PATH OF PATHOLOGY

RAFAŁ FILIPÓW AND JACEK TRYBA

Abstract. We present a few results about (non)pathology of submeasures and ideals.

1. Introduction

The paper consists of two parts. In the first part (Sections 3–7) we examine various kinds of (non)pathological submeasures that are considered in the literature, whereas the second part (Sections 8–14) is devoted to (non)pathological ideals. Below, we briefly describe the content of each section.

In Section 3, we provide various definitions of (non)pathological submeasures that are considered in the literature and in Section 4 we summarize basic properties of these kinds of submeasures. In Sections 5 and 6 we present some examples of (non)pathological submeasures. The relationships and examples from these sections are shown in concise graphical form in Figure 1.

In Section 7, we examine various kinds of degrees of pathology of submeasures that are considered in the literature. The results from this section are summarized in Table 1.

The second part of the paper starts with Section 8 where we present definitions, examples and characterizations of nonpathological ideals.

In Section 9, we prove a characterization of ideals that can be represented as the intersections of matrix summability ideals. The characterization is expressed in terms of the Katětov order and restrictions of the ideal of asymptotic density zero sets. This characterization resembles a characterization of pathological analytic P-ideals given by Hrušák in [15, Corollary 5.6]. As a by-product, we obtain that the Solecki ideal is pathological which answers a question of Martínez, Meza-Alcántara and Uzcátegui [22, Question 3.13].

In Section 10, we solve a problem of Borodulin-Nadzieja and Farkas [4, Problem 4.3]. Namely, we show that the ideal of exponential density zero sets is a nonpathological analytic P-ideal which is not a special variant of a density ideal generated by a family of measures.

In Section 11, we answer two more questions of Martínez, Meza-Alcántara and Uzcátegui [23, Questions 3.6 and 3.10]. Since the second question is quite technical, we describe here only the first one. We construct a submeasure ϕ which has infinite degree of pathology but the ideal $\mathrm{Fin}(\phi)$ is nonpathological and the degrees of pathology of all restrictions of ϕ to sets from the ideal $\mathrm{Fin}(\phi)$ are finite. In the

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same section we also characterize nonpathological F_σ ideals as those which can be represented as the intersection of summable ideals. This result strengthen a known theorem in which summable ideals are replaced by matrix ideals.

In Section 12, we prove that the van der Waerden ideal (which consists of those subsets of integers which do not contain arithmetic progressions of arbitrary finite length) is nonpathological.

In Section 13, we consider the Josefson-Nissenzweig property which has been recently introduced and examined by Marciszewski and Sobota [20]. We prove that the disjoint union of topological spaces has the Josefson-Nissenzweig property if and only if at least one of these spaces has this property. As a corollary, we obtain that a variant of Mrówka-Isbell space defined with the aid of almost disjoint family $\mathcal{A}$ of sets which are not in a given ideal $\mathcal{I}$ has the Josefson-Nissenzweig property if and only if the restriction of $\mathcal{I}$ to some set from the family $\mathcal{A}$ extends to a matrix ideal.

In Section 14, we alter a result of Hrušák and García Ferreira [16, Theorem 3.9] to show that consistently for every tall ideal $\mathcal{I}$ there exists a maximal almost disjoint family $\mathcal{A} \subseteq \mathcal{I}$ such that the ideal $\mathcal{I}(\mathcal{A})$ generated by $\mathcal{A}$ is homogeneous in the sense of Katětov order. Next, we show that this ideal $\mathcal{I}(\mathcal{A})$ is a non-Borel ideal which is the intersection of matrix ideals. This answer the question posed by the authors in [9, Question 3].

2. Preliminaries

We identify an ordinal number α with the set of all ordinal numbers less than α. In particular, the smallest infinite ordinal number $\omega = \{0, 1, \dots\}$ is equal to the set of all natural numbers $\mathbb{N}$, and each natural number $n = \{0, \dots, n-1\}$ is equal to the set of all natural numbers less than n. Using this identification, we can for instance write $n \in k$ instead of $n < k$ and $n < \omega$ instead of $n \in \omega$ or $A \cap n$ instead of $A \cap \{0, 1, \dots, n-1\}$.

2.1. Ideals.

Definition 2.1. An *ideal* on a nonempty set X is a family $\mathcal{I} \subseteq \mathcal{P}(X)$ that satisfies the following properties:

1. $\emptyset \in \mathcal{I}$ and $X \notin \mathcal{I}$,
2. if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$,
3. if $A \subseteq B$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$.

In the second part of the paper (starting at page 17), we alter the definition of an ideal. Namely, we will additionally assume there that an ideal $\mathcal{I}$ on X has to contain all *finite subsets of* X.

For an ideal $\mathcal{I}$, we write $\mathcal{I}^+ = \{A \subseteq \mathbb{N} : A \notin \mathcal{I}\}$ and call it the *coideal of* $\mathcal{I}$, we also write $\mathcal{I}^* = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \in \mathcal{I}\}$ and call it the *dual filter of* $\mathcal{I}$. By Fin we mean the ideal of all finite subsets of $\mathbb{N}$. An ideal $\mathcal{I}$ is a *P-ideal* if for every countable family $\mathcal{A} \subseteq \mathcal{I}$ there is $B \in \mathcal{I}$ such that $A \setminus B$ is finite for every $A \in \mathcal{A}$. An ideal $\mathcal{I}$ is *tall* (a.k.a. *dense*) if for every infinite $A \subseteq X$ there is an infinite $B \in \mathcal{I}$ such that $B \subseteq A$. For an ideal $\mathcal{I}$ and a set $A \notin \mathcal{I}$ we define the *restriction of the ideal* $\mathcal{I}$ to A by $\mathcal{I} \restriction A = \{B \cap A : B \in \mathcal{I}\}$. An ideal $\mathcal{I}$ on X is called *maximal* if there is no ideal $\mathcal{J}$ on X with $\mathcal{I} \subseteq \mathcal{J}$ and $\mathcal{I} \neq \mathcal{J}$. It is known that an ideal $\mathcal{I}$ on X is maximal if and only if for each $A \subseteq X$ either $A \in \mathcal{I}$ or $X \setminus A \in \mathcal{I}$. Dual filters of maximal ideals are called *ultrafilters*.

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on X and Y respectively. We say that $\mathcal{I}$ and $\mathcal{J}$ are *isomorphic* (in short $\mathcal{I} \approx \mathcal{J}$) if there exists a bijection $f : X \to Y$ such that $A \in \mathcal{I} \iff f[A] \in \mathcal{J}$ for every $A \subseteq X$. We say that $\mathcal{J}$ *is below* $\mathcal{I}$ *in Katětov order* (in short $\mathcal{J} \leq_K \mathcal{I}$) if there is a function $f : X \to Y$ such that $A \in \mathcal{J} \implies f^{-1}[A] \in \mathcal{I}$ for every $A \subseteq X$. We say that $\mathcal{I}$ and $\mathcal{J}$ are *Katětov equivalent* (in short $\mathcal{I} \approx_K \mathcal{J}$ or $\leq_K$-equivalent) if $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$. We say that $\mathcal{I}$ is *K-homogeneous* if $\mathcal{I}$ and $\mathcal{I} \restriction A$ are Katětov equivalent for every $A \notin \mathcal{I}$.

By identifying sets of natural numbers with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the topology of the Cantor space $\{0,1\}^\omega$ and therefore we can assign topological complexity to ideals on ω. In particular, an ideal $\mathcal{I}$ is Borel (F_σ, analytic, resp.) if $\mathcal{I}$ is Borel (F_σ, analytic, resp.) as a subset of the Cantor space.

2.2. **Submeasures.** A map $\phi : \mathcal{P}(X) \to [0,\infty]$ is a *measure* (σ*-measure*, resp.) on a nonempty set X if $\phi(\emptyset) = 0$ and ϕ is finitely (countably, resp.) additive.

For a function $f : \omega \to [0,\infty)$, we define a σ-measure μ_f on ω by

$$\mu_f(A) = \sum_{n \in A} f(n).$$

It is not difficult to see that ϕ is a σ-measure on $\omega \iff \phi = \mu_f$ for some function $f : \omega \to [0,\infty]$.

For an ideal $\mathcal{I}$ on ω, we define a measure $\delta^\infty_{\mathcal{I}}$ on ω by

$$\delta^\infty_{\mathcal{I}}(A) = \begin{cases} 0 & \text{if } A \in \mathcal{I}, \\ \infty & \text{otherwise,} \end{cases}$$

and for a maximal ideal $\mathcal{I}$, we define a measure $\delta_{\mathcal{I}}$ by

$$\delta_{\mathcal{I}}(A) = \begin{cases} 0 & \text{if } A \in \mathcal{I}, \\ 1 & \text{otherwise.} \end{cases}$$

To obtain more examples of measures, one can use the so called *ultrafilter limits* (see e.g. [2]). Namely, if $\mathcal{U}$ is an ultrafilter on ω and μ_n is a measure on X for each $n \in \omega$, then

$$\nu(A) = \lim_{n \in \mathcal{U}} \mu_n(A)$$

is a measure on X.

Definition 2.2. A map $\phi : \mathcal{P}(X) \to [0,\infty]$ is a *submeasure* on a set X if

1. $\phi(\emptyset) = 0$,
2. if $A \subseteq B$ then $\phi(A) \leq \phi(B)$,
3. $\phi(A \cup B) \leq \phi(A) + \phi(B)$.

