B_\alpha$ for some $\alpha < \mathfrak{c}$. Moreover, for each $n \in \omega$ we have $\rho_1(A_\alpha \setminus n) = E \setminus \bigcup_{i<n} I_i = \bigcup_{i \in T, i \geq n} D_i \cap C_i \subseteq C_n$.

Assume now that T is finite. Inductively pick $i_n \in \omega$ and $D_n \notin \mathrm{nwd}$ such that $i_{n+1} > i_n$ and $D_n \subseteq I_{i_n} \cap C_n$ for all $n \in \omega$. Define $E = \bigcup_{n \in \omega} D_n$ and note that $E \in \mathcal{B}$. Hence, $E = B_\alpha$ for some $\alpha < \mathfrak{c}$. Moreover, for each $n \in \omega$ we have $\rho_1(A_\alpha \setminus i_n) = E \setminus \bigcup_{i < i_n} I_i = \bigcup_{i \geq n} D_i \subseteq C_n$.

Finally, we will show that $\rho_2 \not\leq_K \rho_1$. Fix any $f : \mathbb{Q} \cap [0,1] \to \mathbb{Q} \cap [0,1]$. For each $n \in \omega$ find $r_n \in \mathbb{Q} \cap [0,1]$ such that $f^{-1}[(r_n - \frac{1}{2^n}, r_n + \frac{1}{2^n})] \cap I_n \notin \mathrm{nwd}$. This is possible as $[0,1]$ can be covered by finitely many intervals of the form $(r - \frac{1}{2^n}, r + \frac{1}{2^n})$ and $I_n \cap (\mathbb{Q} \cap [0,1]) \notin \mathrm{nwd}$. Since $[0,1]$ is sequentially compact, there is an infinite $S \subseteq \omega$ such that $(r_n)_{n \in S}$ converges to some $x \in [0,1]$. Put $F = \bigcup_{n \in S} f^{-1}[(r_n - \frac{1}{2^n}, r_n + \frac{1}{2^n})] \cap I_n$. Then $F \in \mathcal{B}$ (in particular, $F \in \mathcal{I}^+_{\rho_1}$), so $F = B_\alpha$ for some $\alpha < \mathfrak{c}$.

Fix any $E \in \mathrm{nwd}^+$ and enumerate $S = \{s_i : i \in \omega\}$ in such a way that $s_i < s_j$ whenever $i < j$. Observe that $E \cap ((r_{s_i} - \frac{1}{2^{s_i}}, r_{s_i} + \frac{1}{2^{s_i}}) \setminus \bigcup_{j>i}(r_{s_j} - \frac{1}{2^{s_j}}, r_{s_j} + \frac{1}{2^{s_j}}))$ is infinite for some $i \in \omega$ as otherwise E would converge to x, so $E \in \mathrm{nwd}$.

We claim that for every finite set $L \subseteq \mathbb{Q} \cap [0,1]$ we have:

$$\begin{aligned} E \setminus L = \rho_1(E \setminus L) &\not\subseteq f[\rho_2(A_\alpha \setminus (s_i+1))] \\ &= f\left[F \setminus \bigcup_{j \leq i} (f^{-1}[(r_{s_j} - \frac{1}{2^{s_j}}, r_{s_j} + \frac{1}{2^{s_j}})] \cap I_{s_j})\right]. \end{aligned}$$

Let $L \subseteq \mathbb{Q} \cap [0,1]$ be a finite set. We will show that $E \setminus L \not\subseteq f[F \setminus \bigcup_{j \leq i}(f^{-1}[(r_{s_j} - \frac{1}{2^{s_j}}, r_{s_j} + \frac{1}{2^{s_j}})]$. Suppose that $E \setminus L \subseteq f[F \setminus \bigcup_{j \leq i}(f^{-1}[(r_{s_j} - \frac{1}{2^{s_j}}, r_{s_j} + \frac{1}{2^{s_j}})]$. Let

$$x \in E \cap \left(\left(r_{s_i} - \frac{1}{2^{s_i}}, r_{s_i} + \frac{1}{2^{s_i}} \right) \setminus \bigcup_{j>i} \left(r_{s_j} - \frac{1}{2^{s_j}}, r_{s_j} + \frac{1}{2^{s_j}} \right) \right) \setminus L \subseteq E \setminus L.$$

Then

$$\begin{aligned} x &\in f\left[F \setminus \bigcup_{j \leq i} \left(f^{-1}\left[\left(r_{s_j} - \frac{1}{2^{s_j}}, r_{s_j} + \frac{1}{2^{s_j}} \right) \right] \cap I_{s_j} \right) \right] \\ &= f\left[\bigcup_{j>i} \left(f^{-1}\left[\left(r_{s_j} - \frac{1}{2^{s_j}}, r_{s_j} + \frac{1}{2^{s_j}} \right) \right] \cap I_{s_j} \right) \right] \\ &\subseteq f\left[\bigcup_{j>i} \left(f^{-1}\left[\left(r_{s_j} - \frac{1}{2^{s_j}}, r_{s_j} + \frac{1}{2^{s_j}} \right) \right] \right) \right] \\ &= f\left[f^{-1}\left[\bigcup_{j>i} \left(r_{s_j} - \frac{1}{2^{s_j}}, r_{s_j} + \frac{1}{2^{s_j}} \right) \right] \right] \\ &\subseteq \bigcup_{j>i} \left(r_{s_j} - \frac{1}{2^{s_j}}, r_{s_j} + \frac{1}{2^{s_j}} \right). \end{aligned}$$