We say that a submeasure ϕ is *lower semicontinuous* (in short: *lsc*) if

$$\phi(A) = \sup\{\phi(F) : F \text{ is a finite subset of } A\}$$

for each $A \subseteq X$. If $X = \omega$, then ϕ is lsc $\iff$ $\phi(A) = \lim_{n\to\infty} \phi(A \cap n)$.

Every measure is a submeasure, and a measure ϕ is a σ-measure $\iff$ ϕ is lower semicontinuous. Moreover, if ϕ is a nonzero submeasure on X, then

$$\mathcal{Z}_\phi = \{A \subseteq X : \phi(A) = 0\}$$

is an ideal on X, whereas if $\mathcal{I}$ is an ideal on X, then

$$\delta_{\mathcal{I}}(A) = \begin{cases} 0 & \text{if } A \in \mathcal{I}, \\ 1 & \text{otherwise} \end{cases}$$

is a submeasure on X with $\mathcal{Z}_{\delta_{\mathcal{I}}} = \mathcal{I}$.

To obtain more examples of submeasures on ω, one can observe that if ϕ_n is a submeasure on X for every $n \in \omega$, then

$$\phi(A) = \limsup_{n\to\infty} \phi_n(A)$$

is a submeasure on X, and if $\mathcal{S}$ is a family of submeasures (lsc submeasures, resp.) on X, then

$$\psi(A) = \sup\{\phi(A) : \phi \in \mathcal{S}\}$$

is a submeasure (lsc submeasures, resp.) on X.

Part 1. (Non)pathological submeasures

In this part we present thorough analysis of various kinds of (non)pathological submeasures that are considered in the literature.

3. Definition of (non)pathological submeasures

For two submeasures ϕ and ψ on a set X, we say that ψ *dominates* ϕ and write $\phi \leq \psi$ if $\phi(A) \leq \psi(A)$ for every $A \subseteq X$.

For a submeasure ϕ on X, we define two submeasures $\widehat{\phi}$ and $\widehat{\phi}_\sigma$ on X by

$$\widehat{\phi}(A) = \sup\{\mu(A) : \mu \text{ is a } \quad \text{measure on } X \text{ such that } \mu \leq \phi\},$$

$$\widehat{\phi}_\sigma(A) = \sup\{\mu(A) : \mu \text{ is a } \sigma\text{-measure on } X \text{ such that } \mu \leq \phi\}.$$

Definition 3.1. Let ϕ be a submeasure on X.

1. A submeasure ϕ is called *nonpathological* if $\phi = \widehat{\phi}$, otherwise ϕ is called *pathological* (Farah, [5, p. 21]).
2. A submeasure ϕ is called *σ-nonpathological* if $\phi = \widehat{\phi}_\sigma$, otherwise ϕ is called *σ-pathological.*

The following remark shows that there is a mess with definitions of (non)pathological submeasures.

Remark.

1. In [30], Topsøe introduced the notion of ε-pathological submeasures and one can show that a submeasure ϕ is pathological (in the sense of Farah) $\iff$ there exists $A \subseteq X$ and $\varepsilon < 1$ such that the restriction $\phi \restriction \mathcal{P}(A)$ is ε-pathological (in the sense of Topsøe).
2. To make life a bit confusing, we can also find out that in [30] Topsøe has a definition of a nonpathological submeasure which is not coherent with the notion introduced by Farah, namely a submeasure ϕ is nonpathological in the sense of Topsøe if $\widehat{\phi} = 0$ (i.e. there is no nonzero measure dominated by ϕ). It is easy to see that pathology in the sense of Topsøe is stronger than pathology in the sense of Farah (Proposition 4.1(3)). On the other hand, the pathology in the sense of Topsøe is useless in the realm of *lower semicontinuous* submeasure as is shown in Proposition 4.1(4b).

(3) To make life even more complicated, in [23] (see also [21, p. 3] and [22, p. 4]), the authors define a submeasure ϕ to be nonpathological if $\phi = \widehat{\phi}_\sigma$. Obviously, $\phi = \widehat{\phi}_\sigma$ implies $\phi = \widehat{\phi}$. The converse implication does not hold in general (see Proposition 5.1(2)), but it holds for *lower semicontinuous* submeasures (see Proposition 4.1(7a)). Moreover, there exists a pathological lower semicontinuous submeasure ϕ such that $\widehat{\phi}_\sigma \neq \widehat{\phi}$ (see Proposition 6.4).

The following proposition (a sort of the extreme value theorem) seems to be a folklore, but we decided to include the proof for the sake of completeness because we did not find any reference which contains a proof.

Proposition 3.2. *For every submeasure ϕ on X and every $A \subseteq X$ there exists a measure μ on X such that $\mu \leq \phi$ and $\widehat{\phi}(A) = \mu(A)$. In particulary, if ϕ is nonpathological, then for every $A \subseteq X$ there exists a measure μ on X such that $\mu \leq \phi$ and $\phi(A) = \mu(A)$.*

Proof. Take any $A \subseteq X$. We have two cases: $\widehat{\phi}(A) = \infty$ and $\widehat{\phi}(A) < \infty$.

Case: $\widehat{\phi}(A) = \infty$. We define a function $\mu : \mathcal{P}(X) \to [0, \infty]$ by

$$\mu(B) = \begin{cases} \infty & \text{if } \widehat{\phi}(B) = \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Then μ is a measure on X, $\mu \leq \phi$ and $\mu(A) = \widehat{\phi}(A)$, so we are done.

Case: $\widehat{\phi}(A) < \infty$. Let $\mathcal{M}$ be a family of all measures $\mu : \mathcal{P}(A) \to [0, \widehat{\phi}(A)]$ such that $\mu(B) \leq \phi(B)$ for every $B \subseteq A$. We claim that $\mathcal{M}$ is a closed set in the space $[0, \widehat{\phi}(A)]^{\mathcal{P}(A)}$ with the product topology. Indeed, take any $f \in [0, \widehat{\phi}(A)]^{\mathcal{P}(A)} \setminus \mathcal{M}$. Then we have two cases: (a) f is not a measure or (b) f is a measure.

Case (a). Then we can find disjoint sets $A_0, A_1 \subseteq A$ with $f(A_0 \cup A_1) \neq f(A_0) + f(A_1)$. For $\varepsilon = |f(A_0 \cup A_1) - (f(A_0) + f(A_1))|/4 > 0$, we define

$$U = \{g \in [0, \widehat{\phi}(A)]^{\mathcal{P}(A)} : |g(A_0 \cup A_1) - f(A_0 \cup A_1)| < \varepsilon, |g(A_i) - f(A_i)| < \varepsilon \text{ for } i < 2\}.$$

Then U is an open neighborhood of f and $U \cap \mathcal{M} = \emptyset$.

Case (b). Since $f \notin \mathcal{M}$, we can find $B \subseteq A$ with $f(B) > \phi(B)$. For $\varepsilon = (f(B) - \phi(B))/3 > 0$, we define

$$U = \{g \in [0, \widehat{\phi}(A)]^{\mathcal{P}(A)} : |g(B) - f(B)| < \varepsilon\}.$$

Then U is an open neighborhood of f and $U \cap \mathcal{M} = \emptyset$.

In both cases, we find an open neighborhood of f which is disjoint from $\mathcal{M}$, thus $\mathcal{M}$ is closed. Since $[0, \widehat{\phi}(A)]^{\mathcal{P}(A)}$ is a compact space, we obtain that $\mathcal{M}$ is compact as well. For each $n \in \omega$, we define

$$C_n = \{\mu \in \mathcal{M} : |\mu(A) - \widehat{\phi}(A)| \leq 1/(n+1)\}.$$

Then C_n is a decreasing sequence of closed sets, so its intersection is nonempty. Let $\mu \in \bigcap_{n \in \omega} C_n \subseteq \mathcal{M}$. Then μ is a measure such that $\mu \leq \phi$ and $\mu(A) = \widehat{\phi}(A)$, so we are done. □