A contradiction. □

Part 4. Characterizations

In the final part we want to characterize when $\mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2) \neq \emptyset$ in the cases of $\rho_1 \in \{FS, r, \Delta\} \cup \{\rho_\mathcal{I} : \mathcal{I} \text{ is an ideal}\}$. In the realm of partition regular functions that are P^- and have small accretions we were able to obtain a full characterization (Theorem 16.1(1)) using Theorem 15.2. If $\rho_1 = \rho_\mathcal{I}$ for some P^- ideal $\mathcal{I}$, then Theorem 14.3 gives us a complete characterization (Theorem 16.1(2b)) and this problem for $\rho_1 = \rho_\mathcal{I}$ in the case of non-P^- ideals $\mathcal{I}$ is rather complicated (see [61] and Example 16.3). However, for instance in the case of $\rho_1 = FS$ and ρ_2 not being P^-, we needed another construction – we were able to obtain a characterization (Theorem 17.2), but only in the realm of spaces with unique limits of sequences (which are not necessarily Hausdorff).

16. Characterizations of distinguishness between FinBW classes via Katětov order

Theorem 16.1 (Assume CH). *Let ρ_1 and ρ_2 be partition regular functions. Let $\mathcal{I}_1$ be an ideal.*

(1) If ρ_1 is P^- and ρ_2 is P^- with small accretions, then
$$\rho_2 \not\leq_K \rho_1 \iff \mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2) \neq \emptyset.$$
(2) (a) If ρ_1 is P^- and ρ_2 is P^+, then
$$\mathcal{I}_{\rho_2} \not\leq_K \mathcal{I}_{\rho_1} \iff \mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2) \neq \emptyset.$$
(b) If $\mathcal{I}_1$ is P^-, then
$$\mathcal{I}_{\rho_2} \not\leq_K \mathcal{I}_1 \iff \mathrm{FinBW}(\mathcal{I}_1) \setminus \mathrm{FinBW}(\rho_2) \neq \emptyset.$$

Moreover, in every item an example showing that the above difference between FinBW *classes is nonempty is of the form $\Phi(\mathcal{A})$ with $\mathcal{A}$ being almost disjoint and of cardinality $\mathfrak{c}$ (in particular, these examples are Hausdorff, compact, separable and of cardinality $\mathfrak{c}$).*

Proof. (1) The implication " $\Longrightarrow$ " follows from Theorem 15.2, whereas the implication " $\Longleftarrow$ " follows from Theorem 11.1(1).

(2a) The implication " $\Longrightarrow$ " follows from Theorem 14.3, whereas the implication " $\Longleftarrow$ " follows from Theorem 11.1(2a).

(2b) It follows from Theorems 14.3, 11.1(2b), 10.2(4) and Proposition 6.5(2). □

Next two examples show that in Theorem 16.1 we cannot drop the assumption that ρ_1 is P^- and obtain a characterization in the realm of Hausdorff spaces.

Example 16.2. There exist partition regular functions ρ_1 and ρ_2 with small accretions such that:

(1) ρ_1 is not P^- and ρ_2 is P^+,
(2) $\rho_2 \not\leq_K \rho_1$,
(3) there is no *Hausdorff* space in $\mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2)$.

Proof. Let $\rho_1 = \rho_{\mathcal{H}}$ and $\rho_2 = \rho_{\mathcal{I}_{1/n}}$. Then ρ_1 and ρ_2 have small accretions (by Proposition 4.3). By Theorem 6.7, $\mathcal{I}_{1/n}$ is P^+ and $\mathcal{H}$ is not $P^-(\omega)$ (hence, not P^-). Applying Proposition 6.5(2), we see that ρ_1 is not P^- and ρ_2 is P^+. By Theorem 7.7(8), $\mathcal{I}_2 \not\leq_K \mathcal{I}_1$, so $\rho_2 \not\leq_K \rho_1$ (by Proposition 7.5(2b)).

By Theorem 10.5(3), $\mathrm{FinBW}(\mathcal{H})$ contains only finite Hausdorff spaces. On the other hand, $\mathrm{FinBW}(\mathcal{I}_2)$ contains all finite spaces (Theorem 10.5(1)), so there is no Hausdorff space $\mathrm{FinBW}(\mathcal{H}) \setminus \mathrm{FinBW}(\mathcal{I}_{1/n})$. Applying Proposition 10.2(4), we obtain that there is no Hausdorff space in $\mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2)$. □

The above example may not be satisfactory as all Hausdorff spaces from $\mathrm{FinBW}(\rho_1)$ are finite. The next example is more sophisticated.

Example 16.3. There exist partition regular functions ρ_1 and ρ_2 with small accretions such that:

(1) ρ_1 is not P^- and ρ_2 is P^-,
(2) assuming CH, there is a Hausdorff, compact, separable space of cardinality $\mathfrak{c}$ in $\mathrm{FinBW}(\rho_1)$,
(3) $\rho_2 \not\leq_K \rho_1$,
(4) there is no *Hausdorff* space in $\mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2)$.