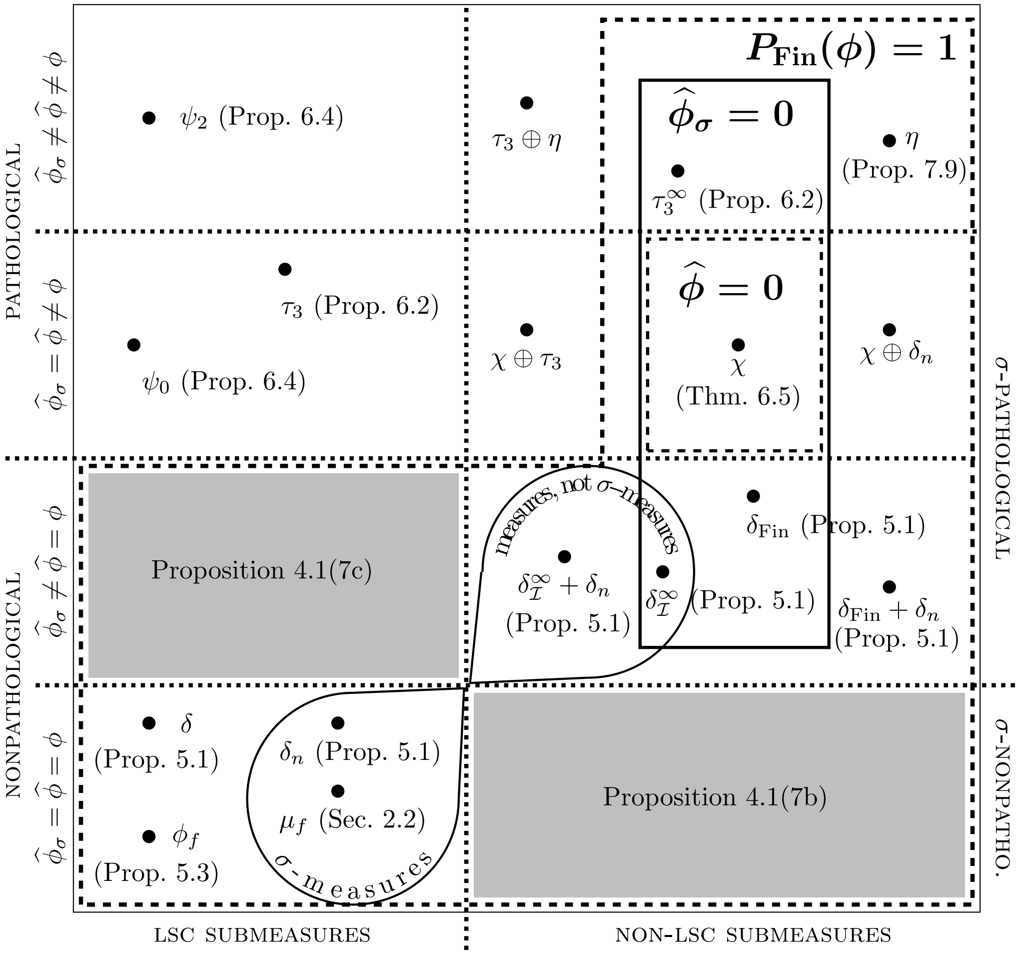

Figure 1. Relationships between (non)lsc and (non)pathological nonzero submeasures on ω (in gray regions there are no submeasures).

4. Properties of (non)pathological submeasures

In Proposition 4.1 we summarize basic properties of (non)pathological submeasures and in Sections 5 and 6 we provide some examples of (non)pathological submeasures. We present these relationships and examples in concise graphical form in Figure 1.

Proposition 4.1. *Let ϕ be a* nonzero *submeasure on ω.*

(1) *$\widehat{\phi}$ and $\widehat{\phi}_\sigma$ are submeasures, and $\widehat{\phi}_\sigma \leq \widehat{\phi} \leq \phi$. Consequently, if ϕ is σ-nonpathological, then ϕ is nonpathological.*
(2) *$\widehat{\phi}_\sigma$ is lower semicontinuous. In general, $\widehat{\phi}$ may be not lower semicontinuous (see Proposition 5.1(4)), even if ϕ is lsc (see Proposition 6.4(4)).*
(3) *If $\widehat{\phi} = 0$, then ϕ is pathological.*
(4) (a) *If $\phi(\omega) = \infty$, then $\widehat{\phi} \neq 0$.*

   (b) *If* $\phi(F) \neq 0$ *for some finite set* F*, then* $\widehat{\phi}_\sigma \neq 0$*. Consequently, if* ϕ *is lower semicontinuous, then* $\widehat{\phi}_\sigma \neq 0$*.*
   (c) *If* $\phi(F) = 0$ *for every finite set* F*, then* ϕ *is not lower semicontinuous and* $\widehat{\phi}_\sigma = 0 \neq \phi$*.*

(5) (a) $\widehat{\phi}(F) = \widehat{\phi}_\sigma(F)$ *for every finite set* F*.*
   (b) *If* $\phi(\omega \setminus F) = 0$ *for some finite set* F*, then* $\widehat{\phi}_\sigma = \widehat{\phi}$*.*

(6) (a) *If* ϕ *is a measure (*σ*-measure, resp.), then* ϕ *is nonpathological (*σ*-nonpathological, resp.).*
   (b) *If* ϕ *is not lower semicontinuous, then* ϕ *is* σ*-pathological.*
   (c) *If* ϕ *is a measure but not a* σ*-measure, then* ϕ *is nonpathological but* σ*-pathological.*

(7) (a) *If* ϕ *is lower semicontinuous, then* ϕ *is nonpathological* $\iff$ ϕ *is* σ*-nonpathological. The assumption that* ϕ *is lower semicontinuous cannot be dropped (see Proposition 5.1(4)).*
   (b) *If* ϕ *is nonpathological but* σ*-pathological, then* ϕ *is not lower semicontinuous.*
   (c) *If* ϕ *is* σ*-nonpathological, then* ϕ *is lower semicontinuous.*

(8) $\widehat{\phi}$ *(*$\widehat{\phi}_\sigma$*, resp.) is the largest nonpathological (*σ*-nonpathological, resp.) submeasure dominated by* ϕ*.*

Proof. (1) Straightforward.

(2) Because $\widehat{\phi}_\sigma$ is the supremum of a family of lsc (sub)measures.

(3) Because we assumed that ϕ is nonzero.

(4a) The function

$$\mu(A) = \begin{cases} 0 & \text{if } \phi(A) < \infty, \\ \infty & \text{if } \phi(A) = \infty \end{cases}$$

is a nonzero measure which is dominated by ϕ. Hence, $\widehat{\phi} \neq 0$.

(4b) Let F be finite with $\phi(F) > 0$. Since ϕ is subadditive, there is $n \in F$ with $\phi(\{n\}) > 0$. Then the function

$$\mu(A) = \begin{cases} \phi(\{n\}) & \text{if } n \in A, \\ 0 & \text{otherwise} \end{cases}$$

is a nonzero σ-measure which is dominated by ϕ. Hence $\widehat{\phi}_\sigma \neq 0$.

(4c) Straightforward.

(5a) Since $\widehat{\phi}_\sigma \leq \widehat{\phi}$, we only need to show $\widehat{\phi}(F) \geq \widehat{\phi}_\sigma(F)$ for finite F. There is a measure ν on ω such that $\nu \leq \phi$ and $\nu(F) = \widehat{\phi}(F)$. We define $\mu : \mathcal{P}(\omega) \to [0, \infty]$ by $\mu(A) = \nu(A \cap F)$. Then μ is a σ-measure, $\mu \leq \phi$ and $\mu(F) = \widehat{\phi}(F)$, thus we obtain $\widehat{\phi}_\sigma(F) \geq \widehat{\phi}(F)$.

(5b) Let F be a finite set such that $\phi(\omega \setminus F) = 0$. For any $A \subseteq \omega$ we have $\widehat{\phi}(A \cap F) \leq \widehat{\phi}(A) \leq \widehat{\phi}(A \cap F) + \widehat{\phi}(A \setminus F) = \widehat{\phi}(A \cap F)$, so $\widehat{\phi}(A \cap F) = \widehat{\phi}(A)$. The same argument shows that $\widehat{\phi}_\sigma(A \cap F) = \widehat{\phi}_\sigma(A)$ for every $A \subseteq \omega$. Since $A \cap F$ is finite, we can use item (5a) to obtain $\widehat{\phi}_\sigma(A) = \widehat{\phi}(A)$ for every $A \subseteq \omega$.

(6a) Obvious.

(6b) It follows from item (2).

(6c) It follows from items (6a)–(6b) and the fact that a lsc measure is a σ-measure.

(7a) Since $\widehat{\phi}_\sigma \leq \widehat{\phi} = \phi$, we only need to show $\widehat{\phi}_\sigma(A) \geq \phi(A)$ for any $A \subseteq \omega$. Take any $M < \widehat{\phi}(A)$. Since ϕ is lsc, we can find a finite set $F \subseteq A$ with $\phi(F) > M$. Using item (5a), we obtain $\phi(F) = \widehat{\phi}(F) = \widehat{\phi}_\sigma(F)$. Then $\widehat{\phi}_\sigma(A) \geq \widehat{\phi}_\sigma(F) = \phi(F) > M$. Hence $\widehat{\phi}_\sigma(A) \geq \widehat{\phi}(A)$.

(7b) It follows from item (7a).

(7c) It follows from item (2).

(8) The proof in the case of $\widehat{\phi}_\sigma$ is almost the same as the one for $\widehat{\phi}$, so we provide only the proof for $\widehat{\phi}$.

First, we show that the submeasure $\widehat{\phi}$ is nonpathological. Since $\widehat{\widehat{\phi}} \leq \widehat{\phi}$, we only need to show $\widehat{\widehat{\phi}}(A) \geq \widehat{\phi}(A)$ for any $A \subseteq \omega$. Take any $M < \widehat{\phi}(A)$. Then there exists a measure $\mu \leq \phi$ such that $\mu(A) > M$. Since $\widehat{\phi}$ is the supremum of all measures dominated by ϕ, we obtain $\mu \leq \widehat{\phi}$. Consequently, $\widehat{\widehat{\phi}}(A) \geq \mu(A) > M$. Since $M < \widehat{\phi}(A)$ was arbitrary, we obtain $\widehat{\widehat{\phi}}(A) \geq \widehat{\phi}(A)$.

Second, we show that $\widehat{\phi}$ is the largest nonpathological submeasure dominated by ϕ. Take any nonpathological submeasure $\psi \leq \phi$. We take any $A \subseteq \omega$ and any $M < \psi(A)$. If we show that $\widehat{\phi}(A) > M$, the proof will be finished. Since ψ is nonpathological, there is a measure $\mu \leq \psi$ with $\mu(A) > M$. Since $\mu \leq \psi \leq \phi$, we obtain $\widehat{\phi}(A) \geq \mu(A) > M$. □

5. Examples of nonpathological submeasures

In Section 2.2, we defined the submeasures $\delta_\mathcal{I}$ and $\delta^\infty_\mathcal{I}$ for any ideal $\mathcal{I}$ on X. In the case of two simple ideals on ω, we will use the following notations.

- For $\mathcal{I} = \{\emptyset\}$, we write $\delta = \delta_\mathcal{I}$ and $\delta^\infty = \delta^\infty_\mathcal{I}$.
- For $\mathcal{I} = \{A \subseteq \omega : n \in A\}$ with a fixed $n \in \omega$, we write $\delta_n = \delta_\mathcal{I}$ and $\delta^\infty_n = \delta^\infty_\mathcal{I}$.

Proposition 5.1. *Let $\mathcal{I}$ be an ideal on ω which contains all finite subsets of ω.*

(1) (a) *δ is a σ-nonpathological and lower semicontinuous submeasures which is not a measure.*
  (b) *δ^∞, δ_n and δ^∞_n are σ-measures (hence lower semicontinuous and σ-nonpathological).*

(2) (a) *$\delta_\mathcal{I}$ is not lower semicontinuous nor a σ-measure.*
  (b) *$\delta_\mathcal{I}$ is a measure $\iff$ $\mathcal{I}$ is a maximal ideal.*
  (c) $0 = \widehat{(\delta_\mathcal{I})}_\sigma \neq \widehat{\delta_\mathcal{I}} = \delta_\mathcal{I}$.

(3) (a) *$\delta + \delta_\mathcal{I}$ is a non-lsc submeasure.*
  (b) $\delta = \widehat{(\delta + \delta_\mathcal{I})}_\sigma \neq \widehat{\delta + \delta_\mathcal{I}} = \delta + \delta_\mathcal{I}$.

(4) (a) *$\delta^\infty_\mathcal{I}$ is a measure which is not lower semicontinuous nor a σ-measure.*
  (b) $0 = \widehat{(\delta^\infty_\mathcal{I})}_\sigma \neq \widehat{\delta^\infty_\mathcal{I}} = \delta^\infty_\mathcal{I}$.

(5) (a) *$\delta^\infty_\mathcal{I} + \delta_n$ is a measure which is not lower semicontinuous nor a σ-measure.*
  (b) $0 \neq \widehat{(\delta^\infty_\mathcal{I} + \delta_n)}_\sigma \neq \widehat{\delta^\infty_\mathcal{I} + \delta_n} = \delta^\infty_\mathcal{I} + \delta_n$.

(6) (a) *$\delta_\mathcal{I} + \delta_n$ is a submeasure which is not lower semicontinuous.*
  (b) *$\delta_\mathcal{I} + \delta_n$ is a measure $\iff$ $\mathcal{I}$ is maximal.*
  (c) $0 \neq \widehat{(\delta_\mathcal{I} + \delta_n)}_\sigma \neq \widehat{\delta_\mathcal{I} + \delta_n} = \delta_\mathcal{I} + \delta_n$.

Proof. (1), (2a), (2b) and (4a) Straightforward.

(2c) The equality $\widehat{(\delta_{\mathcal{I}})}_\sigma = 0$ follows from Proposition 4.1(4c). Since $\widehat{\delta_{\mathcal{I}}} \leq \delta_{\mathcal{I}}$, we only need to show the converse inequality. If $A \in \mathcal{I}$, then $\delta_{\mathcal{I}}(A) = 0 \leq \widehat{\delta_{\mathcal{I}}}(A)$, so we can assume that $A \notin \mathcal{I}$. Then we can find a maximal ideal $\mathcal{J}$ such that $A \notin \mathcal{J}$ and $\mathcal{I} \subseteq \mathcal{J}$. Since $\delta_{\mathcal{J}}$ is a measure, $\delta_{\mathcal{J}} \leq \delta_{\mathcal{I}}$ and $\widehat{\delta_{\mathcal{I}}}(A) \geq \delta_{\mathcal{J}}(A) = 1 \geq \delta_{\mathcal{I}}(A)$, the proof is finished.

(3a) Follows from $(\delta + \delta_{\mathcal{I}})(\omega) = 2$ and $(\delta + \delta_{\mathcal{I}})(F) = 1$ for every finite $F \subseteq \omega$.

(3b) Since δ and $\delta_{\mathcal{I}}$ are nonpathological, $\delta + \delta_{\mathcal{I}}$ is nonpathological as well. To show the first equality, we first observe that $\widehat{(\delta + \delta_{\mathcal{I}})}_\sigma \geq \widehat{\delta}_\sigma = \delta$. On the other hand, if μ is a measure such that $\mu \leq \delta + \delta_{\mathcal{I}}$, then $\mu(F) \leq (\delta + \delta_{\mathcal{I}})(F) = 1$, so by using lsc of $\widehat{(\delta + \delta_{\mathcal{I}})}_\sigma$ we obtain that $\widehat{(\delta + \delta_{\mathcal{I}})}_\sigma(A) \leq 1 = \delta(A)$ for every nonempty A.

(4b) The equality $\widehat{(\delta_{\mathcal{I}}^\infty)}_\sigma = 0$ follows from Proposition 4.1(4c), whereas the equality $\widehat{(\delta_{\mathcal{I}}^\infty)} = \delta_{\mathcal{I}}^\infty$ follows from Proposition 4.1(6a).

(5a) It follows from items (1b) and (4a).

(5b) The first inequality follows from the fact that δ_n is a nonzero σ-measure. The equality follows from the fact that $\delta_{\mathcal{I}}^\infty + \delta_n$ is a measure. The second inequality follows from the fact that $\widehat{(\delta_{\mathcal{I}}^\infty + \delta_n)}_\sigma$ is lower semicontinuous (by Proposition 4.1(2)), whereas $\widehat{\delta_{\mathcal{I}}^\infty + \delta_n} = \delta_{\mathcal{I}}^\infty + \delta_n$ is not lower semicontinuous.

(6a) It follows from items (1b) and (2a).

(6b) It follows from items (1b) and (2b).

(6c) The first inequality follows from the fact that δ_n is a nonzero σ-measure. The equality follows from the fact that $\widehat{\delta_{\mathcal{I}}} = \delta_{\mathcal{I}}$ and δ_n is a measure. The second inequality follows from the fact that $\widehat{(\delta_{\mathcal{I}} + \delta_n)}_\sigma$ is lower semicontinuous (by Proposition 4.1(2)), whereas $\widehat{\delta_{\mathcal{I}} + \delta_n} = \delta_{\mathcal{I}} + \delta_n$ is not lower semicontinuous. □

Proposition 5.2. *If μ_n is a measure on X for every $n \in \omega$, then the submeasure*

$$\phi(A) = \limsup_{n \to \infty} \mu_n(A)$$

is nonpathological.

Proof. Take any $A \subseteq X$. Then there exists an increasing sequence $k_0 < k_1 < \dots$ such that $\phi(A) = \lim_{n \to \infty} \mu_{k_n}(A)$. Let $\mathcal{U}$ be an ultrafilter on ω such that $\{k_n : n \in \omega\} \in \mathcal{U}$. Then the measure

$$\nu(A) = \lim_{n \in \mathcal{U}} \mu_n(A)$$

is dominated by ϕ and $\mu(A) = \phi(A)$. Hence $\widehat{\phi}(A) = \phi(A)$. □

For a function $f : \omega \to [0, \infty)$ such that $f(0) \neq 0$,

$$\sum_{i \in \omega} f(i) = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{f(n)}{\sum_{i \in n} f(i)} = 0,$$

and for a fixed $n \in \omega \setminus \{0\}$, we define a σ-measure $\phi_{f,n}$ on ω by

$$\phi_{f,n}(A) = \frac{\sum_{i \in A \cap n} f(i)}{\sum_{i \in n} f(i)},$$

and two submeasures $\phi_f, \overline{\phi}_f$ on ω by

$$\phi_f(A) = \sup \{\phi_{f,n}(A) : n \in \omega\} \quad \text{and} \quad \overline{\phi}_f(A) = \limsup_{n \to \infty} \phi_{f,n}(A).$$

For a constant function $f = 1$ we obtain the *asymptotic density*

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap n|}{n}.$$

Proposition 5.3. *The submeasure ϕ_f is σ-nonpathological and lower semicontinuous, whereas $\overline{\phi}_f$ is nonpathological, but it is not lower semicontinuous.*

Proof. The nonpathology of $\overline{\phi}_f$ follows from Proposition 5.2 and the rest of properties are straightforward. □

6. Examples of pathological submeasures

We start with a well known example of a pathological submeasures on a three-element set. This submeasure can be considered as a blueprint for an example from Proposition 6.2.