Proof. Let $\mathcal{I}$ and $\mathcal{J}$ be the ideals from [61, Example 8.9] and define $\rho_1 = \rho_{\mathcal{I}}$ and $\rho_2 = \rho_{\mathcal{J}}$. Then ρ_1 and ρ_2 have small accretions (by Proposition 4.3) and $\mathcal{J} \not\leq_K \mathcal{I}$, so $\rho_2 \not\leq_K \rho_1$ (by Proposition 7.5(2b)). By [61, Example 10.6] and Proposition 10.2(4), there is no *Hausdorff* space in $\mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2)$. Applying [61, Theorem 6.6] and Proposition 10.2(4) we see that, assuming CH, there is a Hausdorff, compact, separable space of cardinality $\mathfrak{c}$ in $\mathrm{FinBW}(\rho_1)$. Since $\mathcal{J}$ is P^-, ρ_2 is P^- (by Proposition 6.5(2)) and ρ_1 cannot be P^- as it would contradict Theorem 16.1(1). □

Question 16.4. Can we drop the assumption that ρ_2 is P^- in Theorem 16.1 and obtain the characterization in the realm of Hausdorff spaces?

In Theorem 17.2, we show that we can drop the assumption that ρ_2 is P^- in Theorem 16.1(1) and obtain a characterization in the realm of *non*-Hausdorff spaces with *unique limits* of sequences, but at the cost of requiring that ρ_1 is weak P^+ instead of P^-.

Now we present some applications of Theorem 16.1.

Corollary 16.5 ([61, Theorem 10.4])**.** *Assume CH. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be ideals. If $\mathcal{I}_1$ is P^-, then the following are equivalent:*

(1) $\mathcal{I}_2 \not\leq_K \mathcal{I}_1$,

(2) $\mathrm{FinBW}(\mathcal{I}_1) \setminus \mathrm{FinBW}(\mathcal{I}_2) \neq \emptyset$.

Moreover, an example showing that the above difference between FinBW *classes is nonempty is of the form $\Phi(\mathcal{A})$ with $\mathcal{A}$ being almost disjoint and of cardinality $\mathfrak{c}$ (in particular, these examples are Hausdorff, compact, separable and of cardinality $\mathfrak{c}$).*

Proof. It follows from Theorems 16.1(2b) and 10.2(4). □

Question 16.6. Is every $\mathcal{I}_{1/n}$-space (Hindman space, Ramsey space) a van der Waerden space?

Note that under CH Theorem 16.1 reduces the above question to Question 7.8.

Corollary 16.7 (Assume CH)**.** *Let $\mathcal{I}$ be an ideal.*

(1) If $\mathcal{I}$ is P^-, then the following conditions are equivalent:
   - *(a) $\rho_\mathcal{I} \not\leq_K \mathrm{FS}$ ($\rho_\mathcal{I} \not\leq_K r$, $\rho_\mathcal{I} \not\leq_K \Delta$, resp.).*
   - *(b) There exists a Hindman (Ramsey, differentially compact, resp.) space that is not in* $\mathrm{FinBW}(\mathcal{I})$.

   Moreover, if $\mathcal{I}$ is P^+ then the above are equivalent to $\mathcal{I} \not\leq_K \mathcal{H}$.

(2) If $\mathcal{I}$ is P^-, then the following conditions are equivalent:
   - *(a) $\mathcal{H} \not\leq_K \mathcal{I}$ ($\mathcal{R} \not\leq_K \mathcal{I}$, $\mathcal{D} \not\leq_K \mathcal{I}$, resp.).*
   - *(b) $\mathrm{FS} \not\leq_K \rho_\mathcal{I}$ ($r \not\leq_K \rho_\mathcal{I}$, $\Delta \not\leq_K \rho_\mathcal{I}$, resp.).*
   - *(c) There exists a space in* $\mathrm{FinBW}(\mathcal{I})$ *that is not a Hindman (Ramsey, differentially compact, resp.) space.*

   Moreover, in every item an example showing that the above difference between FinBW *classes is nonempty is of the form $\Phi(\mathcal{A})$ with $\mathcal{A}$ being almost disjoint and of cardinality $\mathfrak{c}$ (in particular, these examples are Hausdorff, compact, separable and of cardinality $\mathfrak{c}$).*

Proof. It follows from Theorem 16.1(1) and Propositions 4.3, 6.5(2), 6.7(3), 7.5(2a)(2b) and 10.2(4). □

Remark. In [22, Corollary 2.8], the authors obtained Corollary 16.7(1) in the case of Hindman spaces and P^+ ideals but in the realm of *non*-Hausdorff spaces.

Corollary 16.8 (Assume CH)**.** *Let $\mathcal{I}$ be an ideal.*

(1) The following conditions are equivalent:
   - *(a) $\mathcal{I} \not\leq_K \mathcal{W}$ ($\mathcal{I} \not\leq_K \mathcal{I}_{1/n}$, resp.).*
   - *(b) $\rho_\mathcal{I} \not\leq_K \rho_\mathcal{W}$ ($\rho_\mathcal{I} \not\leq_K \rho_{\mathcal{I}_{1/n}}$, resp.).*
   - *(c) There exists a van der Waerden space ($\mathcal{I}_{1/n}$-space) that is not in* $\mathrm{FinBW}(\mathcal{I})$.

(2) If $\mathcal{I}$ is a P^- ideal, then the following conditions are equivalent:
   - *(a) $\mathcal{W} \not\leq_K \mathcal{I}$ ($\mathcal{I}_{1/n} \not\leq_K \mathcal{I}$, resp.).*
   - *(b) $\rho_\mathcal{W} \not\leq_K \rho_\mathcal{I}$ ($\rho_{\mathcal{I}_{1/n}} \not\leq_K \rho_\mathcal{I}$, resp.).*
   - *(c) There exists a space in* $\mathrm{FinBW}(\mathcal{I})$ *that is not a van der Waerden space ($\mathcal{I}_{1/n}$-space).*

Moreover, in every item an example showing that the above difference between FinBW *classes is nonempty is of the form* $\Phi(\mathcal{A})$ *with* $\mathcal{A}$ *being almost disjoint and of cardinality* $\mathfrak{c}$ *(in particular, these examples are Hausdorff, compact, separable and of cardinality* $\mathfrak{c}$*).*

Proof. It follows from Theorem 16.1(1) and Propositions 4.3, 6.5(2), 6.7(1), 7.5(2b) and 10.2(4). □

17. Non-Hausdorff world

Proposition 17.1. *The following conditions are equivalent for every topological space* X*.*

1. *(1)* X *has unique limits of sequences.*
2. *(2)* ρ*-limits of sequences in* X *are unique for every* ρ*.*

Proof. Since ρ_{Fin}-convergence is convergence, (2) $\implies$ (1) is obvious.

(1) $\implies$ (2): We will show that the negation of (2) implies the negation of (1). Suppose that there are partition regular $\rho : \mathcal{F} \to [\Lambda]^\omega$ with $\mathcal{F} \subseteq [\Omega]^\omega$, $F \in \mathcal{F}$ and $\{x_n : n \in \rho(F)\} \subseteq X$ which ρ-converges to x and to y, for some $x, y \in X$, $x \neq y$. Let $\{K_n : n \in \omega\} \subseteq [\Omega]^{<\omega}$ be nondecreasing and such that $\bigcup_{n\in\omega} K_n = \Omega$. For each $n \in \omega$ inductively find any $m_n \in \rho(F \setminus K_n) \setminus \{m_i : i < n\}$ (this is possible as $\rho(F \setminus K_n)$ is infinite). Observe that the sequence $(x_{m_n})_{n\in\omega}$ is convergent to x and to y. □

The main result of this section is as follows.

Theorem 17.2 (Assume CH)**.** *Let* $\rho_i : \mathcal{F}_i \to [\Lambda_i]^\omega$ *be partition regular for each* $i = 1, 2$*. If* ρ_1 *is weak* P^+ *and has small accretions, then the following conditions are equivalent.*

1. *(1)* $\rho_2 \not\leq_K \rho_1$*.*
2. *(2) There exists a separable space* X *with unique limits of sequences such that* $X \in \mathrm{FinBW}(\rho_1) \setminus \mathrm{FinBW}(\rho_2)$*.*

Proof. (2) $\implies$ (1). It follows from Theorem 11.1(1).

(1) $\implies$ (2). Fix a list $\{f_\alpha : \alpha < \mathfrak{c}\}$ of all $\mathcal{I}_{\rho_1}$-to-one functions $f : \Lambda_1 \to \Lambda_2$.

We will construct a sequence $\{D_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathcal{F}_1$ such that for every $\alpha < \mathfrak{c}$ we have

$$\forall E \in \mathcal{F}_2 \, \exists K \in [\Omega_1]^{<\omega} \, \forall L \in [\Omega_2]^{<\omega} \, \left(\rho_2(E \setminus L) \not\subseteq f_\alpha[\rho_1(D_\alpha \setminus K)]\right)$$

and one of the following conditions holds:

(W1)
$$\forall \beta < \alpha \, \forall M \in [\Lambda_2]^{<\omega} \, \exists K \in [\Omega_1]^{<\omega} \, \left(f_\alpha[\rho_1(D_\alpha \setminus K)] \cap (M \cup f_\beta[\rho_1(D_\beta \setminus K)]) = \emptyset\right)$$

or

$$\text{(W2)} \quad \exists \beta < \alpha \, \forall K \in [\Omega_1]^{<\omega} \, \forall M \in [\Lambda_2]^{<\omega} \, \exists L \in [\Omega_1]^{<\omega} \\ \left(f_\alpha[\rho_1(D_\alpha \setminus L)] \subseteq f_\beta[\rho_1(D_\beta \setminus K)] \setminus M\right).$$

Suppose that $\alpha < \mathfrak{c}$ and that D_β have been chosen for all $\beta < \alpha$. Since $\rho_2 \not\leq_K \rho_1$, there is $D^0 \in \mathcal{F}_1$ such that:

$$\text{(A1)} \qquad \forall E \in \mathcal{F}_2 \, \exists K \in [\Omega_1]^{<\omega} \, \forall L \in [\Omega_2]^{<\omega} \, \left(\rho_2(E \setminus L) \not\subseteq f_\alpha[\rho_1(D^0 \setminus K)]\right).$$

Since ρ_1 has small accretions, there is $D^1 \in \mathcal{F}_1$, $D^1 \subseteq D^0$, such that for every $K \in [\Omega_1]^{<\omega}$ we have $\rho_1(D^1) \setminus \rho_1(D^1 \setminus K) \in \mathcal{I}_{\rho_1}$. Observe that D^1 also has the property (A1) as $f_\alpha[\rho_1(D^0 \setminus K)] \supseteq f_\alpha[\rho_1(D^1 \setminus K)]$ for every $K \in [\Omega_1]^{<\omega}$.