Proposition 6.1 (Folklore)**.** *The submeasure $\tau : \{0,1,2\} \to [0,2]$ given by*

$$\tau(A) = \begin{cases} 0 & \text{if } |A| = 0, \\ 1 & \text{if } 1 \leq |A| \leq 2, \\ 2 & \text{if } |A| = 3 \end{cases}$$

is pathological and $0 \neq \widehat{\tau}_\sigma = \widehat{\tau} \neq \tau$. Moreover,

$$\widehat{\tau}(A) = \begin{cases} 0 & \text{if } \tau(A) = 0, \\ 1 & \text{if } \tau(A) = 1, \\ 3/2 & \text{if } \tau(A) = 2. \end{cases}$$

Proof. Using Proposition 4.1(5a), we obtain $\widehat{\tau}_\sigma = \widehat{\tau}$. Since δ_0 is a nonzero σ-measure (Proposition 5.1(1b)) and $\delta_0 \leq \tau$, we obtain $\widehat{\tau}_\sigma \neq 0$. Finally we show that $\widehat{\tau} \neq \tau$ and $\widehat{\tau}(\{0,1,2\}) = 3/2$.

Let $\mu \leq \tau$ be a measure on $\{0,1,2\}$. Then $\mu(\{0\}) + \mu(\{1\}) \leq \tau(\{0,1\}) = 1$, $\mu(\{0\}) + \mu(\{2\}) \leq \tau(\{0,1\}) = 1$ and $\mu(\{1\}) + \mu(\{2\}) \leq \tau(\{0,1\}) = 1$, hence we obtain $2\mu(\{0,1,2\}) \leq 3$. Thus $\mu(\{0,1,2\}) \leq 3/2$, so $\widehat{\tau} \neq \tau$ and $\widehat{\tau}(\{0,1,2\}) \leq 3/2$.

Let μ be a measure on $\{0,1,2\}$ such that $\mu(\{0\}) = \mu(\{1\}) = \mu(\{2\}) = 1/2$. Then $\mu \leq \tau$ and $\mu(\{1,2,3\}) = 3/2$, so $\widehat{\tau}(\{0,1,2\}) \geq 3/2$. □

Proposition 6.2. *Let τ be a submeasure from Proposition 6.1. Let $A_0, A_1, A_2 \subseteq \omega$ be infinite and pairwise disjoint.*

1. *The submeasure $\tau_3 : \mathcal{P}(\omega) \to [0,2]$ given by*

   $$\tau_3(A) = \tau(A \cap \{0,1,2\})$$

   is pathological, lower semicontinuous and $0 \neq \widehat{(\tau_3)}_\sigma = \widehat{\tau_3} \neq \tau_3$. Moreover,

   $$\widehat{\tau_3}(A) = \begin{cases} 0 & \text{if } \tau_3(A) = 0, \\ 1 & \text{if } \tau_3(A) = 1, \\ 3/2 & \text{if } \tau_3(A) = 2. \end{cases}$$

2. *The submeasure $\tau_3^\infty : \mathcal{P}(\omega) \to [0,2]$ given by*

   $$\tau_3^\infty(A) = \begin{cases} 0 & \text{if } |\{i < 3 : |A \cap A_i| = \omega\}| = 0, \\ 1 & \text{if } 1 \leq |\{i < 3 : |A \cap A_i| = \omega\}| \leq 2, \\ 2 & \text{if } |\{i < 3 : |A \cap A_i| = \omega\}| = 3 \end{cases}$$

is pathological, not lower semicontinuous and $0 = \widehat{(\tau_3^\infty)_\sigma} \neq \widehat{\tau_3^\infty} \neq \tau_3^\infty$. *Moreover,*

$$\widehat{\tau_3^\infty}(A) = \begin{cases} 0 & \text{if } \tau_3^\infty(A) = 0, \\ 1 & \text{if } \tau_3^\infty(A) = 1, \\ 3/2 & \text{if } \tau_3^\infty(A) = 2. \end{cases}$$

Proof. Obviously τ_3 is lower semicontinuous while τ_3^∞ is not lower semicontinuous because it vanishes on finite sets (see Proposition 4.1(4c)).

To show that $\widehat{\tau_3} \neq \tau_3$, $\widehat{\tau_3^\infty} \neq \tau_3^\infty$ and obtain the values of $\widehat{\tau_3}$, it is enough to imitate the proof of Proposition 6.1, so we omit the details.

Since τ_3^∞ vanishes on finite sets, we obtain $\widehat{(\tau_3^\infty)_\sigma} = 0$ by Proposition 4.1(4c), whereas by Proposition 4.1(4b), we obtain that $\widehat{(\tau_3)_\sigma} \neq 0$.

Since $\tau_3(\omega \setminus \{0,1,2\}) = 0$, we can use Proposition 4.1(5b) to obtain $\widehat{(\tau_3)_\sigma} = \widehat{\tau_3}$.

Finally we calculate the value $\widehat{\tau_3^\infty}(A)$ for every set A.

If $\tau_3^\infty(A) = 0$, then $\widehat{\tau_3^\infty}(A) \leq \tau_3^\infty(A) = 0$, so we are done.

If $\tau_3^\infty(A) = 1$, then there is $i < 3$ such that $A \cap A_i$ is infinite. Let $\mathcal{I}$ be a maximal ideal on ω which contains all finite sets and $A \cap A_i \notin \mathcal{I}$. Then $\delta_\mathcal{I}$ is a nonzero measure (by Proposition 5.1(2b)). We show that $\delta_\mathcal{I} \leq \tau_3^\infty$. If $B \in \mathcal{I}$, then $\delta_\mathcal{I}(B) = 0 \leq \tau_3^\infty(B)$. Assume that $B \notin \mathcal{I}$. Since $\mathcal{I}$ is maximal, $B \cap A_i \notin \mathcal{I}$. Consequently, $A_i \cap B$ is infinite, so $\tau_3^\infty(B) \geq 1 = \delta_\mathcal{I}(B)$. Thus $1 \leq \widehat{\tau_3^\infty}(A) \leq \tau_3^\infty(A) = 1$.

If $\tau_3^\infty(A) = 2$, we first observe that the inequality $\widehat{\tau_3^\infty}(A) \leq 3/2$ can be shown as in the proof of Proposition 6.1. To show the reverse inequality, we take three maximal ideals $\mathcal{I}_0, \mathcal{I}_1$ and $\mathcal{I}_2$ on ω which contain all finite sets and $A \cap A_i \notin \mathcal{I}_i$ for $i = 0,1,2$. Then we define a measure $\mu = (\delta_{\mathcal{I}_0} + \delta_{\mathcal{I}_1} + \delta_{\mathcal{I}_2})/2$ and repeating the argument from the previous paragraph, we show that $\mu \leq \tau_3^\infty$. Since $\mu(A) = 3/2$, the proof is finished. $\square$

The following theorem shows that there are ε-pathological submeasures on finite sets for arbitrarily small ε. Then using this theorem we can construct three types of pathological submeasures on ω (see Proposition 6.4).

Theorem 6.3 ([12, Theorem 1]). *Let* $\varepsilon > 0$ *be an arbitrary positive number. There exist a finite set* X *and a submeasure* ϕ *on* X *such that* $\phi(X) = 1$ *and* $\mu(X) \leq \varepsilon$ *for any measure* $\mu \leq \phi$.

Proposition 6.4. *Using Theorem 6.3, we construct a partition* $\{I_n : n < \omega\}$ *of* ω *into finite intervals and submeasures* ϕ_n *on* I_n *such that* $\phi_n(I_n) = 1$ *and* $\mu(I_n) \leq 1/2^{n+1}$ *for every measure* $\mu_n \leq \phi_n$ *and each* $n < \omega$.

(1) *The lower semicontinuous submeasure* ψ_0 *on* ω *given by*

$$\psi_0(A) = \phi_0(A \cap I_0)$$

is pathological and $0 \neq \widehat{(\psi_0)_\sigma} = \widehat{\psi_0} \neq \psi_0$.

(2) *The lower semicontinuous submeasure* ψ_1 *on* ω *given by*

$$\psi_1(A) = \sup\{\phi_n(A \cap I_n) : n < \omega\}$$

is pathological.

(3) *The lower semicontinuous submeasure* ψ_2 *on* ω *given by*

$$\psi_2(A) = \sum_{n<\omega} \phi_n(A \cap I_n)$$

is pathological, $0 \neq \widehat{(\psi_2)}_\sigma \neq \widehat{\psi_2} \neq \psi_2$ *and* $\widehat{(\psi_2)}_\sigma(\omega) < \infty = \widehat{\psi_2}(\omega)$.

(4) *The lower semicontinuous submeasure* ψ_3 *on* ω *given by*

$$\psi_3(A) = \sup\{(n+1)\phi_n(A \cap I_n) : n < \omega\}$$

is pathological, $0 \neq \widehat{(\psi_3)}_\sigma \neq \widehat{\psi_3} \neq \psi_3$, $\widehat{(\psi_3)}_\sigma(\omega) < \infty = \widehat{\psi_3}(\omega)$ *and* $\widehat{\psi_3}$ *is* not *lower semicontinuous.*

Proof. Since $\psi_i(I_0) = 1 > 0$ for each i, we obtain $\widehat{(\psi_i)}_\sigma \neq 0$ by Proposition 4.1(4b).

We show that ψ_i is pathological for each i. We fix $i < 4$ and take any measure $\mu \leq \psi_i$. Since $\mu \restriction \mathcal{P}(I_0)$ is a measure on I_0 which is dominated by ϕ_0, we obtain that $\mu(I_0) \leq 1/2 < 1 = \psi_i(I_0)$. Thus $\widehat{\psi_i} \neq \psi_i$.

We show $\widehat{(\psi_0)}_\sigma = \widehat{\psi_0}$. It is enough to observe that for any measure $\mu \leq \psi_0$ we have $\mu(\omega \setminus I_0) \leq \psi_0(\omega \setminus I_0) = 0$. Thus, μ takes nonzero values only for subsets of a finite set I_0, and consequently μ is a σ-measure.

We show $\widehat{(\psi_i)}_\sigma \neq \widehat{\psi_i}$ and $\widehat{\psi_i}(\omega) = \infty$ for $i \in \{2, 3\}$. Since $\phi_n(I_n) = 1$ for every n, we obtain that $\psi_i(\omega) = \infty$. Let $\mu : \mathcal{P}(\omega) \to [0, \infty]$ be given by

$$\mu(A) = \begin{cases} \infty & \text{if } \psi_i(A) = \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Since μ is a measure and $\mu \leq \psi_i$, we obtain $\widehat{\psi_i}(\omega) \geq \mu(\omega) = \infty$. Once we show that $\widehat{(\psi_i)}_\sigma(\omega) < \infty$, the proof of this case will be finished. Let $\mu \leq \psi_i$ be a σ-measure. Since $\mu \restriction \mathcal{P}(I_k)$ is a measure on I_k dominated by $(k+1)\phi_k$, we obtain that

$$\mu\left(\bigcup_{k<n} I_k\right) = \sum_{k<n} \mu(I_k) \leq \sum_{k<n} \frac{k+1}{2^{k+1}} < \sum_{k<\omega} \frac{k+1}{2^{k+1}} = 2 < \infty$$

for every $n \in \omega$. Since $\widehat{(\psi_i)}_\sigma$ is lower semicontinuous (by Proposition 4.1(2)), we have

$$\widehat{(\psi_i)}_\sigma(\omega) = \lim_{n\to\infty} \widehat{(\psi_i)}_\sigma\left(\bigcup_{k<n} I_k\right) \leq 2 < \infty.$$

Finally, we show that $\widehat{\psi_3}$ is not lower semicontinuous. In the previous paragraph, we showed that $\widehat{\psi_3}(\omega) = \infty$. By Proposition 4.1(5a), we know that $\widehat{\psi_3}(F) = \widehat{(\psi_3)}_\sigma(F)$ for every finite set F, so

$$\widehat{\psi_3}(F) = \widehat{(\psi_3)}_\sigma(F) \leq 2,$$

and consequently

$$\lim_{n\to\infty} \widehat{\psi_3}(n) \leq 2 < \infty = \widehat{\psi_3}(\omega).$$

□

Theorem 6.5 ([12, Theorem 2] (see also [30, Example 3]))**.** *There exists a nonzero submeasure* χ *on* ω *such that* $\widehat{\chi} = 0$, $\chi(\omega) = 1$ *and* $\chi(F) = 0$ *for every finite set* $F \subseteq \omega$.

7. Degree of pathology of submeasures

As in the case of definitions of pathological submeasure, there is also a mess with definitions of degrees of pathology of submeasures. In this section we try to clean it up. The results we obtained are summarized in Table 1.

Definition 7.1.

(1) The *degree of pathology* ([5, p. 21]) of a submeasure ϕ on X is given by

$$P(\phi) = \sup\left\{ \frac{\phi(A)}{\widehat{\phi}(A)} : A \subseteq X \right\}$$

with the convention that $\infty/\infty = 0/0 = 1$ and $a/0 = \infty/a = \infty$ for a positive $a \in \mathbb{R}$.

(2) We also define a "σ-version" of the degree of pathology by

$$P_\sigma(\phi) = \sup\left\{ \frac{\phi(A)}{\widehat{\phi}_\sigma(A)} : A \subseteq X \right\}.$$

(3) In [23, Section 3.1] (see also [22, p. 5] or [21, p. 3]), the authors introduced a "Fin-variant" of the degree of pathology by

$$P_{\mathrm{Fin}}(\phi) = \sup\left\{ \frac{\phi(F)}{\widehat{\phi}(F)} : F \text{ is a finite subset of } X \right\}.$$

(4) Since $\widehat{\phi}(F) = \widehat{\phi}_\sigma(F)$ for every finite set $F \subseteq \omega$ (by Proposition 4.1(5a)), a "σ-Fin-variation" of the degree of pathology coincides with P_{Fin}.

The following proposition summarize basic properties of these three versions of the degree of pathology.

Proposition 7.2. *Let ϕ be a submeasure on X.*

(1) $1 \leq P_{\mathrm{Fin}}(\phi) \leq P(\phi) \leq P_\sigma(\phi) \leq \infty$.

(2) (a) *ϕ is nonpathological* $\iff$ $P(\phi) = 1$.
    (b) *ϕ is σ-nonpathological* $\iff$ $P_\sigma(\phi) = 1$.

(3) *If ϕ is a lower semicontinuous submeasure on ω, then*
    (a) $P_{\mathrm{Fin}}(\phi) = P(\phi) = P_\sigma(\phi)$,
    (b) *ϕ is nonpathological* $\iff$ $P_{\mathrm{Fin}}(\phi) = 1$.

Proof. Straightforward with the aid of Proposition 4.1. □

In Table 1, we present examples which show that almost all configurations of "<" and "=" signs in item (1) of Proposition 7.2 are possible. Below we provide some proofs for the examples presented in Table 1.

Proposition 7.3. $P_{\mathrm{Fin}}(\psi_1) = P_{\mathrm{Fin}}(\psi_2) = \infty$.

Proof. It is enough to notice that $\psi_i(I_n) = 1$ and $\widehat{\psi}_i(I_n) \leq 1/2^{n+1}$ for each $n \in \omega$ and $i = 1, 2$. □

Proposition 7.4. $P(\delta + \delta_{\mathrm{Fin}}) = 1$ *and* $P_\sigma(\delta + \delta_{\mathrm{Fin}}) = 2$.

Proof. Because $\delta + \delta_{\mathrm{Fin}}$ is nonpathological (Prop. 5.1(3)), we obtain $P(\delta + \delta_{\mathrm{Fin}}) = 1$ by Prop. 7.2(2a). The equality $P_\sigma(\delta + \delta_{\mathrm{Fin}}) = 2$ follows from the fact that $(\widehat{\delta + \delta_{\mathrm{Fin}}})_\sigma = \delta$ (Prop. 5.1(3)). □

1	$\leq$	$P_{\mathrm{Fin}}(\phi)$	$\leq$	$P(\phi)$	$\leq$	$P_\sigma(\phi)$	$\leq$	∞	Example
1	$=$	1	$=$	1	$=$	1	$<$	∞	Every σ-measure.
1	$=$	1	$=$	1	$<$	∞	$=$	∞	$\delta_{\mathrm{Fin}}^\infty$ (Prop. 5.1(4b))
1	$=$	1	$<$	∞	$=$	∞	$=$	∞	χ (Thm. 6.5)
1	$<$	∞	$=$	∞	$=$	∞	$=$	∞	ψ_1, ψ_2 (Prop. 7.3)
1	$=$	1	$=$	1	$<$	2	$<$	∞	$\delta+\delta_{\mathrm{Fin}}$ (Prop. 7.4)
1	$=$	1	$<$		$=$	2	$<$	∞	**?** $\delta+\chi$ (Prop. 7.5)
1	$=$	1	$<$	$4/3$	$<$	∞	$=$	∞	τ_3^∞ (Prop. 7.6))
1	$<$	$4/3$	$=$	$4/3$	$=$	$4/3$	$<$	∞	τ_3 (Prop. 6.2(1))
1	$<$	$4/3$	$=$	$4/3$	$<$	∞	$=$	∞	$\tau_3 \oplus \tau_3^\infty$ (Prop. 7.8)
1	$<$	$4/3$	$<$	∞	$=$	∞	$=$	∞	$\tau_3 \oplus \chi$ (Prop. 7.8)
1	$=$	1	$<$	$3/2$	$<$	6	$<$	∞	η (Prop. 7.9)
1	$<$	$4/3$	$=$	$4/3$	$<$	2	$<$	∞	$(\delta+\delta_{\mathrm{Fin}}) \oplus \tau_3$ (Prop. 7.8)
1	$<$	$4/3$	$<$		$=$	2	$<$	∞	**?** $\tau_3 \oplus (\delta+\chi)$ (Prop. 7.8)
1	$<$	$4/3$	$<$	$3/2$	$<$	∞	$=$	∞	$\tau_3 \oplus \tau_3^\infty \oplus \eta$ (Prop. 7.8)
1	$<$	$4/3$	$<$	$3/2$	$<$	6	$<$	∞	$\eta \oplus \tau_3$ (Prop. 7.8)

TABLE 1. Possible configurations of "<" and "=" signs in Proposition 7.2(1).