Since ρ_1 is weak P^+, there is $D \in \mathcal{F}_1$ such that $\rho_1(D) \subseteq \rho_1(D^1)$ and satisfying property:

$$\text{(A2)} \qquad \begin{aligned} &\forall \{F_n : n \in \omega\} \subseteq \mathcal{F}_1 \,(\forall n \in \omega \,(\rho_1(F_{n+1}) \subseteq \rho_1(F_n) \subseteq \rho_1(D)) \\ &\implies \exists E' \in \mathcal{F}_1 \,\forall n \in \omega \,\exists K \in [\Omega_1]^{<\omega} \,(\rho_1(E' \setminus K) \subseteq \rho_1(F_n)). \end{aligned}$$

Now we have 2 cases:

$$\text{(P1)} \qquad \begin{aligned} &\forall D' \in \mathcal{F}_1 \,\forall \beta < \alpha \,(\rho_1(D') \subseteq \rho_1(D) \\ &\implies \exists K \in [\Omega_1]^{<\omega} \,(\rho_1(D') \setminus f_\alpha^{-1}[f_\beta[\rho_1(D_\beta \setminus K)]] \notin \mathcal{I}_{\rho_1}) \end{aligned}$$

or

$$\text{(P2)} \qquad \begin{aligned} &\exists D' \in \mathcal{F}_1 \,\exists \beta < \alpha \,(\rho_1(D') \subseteq \rho_1(D) \\ &\wedge \forall K \in [\Omega_1]^{<\omega} \,(\rho_1(D') \setminus f_\alpha^{-1}[f_\beta[\rho_1(D_\beta \setminus K)]] \in \mathcal{I}_{\rho_1}). \end{aligned}$$

In the first case, let $\{K_n : n \in \omega\} \subseteq [\Omega_1]^{<\omega}$ be such that $\bigcup_{n\in\omega} K_n = \Omega_1$ and let $\alpha \times [\Lambda_2]^{<\omega} = \{(\beta_n, M_n) : n \in \omega\}$, taking into account that α is countable (as we assumed CH). Using condition (P1) repeatedly and the facts that $f_\alpha^{-1}[\{\lambda\}] \in \mathcal{I}_{\rho_1}$, for every $\lambda \in \Lambda_2$, and $\rho_1(D^1) \setminus \rho_1(D^1 \setminus K) \in \mathcal{I}_{\rho_1}$, for all $K \in [\Omega_1]^{<\omega}$, one can easily construct a sequence $\{E_n : n \in \omega\} \subseteq \mathcal{F}_1$ such that

(1) $\rho_1(E_0) \subseteq \rho_1(D)$,
(2) $\forall n \in \omega \,(\rho_1(E_{n+1}) \subseteq \rho_1(E_n))$,
(3) $\forall n \in \omega \,\exists K \in [\Omega_1]^{<\omega} \,\big(\rho_1(E_n) \cap f_\alpha^{-1}[M_n \cup f_{\beta_n}[\rho_1(D_{\beta_n} \setminus K)]] = \emptyset\big)$,
(4) $\forall n \in \omega \,\rho_1(E_n) \cap \rho_1(D^1) \setminus \rho_1(D^1 \setminus K_n) = \emptyset$.

Now using property (A2) we find $E' \in \mathcal{F}_1$ such that $\rho_1(E') \subseteq \rho_1(D) \subseteq \rho_1(D^1)$ and for every $n \in \omega$ there is $K \in [\Omega_1]^{<\omega}$ with $\rho_1(E' \setminus K) \subseteq \rho_1(E_n)$.

It is not difficult to see that $D_\alpha = E'$ satisfies (A1) and (W1), i.e., it is as needed.

Consider the second case. Let $D' \in \mathcal{F}_1$ and $\beta < \alpha$ be such that $\rho_1(D') \subseteq \rho_1(D)$ and $\rho_1(D') \setminus f_\alpha^{-1}[f_\beta[\rho_1(D_\beta \setminus K)]] \in \mathcal{I}_{\rho_1}$ for each $K \in [\Omega_1]^{<\omega}$. Since $f_\alpha^{-1}[\{\lambda\}] \in \mathcal{I}_{\rho_1}$ for every $\lambda \in \Lambda_2$, we also have $\rho_1(D') \setminus f_\alpha^{-1}[f_\beta[\rho_1(D_\beta \setminus K)] \setminus M] \in \mathcal{I}_{\rho_1}$ for each $K \in [\Omega_1]^{<\omega}$ and $M \in [\Lambda_2]^{<\omega}$. Recall also that $\rho_1(D^1) \setminus \rho_1(D^1 \setminus K) \in \mathcal{I}_{\rho_1}$, for all $K \in [\Omega_1]^{<\omega}$. Since ρ_1 is P^- (by Proposition 6.5(1), as ρ_1 is weak P^+), we find an infinite set $D'' \in \mathcal{F}_1$ such that

- $\rho_1(D'') \subseteq \rho_1(D')$,
- for every $K \in [\Omega_1]^{<\omega}$ there is $L \in [\Omega_1]^{<\omega}$ with $\rho_1(D'' \setminus L) \subseteq \rho_1(D^1 \setminus K)$,
- for every $K \in [\Omega_1]^{<\omega}$ and $M \in [\Lambda_2]^{<\omega}$ there is $L \in [\Omega_1]^{<\omega}$ with $\rho_1(D'' \setminus L) \cap (\rho_1(D') \setminus f_\alpha^{-1}[f_\beta[\rho_1(D_\beta \setminus K)] \setminus M]) = \emptyset$.

It is not difficult to see that $D_\alpha = D''$ satisfies (A1) and (W2).

The construction of sets D_α is finished.

We are ready to define the required space. Let $T = \{\alpha < \mathfrak{c} : D_\alpha \text{ satisfies (W1)}\}$ and

$$X = \Lambda_2 \cup \{\rho_1(D_\alpha) : \alpha \in T\} \cup \{\infty\}.$$

For every $x \in X$ we define the family $\mathcal{B}(x) \subseteq \mathcal{P}(X)$ as follows:

- $\mathcal{B}(\lambda) = \{\{\lambda\}\}$ for $\lambda \in \Lambda_2$,
- $\mathcal{B}(\rho_1(D_\alpha)) = \{\{\rho_1(D_\alpha)\} \cup f_\alpha[\rho_1(D_\alpha \setminus K)] \setminus M : K \in [\Omega_1]^{<\omega}, M \in [\Lambda_2]^{<\omega}\}$ for $\alpha \in T$,
- $\mathcal{B}(\infty) = \{\{\infty\} \cup \bigcup_{\alpha \in T \setminus F} U_\alpha : F \in [T]^{<\omega} \wedge U_\alpha \in \mathcal{B}(\rho_1(D_\alpha)) \text{ for } \alpha \in T \setminus F\}$.

It is not difficult to check that the family $\mathcal{N} = \{\mathcal{B}(x) : x \in X\}$ is a neighborhood system (see e.g. [15, Proposition 1.2.3]). We claim that X with the topology generated by $\mathcal{N}$ is a topological space that we are looking for.

First we will show that X has unique limits of sequences. It is not difficult to see that $X \setminus \{\infty\}$ is Hausdorff. Thus, it suffices to check that if $\{x_n : n \in \omega\} \subseteq X$ converges to ∞ then it cannot converge to any other point in X. Indeed, if $\{x_n :$

$n \in \omega\}$ would converge to some $\lambda \in \Lambda_2$ then it would have to be constant from some point on, so $\{\infty\} \cup \bigcup_{\alpha \in T}(\{\rho_1(D_\alpha)\} \cup f_\alpha[\rho_1(D_\alpha)] \setminus \{\lambda\})$ would be an open neighborhood of ∞ omitting almost all x_n's. On the other hand, if $(x_n)_{n\in\omega}$ would converge to some $\rho_1(D_\alpha)$ for $\alpha \in T$ then using (W1) for each $\beta \in T \setminus \{\alpha\}$ we could find $K_\beta \in [\Omega]^{<\omega}$ with $f_\alpha[\rho_1(D_\alpha \setminus K_\beta)] \cap f_\beta[\rho_1(D_\beta \setminus K_\beta)] = \emptyset$. Then, denoting $M_\beta = \{x_n : n \in \omega\} \setminus f_\alpha[\rho_1(D_\alpha \setminus K_\beta)]$ (which is a finite set, as $\{\rho_1(D_\alpha)\} \cup f_\alpha[\rho_1(D_\alpha \setminus K_\beta)]$ is an open neighborhood of $\rho_1(D_\alpha)$ and $(x_n)_{n\in\omega}$ converges to $\rho_1(D_\alpha)$), the set $\{\infty\} \cup \bigcup_{\beta \in T\setminus\{\alpha\}}(\{\rho_1(D_\beta)\} \cup f_\beta[\rho_1(D_\beta \setminus K_\beta)] \setminus M_\beta)$ would be an open neighborhood of ∞ omitting all x_n's. Hence, X has unique limits of sequences.

Now we show that $X \in \mathrm{FinBW}(\rho_1)$. Fix any $f : \Lambda_1 \to X$. If there is $x \in X$ with $f^{-1}[\{x\}] \notin \mathcal{I}_{\rho_1}$ then find $F \in \mathcal{F}_1$ with $\rho_1(F) \subseteq f^{-1}[\{x\}]$ and observe that $(f(n))_{n\in\rho_1(F)}$ is ρ_1-convergent to x. Thus, we can assume that $f^{-1}[\{x\}] \in \mathcal{I}_{\rho_1}$ for all $x \in X$. There are two possible cases: $f^{-1}[X \setminus \Lambda_2] \notin \mathcal{I}_{\rho_1}$ or $f^{-1}[X \setminus \Lambda_2] \in \mathcal{I}_{\rho_1}$.

If $f^{-1}[X \setminus \Lambda_2] \notin \mathcal{I}_{\rho_1}$ then we find $F \in \mathcal{F}_1$ with $\rho_1(F) \subseteq f^{-1}[X \setminus \Lambda_2]$. As $f^{-1}[\{x\}] \in \mathcal{I}_{\rho_1}$ for all $x \in X$ and $f^{-1}[\{x\}] \neq \emptyset$ only for countably many $x \in X$, using the fact that ρ_1 is P^- (by Proposition 6.5(1)) we can find $E \in \mathcal{F}_1$ with $\rho_1(E) \subseteq \rho_1(F)$ and such that for each $x \in X \setminus \Lambda_2$ there is $K \in [\Omega_1]^{<\omega}$ with $\rho_1(E \setminus K) \cap f^{-1}[\{x\}] = \emptyset$. Since for each $U \in \mathcal{B}(\infty)$ there are only finitely many $\alpha \in T$ with $\rho_1(D_\alpha) \notin U$, $(f(n))_{n\in\rho_1(E)}$ ρ_1-converges to ∞.

If $f^{-1}[X \setminus \Lambda_2] \in \mathcal{I}_{\rho_1}$ then define $g : \Lambda_1 \to \Lambda_2$ by $g(\lambda) = f(\lambda)$ for all $\lambda \in \Lambda_1 \setminus f^{-1}[X \setminus \Lambda_2]$ and $g(\lambda) = x$ for all $\lambda \in f^{-1}[X \setminus \Lambda_2]$, where $x \in \Lambda_2$ is a fixed point. Then there is $\alpha < \mathfrak{c}$ with $f_\alpha = g$. We have two subcases: $\alpha \in T$ and $\alpha \notin T$.