Proposition 7.5. *$P_{\mathrm{Fin}}(\delta+\chi) = 1$ and $P_\sigma(\delta+\chi) = 2$*

Proof. To show that $P_{\mathrm{Fin}}(\delta+\chi) = 1$, it is enough to observe that for every finite set F, $(\delta+\chi)(F) = \delta(F)$, so $\widehat{(\delta+\chi)}(F) = \widehat{\delta}(F)$

To show that $P_\sigma(\delta+\chi) = 2$, we first observe that $\widehat{(\delta+\chi)}_\sigma \geq \widehat{\delta}_\sigma = \delta$. On the other hand, if we have a measure μ such that $\mu \leq \delta+\chi$, then $\mu(F) \leq (\delta+\chi)(F) = 1$, so by using lsc of $\widehat{(\delta+\chi)}_\sigma$ we obtain that $\widehat{(\delta+\chi)}_\sigma(A) \leq 1 = \delta(A)$ for every nonempty A. □

Proposition 7.6. *$P_{\mathrm{Fin}}(\tau_3^\infty) = 1$, $P(\tau_3^\infty) = 4/3$ and $P_\sigma(\tau_3^\infty) = \infty$.*

Proof. Since $\tau_3^\infty(F) = 0$ for every finite set F, we obtain $P_{\mathrm{Fin}}(\tau_3^\infty) = 1$. By Prop. 6.2, $\widehat{\tau_3^\infty} = 0$, so we obtain $P_\sigma(\tau_3^\infty) = \infty$. Lastly, $P(\tau_3^\infty) = 4/3$ follows from Proposition 6.2(2). □

For submeasures ϕ and ψ on ω, we define two submeasures $\phi \oplus_m \psi$ and $\phi \oplus_s \psi$ on $X = \{0,1\} \times \omega$ by

$$(\phi \oplus_m \psi)(A) = \max\{\phi(A), \psi(A)\}$$
$$(\phi \oplus_s \psi)(A) = \phi(A) + \psi(A),$$

where $\phi(A)$ and $\psi(A)$ mean $\phi(\{n \in \omega : (0,n) \in A\})$ and $\psi(\{n \in \omega : (1,n) \in A\})$, respectively.

Lemma 7.7. *For every submeasures ϕ and ψ on ω and every $C \subseteq \{0,1\} \times \omega$, we have*

1. $\widehat{\phi \oplus_m \psi}(C) \geq \max\{\widehat{\phi}(C), \widehat{\psi}(C)\}$,
2. $\widehat{\phi \oplus_s \psi}(C) = \widehat{\phi}(C) + \widehat{\psi}(C)$.

The same properties hold for $\widehat{\phi}_\sigma$ and $\widehat{\psi}_\sigma$.

Proof. (1) Take any $C \subseteq \{0,1\} \times \omega$. Without loss of generality, we can assume $\widehat{\phi(C)} \geq \widehat{\psi(C)}$. Then there is a measure μ on ω such that $\mu \leq \phi$ and $\mu(C) > M$. Let ν be a measure on $\{0,1\} \times \omega$ such that $\nu(\{0\} \times A) = \mu(A)$ and $\nu(\{1\} \times A) = 0$ for every set $A \subseteq \omega$. Then $\nu \leq \phi \oplus_m \psi$ and $\nu(C) = \mu(C) > M$. Hence $\widehat{\phi \oplus_m \psi}(C) > M$.

(2,$\leq$) Take any measure $\mu \leq \widehat{\phi \oplus_s \psi}$. Let $\mu_i(A) = \mu(\{(i,n) : n \in A\})$ for $A \subseteq \omega$ and $i = 0, 1$. Then μ_0 and μ_1 are measures on ω and $\mu_0 \leq \phi$, $\mu_1 \leq \psi$. Hence $\mu_0(A) \leq \widehat{\phi}(A)$ and $\mu_1(A) \leq \widehat{\psi}(A)$ for every $A \subseteq \omega$. Then $\mu(C) = \mu_0(C) + \mu_1(C) \leq \widehat{\phi}(C) + \widehat{\psi}(C)$.

(2,$\geq$) Take any $C \subseteq \{0,1\} \times \omega$. Take any $M_1 < \widehat{\phi}(C)$ and $M_2 < \widehat{\psi}(C)$. Then there are measures μ_1 and μ_2 on ω such that $\mu_1 \leq \phi$, $\mu_2 \leq \psi$ and $\mu_i(C) > M_i$ for $i = 1, 2$. Let $\nu = \mu_1 \oplus_s \mu_2$. Then $\nu \leq \phi \oplus_s \psi$ and $\nu(C) = \mu_1(C) + \mu_2(C) > M_1 + M_2$. Hence $\widehat{\phi \oplus_s \psi}(C) > M_1 + M_2$. □

Proposition 7.8. *For every submeasures ϕ and ψ, we have*

(1) $P(\phi \oplus_m \psi) = \max\{P(\phi), P(\psi)\}$,
(2) $P(\phi \oplus_s \psi) = \max\{P(\phi), P(\psi)\}$.

The same equalities hold for P_σ and P_{Fin}.

Proof. (1, $\leq$) It is enough to notice, using Lemma 7.7, that for every set C we have

$$\frac{\phi \oplus_m \psi(C)}{\widehat{\phi \oplus_m \psi}(C)} \leq \frac{\max\{\phi(C), \psi(C)\}}{\max\{\widehat{\phi}(C), \widehat{\psi}(C)\}} \leq \max\left\{\frac{\phi(C)}{\widehat{\phi}(C)}, \frac{\psi(C)}{\widehat{\psi}(C)}\right\} \leq \max\{P(\phi), P(\psi)\}.$$

(1, $\geq$) Without loss of generality, we can assume that $P(\phi) \geq P(\psi)$. Take any $M < P(\phi)$. Then there is $C \subseteq \{0\} \times \omega$ such that $\phi(C)/\widehat{\phi}(C) > M$. Then $(\phi \oplus_m \psi)(C) = \max\{\phi(C), 0\} = \phi(C)$, and consequently $\widehat{\phi \oplus_m \psi}(C) = \widehat{\phi}(C)$. Thus,

$$\frac{(\phi \oplus_m \psi)(C)}{\widehat{\phi \oplus_m \psi}(C)} = \frac{\phi(C)}{\widehat{\phi}(C)} > M.$$

Hence, $P(\phi \oplus_m \psi) > M$.

(2) The inequality $\geq$ can be done as in item (1), and to show the inequality $\leq$ it is enough to notice, using Lemma 7.7, that for every set C we have

$$\frac{\phi \oplus_s \psi(C)}{\widehat{\phi \oplus_s \psi}(C)} = \frac{\phi(C) + \psi(C)}{\widehat{\phi}(C) + \widehat{\psi}(C)} \leq \max\left\{\frac{\phi(C)}{\widehat{\phi}(C)}, \frac{\psi(C)}{\widehat{\psi}(C)}\right\} \leq \max\{P(\phi), P(\psi)\}.$$

□

Proposition 7.9. *Let $A_0, A_1, A_2, A_3 \subseteq \omega$ be infinite and pairwise disjoint. The submeasure η on ω given by*

$$\eta(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A \neq \emptyset \text{ and } |\{i < 4 : |A \cap A_i| = \omega\}| = 0, \\ 3 & \text{if } 1 \leq |\{i < 4 : |A \cap A_i| = \omega\}| \leq 3, \\ 6 & \text{if } |\{i < 4 : |A \cap A_i| = \omega\}| = 4 \end{cases}$$

is pathological, not lower semicontinuous and $0 \neq \delta = \widehat{(\eta)}_\sigma \neq \widehat{\eta} \neq \eta$*. Moreover,*

$$\widehat{\eta}(A) = \begin{cases} 0 & \text{if } \eta(A) = 0, \\ 1 & \text{if } \eta(A) = 1, \\ 3 & \text{if } \eta(A) = 3, \\ 4 & \text{if } \eta(A) = 6. \end{cases}$$

Proof. Clearly, $\eta(F) \leq 1$ for every finite $F \subseteq \omega$ and $\eta(\omega) = 6$, so η is not lower semicontinuous. It is also an easy observation that $\widehat{(\eta)}_\sigma = \delta$.

We will now calculate the value $\widehat{\eta}(A)$ for every set A, which will also prove that $\widehat{(\eta)}_\sigma \neq \widehat{\eta} \neq \eta$.

If $\eta(A) = 0$, then $\widehat{\eta}(A) \leq \eta(A) = 0$, so we are done.

If $\eta(A) = 1$, then for any $n \in A$ take the measure δ_n in order to obtain $\delta_n \leq \eta$ and $\delta_n(A) = 1$.

If $\eta(A) = 3$, then there is $i < 4$ such that $A \cap A_i$ is infinite. Let $\mathcal{I}$ be a maximal ideal on ω which contains all finite sets and $A \cap A_i \notin \mathcal{I}$. Then $3\delta_{\mathcal{I}}$ is a nonzero measure (by Proposition 5.1(2b)). We show that $3\delta_{\mathcal{I}} \leq \eta$. If $B \in \mathcal{I}$, then $3\delta_{\mathcal{I}}(B) = 0 \leq \eta(B)$. Assume that $B \notin \mathcal{I}$. Since $\mathcal{I}$ is maximal, $B \cap A_i \notin \mathcal{I}$. Consequently $A_i \cap B$ is infinite, so $\eta(B) \geq 3 = 3\delta_{\mathcal{I}}(B)$. Thus $3 \leq \widehat{\eta}(A) \leq \eta(A) = 3$.