Assume $\alpha \in T$. Since ρ_1 has small accretions, there is $E \subseteq D_\alpha$, $E \in \mathcal{F}_1$ such that $\rho_1(E) \setminus \rho_1(E \setminus K) \in \mathcal{I}_{\rho_1}$ for all $K \in [\Omega_1]^{<\omega}$. Using that ρ_1 is P^- (by Proposition 6.5(1)), we find $D \in \mathcal{F}_1$ such that

- $\rho_1(D) \subseteq \rho_1(E) \setminus f^{-1}[X \setminus \Lambda_2]$,
- for each $K \in [\Omega_1]^{<\omega}$ there is $L \in [\Omega_1]^{<\omega}$ with $\rho_1(D \setminus L) \subseteq \rho_1(E \setminus K) \subseteq \rho_1(D_\alpha \setminus K)$,
- for each $M \in [\Lambda_2]^{<\omega}$ there is $L \in [\Omega_1]^{<\omega}$ with $\rho_1(D \setminus L) \cap f^{-1}[M] = \emptyset$.

Since each $U \in \mathcal{B}(\rho_1(D_\alpha))$ is of the form $\{\rho_1(D_\alpha)\} \cup f_\alpha[\rho_1(D_\alpha \setminus K)] \setminus M$ for some $K \in [\Omega_1]^{<\omega}$ and $M \in [\Lambda_2]^{<\omega}$, the subsequence $(f(n))_{n\in\rho_1(D)}$ ρ_1-converges to $\rho_1(D_\alpha)$.

Assume $\alpha \notin T$. Then there is $\beta < \alpha$, $\beta \in T$ such that

$$\forall K \in [\Omega_1]^{<\omega}\, \forall M \in [\Lambda_2]^{<\omega}\, \exists L \in [\Omega_1]^{<\omega}\ \ (f_\alpha[\rho_1(D_\alpha \setminus L)] \subseteq f_\beta[\rho_1(D_\beta \setminus K)] \setminus M)$$

(we take the minimal $\beta < \alpha$ satisfying property (W2)). Since each open neighborhood of $\rho_1(D_\beta)$ is of the form $\{\rho_1(D_\beta)\} \cup f_\beta[\rho_1(D_\beta \setminus K)] \setminus M$ for some $K \in [\Omega_1]^{<\omega}$ and $M \in [\Lambda_2]^{<\omega}$, $(f_\alpha(n))_{n\in\rho_1(D_\alpha)}$ ρ_1-converges to $\rho_1(D_\beta) \in X$. Since ρ_1 has small accretions, there is $E \subseteq D_\alpha$, $E \in \mathcal{F}_1$ such that $\rho_1(E) \setminus \rho_1(E \setminus K) \in \mathcal{I}_{\rho_1}$ for all $K \in [\Omega_1]^{<\omega}$. Then also $(f_\alpha(n))_{n\in\rho_1(E)}$ ρ_1-converges to $\rho_1(D_\beta) \in X$. Finally, since $f^{-1}[X \setminus \Lambda_2] \in \mathcal{I}_{\rho_1}$, using that ρ_1 is P^- (by Proposition 6.5(1)), we get $E' \in \mathcal{F}_1$ such that $\rho_1(E') \subseteq \rho_1(E) \setminus f^{-1}[X \setminus \Lambda_2]$ and for each $K \in [\Omega_1]^{<\omega}$ there is $L \in [\Omega_1]^{<\omega}$ with $\rho_1(E' \setminus L) \subseteq \rho_1(E \setminus K)$. It is easy to see that $f_\alpha \restriction \rho_1(E') = f \restriction \rho_1(E')$ and $(f_\alpha(n))_{n\in\rho_1(E')}$ ρ_1-converges to $\rho_1(D_\beta) \in X$.

Finally, we check that $X \notin \mathrm{FinBW}(\rho_2)$. Define $f : \Lambda_2 \to X$ by $f(\lambda) = \lambda$ for all $\lambda \in \Lambda_2$ and fix any $E \in \mathcal{F}_2$. We claim that $(f(n))_{n\in\rho_2(E)}$ does not ρ_2-converge. Clearly, it cannot converge to any $x \in \Lambda_2$. Moreover, it cannot converge to any $\rho_1(D_\alpha)$ for $\alpha \in T$ as property (A1) guarantees that for some $K \in [\Omega_1]^{<\omega}$ we have $\rho_2(E \setminus L) \not\subseteq f_\alpha[\rho_1(D_\alpha \setminus K)]$ for all $L \in [\Omega_2]^{<\omega}$, so $U = \{\rho_1(D_\alpha)\} \cup f_\alpha[\rho_1(D_\alpha \setminus K)]$ would be an open neighborhood of $\rho_1(D_\alpha)$ such that $\rho_2(E \setminus L) \subseteq U$ for no $L \in [\Omega_2]^{<\omega}$.

We will show that $(f(n))_{n\in\rho_2(E)}$ cannot ρ_2-converge to ∞. Suppose otherwise, let $\{L_n : n \in \omega\} \subseteq [\Omega_2]^{<\omega}$ be such that $\bigcup_{n\in\omega} L_n = \Omega_2$ and inductively pick $m_n \in \rho_2(E \setminus L_n) \setminus \{m_i : i < n\}$. Then $(f(m_n))_{n\in\omega}$ is convergent to ∞. However, if $g : \Lambda_1 \to \{f(m_n) : n \in \omega\}$ is any bijection (the set $\{f(m_n) : n \in \omega\}$ is infinite since f is one-to-one) then $g = f_\alpha$ for some α. If $\alpha \in T$ then in $(f(m_n))_{n\in\omega}$ we could find a subsequence converging to $\rho_1(D_\alpha)$ (in the same way as above when showing that $X \in \mathrm{FinBW}(\rho_1)$ in the case of $\alpha \in T$) which contradicts that X has unique limits of sequences. If $\alpha \notin T$ then in $(f(m_n))_{n\in\omega}$ we could find a subsequence converging to $\rho_1(D_\beta)$ for some $\beta < \alpha$, $\beta \in T$ (in the same way as above when showing that $X \in \mathrm{FinBW}(\rho_1)$ in the case of $\alpha \notin T$) which also contradicts that X has unique limits of sequences. $\square$

18. Hindman (Ramsey, differentially compact) spaces that are not in FinBW($\mathcal{I}$) and vice versa

Now we turn our attention to the question when there is a space in $\mathrm{FinBW}(\mathcal{I})$ that is not Hindman (Ramsey, differentially compact, resp.) and vice versa in the case when $\mathcal{I}$ is an arbitrary ideal.

Corollary 18.1 (Assume CH). *For each ideal $\mathcal{I}$ and $\rho \in \{\mathrm{FS}, r, \Delta\}$ the following conditions are equivalent.*

(1) $\rho_{\mathcal{I}} \not\leq_K \rho$.
(2) There exists a Hindman (Ramsey, differentially compact, resp.) space that is not in $\mathrm{FinBW}(\mathcal{I})$.

Moreover, an example showing that the above difference between FinBW *classes is nonempty is separable and has unique limits of sequences. If $\mathcal{I}$ is P^-, then this example is of the form $\Phi(\mathcal{A})$ with $\mathcal{A}$ being almost disjoint of cardinality $\mathfrak{c}$ (in particular, it is Hausdorff, compact, separable and of cardinality $\mathfrak{c}$).*

Proof. It follows from Theorem 17.2 and Proposition 10.2(4) as each $\rho \in \{\mathrm{FS}, r, \Delta\}$ is weak P^+ (by Theorem 6.7(3)) and has small accretions (by Proposition 4.3). The case of P^- ideals $\mathcal{I}$ follows from Corollary 16.7(1). $\square$

In [61, Definition 4.1], the author introduced the following ideal

$$\mathcal{BI} = \left\{A \subseteq \omega^3 : \exists k \left[\forall i < k\, (A_{(i)} \in \mathrm{Fin}^2) \wedge \forall i \geq k\, (A_{(i)} \in \mathrm{Fin}(\omega^2))\right]\right\}$$

where $A_{(i)} = \{(x, y) \in \omega^2 : (i, x, y) \in A\}$. The ideal $\mathcal{BI}$ proved to be useful in research of $\mathrm{FinBW}(\mathcal{I})$ spaces (see [61] for more details).

Corollary 18.2 (Assume CH). *For each ideal $\mathcal{I}$, the following conditions are equivalent.*

(1) $\mathcal{BI} \not\leq_K \mathcal{I}$.
(2) There exists a space in $\mathrm{FinBW}(\mathcal{I})$ *that is not a Hindman (Ramsey, differentially compact, resp.) space.*

Moreover, an example showing that the above difference between FinBW *classes is nonempty is of the form $\Phi(\mathcal{A})$ with $\mathcal{A}$ being infinite maximal almost disjoint (in particular, it is Hausdorff, compact, separable and of cadinality $\mathfrak{c}$).*

Proof. (1) $\implies$ (2) In [61, Theorem 5.3], the author proved that if $\mathcal{BI} \not\leq_K \mathcal{I}$ then there exists an infinite maximal almost disjoint family $\mathcal{A}$ such that $\Phi(\mathcal{A}) \in \mathrm{FinBW}(\mathcal{I})$. Then Corollary 13.3 shows that $\Phi(\mathcal{A})$ is not Hindman (Ramsey nor differentially compact).

(2) $\implies$ (1) Using [61, Proposition 6.3 and Lemma 3.2(ii)], it is not difficult to see that if $\mathcal{BI} \leq_K \mathcal{I}$ then each space in $\mathrm{FinBW}(\mathcal{I})$ satisfies property $(*)$. On the

other hand, we know that spaces with $(*)$ property are Hindman, Ramsey and differentially compact (see [55, Theorem 11], [59, Corollary 3.2] and [20, Corollary 4.8], resp.). □

Corollary 18.3 (Assume CH)**.** *If* $\rho \in \{FS, r, \Delta\}$ *then* $\mathrm{FinBW}(\rho) \neq \mathrm{FinBW}(\mathcal{I})$ *for every ideal* $\mathcal{I}$*.*

Proof. Let $\rho \in \{FS, r, \Delta\}$ and $\mathcal{I}$ be an ideal. If $\mathcal{BI} \not\leq_K \mathcal{I}$ then $\mathrm{FinBW}(\mathcal{I}) \setminus \mathrm{FinBW}(\rho) \neq \emptyset$ by Corollary 18.2. On the other hand, if $\mathcal{BI} \leq_K \mathcal{I}$ then the interval $[0,1]$ is in $\mathrm{FinBW}(\rho)$ (by Theorem 10.5(6) and Propositions 10.2(2) and 6.7(3)) and it is not in $\mathrm{FinBW}(\mathcal{I})$ (by [61, Proposition 4.6], [2, Example 4.1] and [65, Section 2.7]). □

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