If $\eta(A) = 6$, to show that $\widehat{\eta}(A) \geq 4$, we take four maximal ideals $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$ on ω such that $A \cap A_i \notin \mathcal{I}_i$ for $i = 0, 1, 2, 3$. Then we define a measure $\mu = (\delta_{\mathcal{I}_0} + \delta_{\mathcal{I}_1} + \delta_{\mathcal{I}_2} + \delta_{\mathcal{I}_3})$ and repeating the argument from the previous paragraph, we show that $\mu \leq \eta$. Since $\mu(A) = 4$, we obtain $\widehat{\eta}(A) \geq 4$. To obtain the reverse inequality, we can proceed analogously to the proof of Proposition 6.1. $\square$

Proposition 7.10. *Suppose that* ϕ *is a submeasure on* ω *such that*

$$1 = P_{\text{Fin}}(\phi) < P(\phi) = P_\sigma(\phi) < \infty.$$

Let $\alpha \in \mathbb{R}$ *be such that* $\alpha \leq 2$ *and* $3/2 < \alpha < (3/2) \cdot P(\phi)$*. Let* ψ *be a submeasure on* ω *given by*

$$\psi(A) = \begin{cases} 0 & \text{if } A \cap \{0,1,2\} = \emptyset, \\ 1 & \text{if } 1 \leq |A \cap \{0,1,2\}| \leq 2, \\ \alpha & \text{if } |A \cap \{0,1,2\}| = 3. \end{cases}$$

Then

$$P_{\text{Fin}}(\psi) = P(\psi) = P_\sigma(\psi) = (2/3) \cdot \alpha,$$

and consequently,

$$1 < P_{\text{Fin}}(\phi \oplus \psi) < P(\phi \oplus \psi) = P_\sigma(\phi \oplus \psi) < \infty.$$

Proof. The degree of pathology of ψ can be calculated the same way as for τ_3 (using an appropriate modification of Proposition 6.2). Whereas the degree of pathology of $\psi \oplus \psi$ can be easily calculated with the aid of Proposition 7.8. $\square$

Question 7.11.

1. What is the value of $P(\delta + \chi)$?
2. Is there a submeasure ϕ with $1 = P_{\text{Fin}}(\phi) < P(\phi) = P_\sigma(\phi) < \infty$?

Part 2. (Non)pathological ideals

In this part of the paper, we alter the definition of an ideal. Namely, from this point on we additionally assume that *an ideal $\mathcal{I}$ on X has to contain all finite subsets of X.*

8. (Non)pathological ideals

In the literature, the (non)pathology of ideals is only considered in the realm of analytic P-ideals and F_σ ideals so far. Let us recall well-known characterizations of analytic P-ideals and F_σ ideals that are expressed in terms of submeasures and are necessary to phrase the definition of pathological ideals.

Theorem 8.1 ([27], see also [5, Theorem 1.2.5]). *The following conditions are equivalent.*

1. *$\mathcal{I}$ is an analytic P-ideal on ω.*
2. *There is a lower semicontinuous submeasure ϕ on ω such that* $\lim_{n\to\infty}\phi(\omega\setminus n) > 0$ *and*
$$\mathcal{I} = \mathrm{Exh}(\phi) = \left\{A \subseteq \omega : \lim_{n\to\infty}\phi(A\setminus n) = 0\right\}.$$

Theorem 8.2 ([26, Lemma 1.2], see also [5, Theorem 1.2.5]). *The following conditions are equivalent.*

1. *$\mathcal{I}$ is an F_σ ideal on ω.*
2. *There is a lower semicontinuous submeasure ϕ on ω such that $\phi(n) < \infty$ for each $n \in \omega$, $\phi(\omega) = \infty$ and*
$$\mathcal{I} = \mathrm{Fin}(\phi) = \{A \subseteq \omega : \phi(A) < \infty\}.$$

Having the above characterizations we are prepared to present the definitions of (non)pathological ideals.

Definition 8.3 ([5, pp. 25 and 53]). An analytic P-ideal (F_σ ideal, resp.) $\mathcal{I}$ is *nonpathological* if there exists a nonpathological, lower semicontinuous submeasure ϕ such that $\mathcal{I} = \mathrm{Exh}(\phi)$ ($\mathcal{I} = \mathrm{Fin}(\phi)$, resp.). Otherwise, we say that $\mathcal{I}$ is *pathological.*

By the definition, every submeasure which defines a pathological ideal has to be pathological. On the other hand, every nonpathological ideal can be always expressed with the aid of some pathological submeasure ([23, Proposition 3.4]). Indeed, take any lsc submeasure ϕ with $\mathcal{I} = \mathrm{Fin}(\phi)$ ($\mathcal{I} = \mathrm{Exh}(\phi)$, resp.), then the submeasure $\psi(A) = \tau(A \cap \{0,1,2\}) + \phi(A \setminus \{0,1,2\})$ (where τ is defined in Proposition 6.1) is pathological (as $P(\psi) \geq P(\tau) = 4/3 > 1$ by Proposition 7.8) and $\mathrm{Fin}(\psi) = \mathrm{Fin}(\phi)$ ($\mathrm{Exh}(\psi) = \mathrm{Exh}(\phi)$, resp.).

It is known ([5, Lemma 1.2.2]) that $\mathrm{Exh}(\phi) \subseteq \mathrm{Fin}(\phi)$ for every lsc submeasure ϕ, however there is no relation between pathology of $\mathrm{Exh}(\phi)$ and pathology of $\mathrm{Fin}(\phi)$ in general. Indeed, first we observe that if $\mathrm{Exh}(\phi)$ is a pathological ideal, then $\psi = \min\{1, \phi\}$ is a lsc submeasure with $\mathrm{Exh}(\phi) = \mathrm{Exh}(\psi)$ and $\mathrm{Fin}(\psi) = \mathcal{P}(\omega)$, so $\mathrm{Exh}(\psi)$ is pathological whereas $\mathrm{Fin}(\psi)$ is not even an ideal. If we now take $\zeta(A) = \sup\{\psi(\{n \in \mathbb{N} : 2n \in A\}), |A \cap (2\mathbb{N}+1)|\}$ then $\mathrm{Fin}(\zeta) = \mathcal{P}(\mathbb{N}) \oplus \mathrm{Fin}$ would be a proper nonpathological ideal while $\mathrm{Exh}(\zeta) = \mathrm{Exh}(\psi) \oplus \mathrm{Fin}$ would be pathological. On the other hand, if we take any pathological ideal $\mathrm{Fin}(\phi)$ and put $\psi = \sup\{\phi, \delta\}$ then $\mathrm{Fin}(\psi) = \mathrm{Fin}(\phi)$ would still be pathological while $\mathrm{Exh}(\psi) = \mathrm{Fin}$ would be nonpathological.

Let us describe some important classes of nonpahological ideals.

Using submeasures ϕ_f and $\overline{\phi}_f$ which we defined at page 9 and using Proposition 5.3, we see that the *Erdős-Ulam ideal generated by f* ([17]),

$$\mathcal{EU}_f = \text{Exh}(\phi_f) = \left\{A \subseteq \omega \ : \overline{\phi}_f(A) = 0\right\},$$

is a nonpathological analytic P-ideal. In particular, the the ideal

$$\mathcal{I}_d = \left\{A \subseteq \omega : \limsup_{n\to\infty} \frac{|A \cap n|}{n} = 0\right\}$$

of all sets of the *asymptotic density* zero is a nonpathological analytic P-ideal.

Using measures μ_f which were defined at page 3 we see that the *summable ideal generated by f* ([24, Example 3, p.206]),

$$\mathcal{I}_f = \text{Fin}(\mu_f) = \left\{A \subseteq \omega : \sum_{n\in A} f(n) < \infty\right\},$$

is a nonpathological F_σ ideal. In particular, the the ideal

$$\mathcal{I}_{1/n} = \left\{A \subseteq \omega : \sum_{n\in A} \frac{1}{n+1}\right\}$$

is a nonpathological F_σ ideal.

The following theorem gives a useful characterization of nonpathologicity for analytic P-ideals. One of the items of this characterisation involves the *Solecki ideal* $\mathcal{S}$ introduced in [28] (see also [14, Section 3.6]).

Theorem 8.4 ([15, Corollary 5.6])**.** *For an analytic P-ideal $\mathcal{I}$ the following conditions are equivalent.*

1. *$\mathcal{I}$ is nonpathological.*
2. *$\mathcal{I} \restriction A \leq_K \mathcal{I}_d$ for every $A \notin \mathcal{I}$.*
3. *$\mathcal{S} \not\leq_K \mathcal{I} \restriction A$ for every $A \notin \mathcal{I}$.*

Another class of nonpathological ideals can be defined with the aid of infinite matrices of reals. Namely, if a nonnegative matrix $A = (a_{i,k})_{i,k\in\omega}$ is *regular*, i.e.

$$\sup_{i\in\omega} \sum_{k\in\omega} a_{i,k} < \infty, \quad \lim_{i\to\infty} \sum_{k\in\omega} a_{i,k} = 1 \quad \text{and} \quad \lim_{i\to\infty} a_{i,k} = 0 \text{ for every } k \in \omega,$$

then the *matrix summability ideal generated by A* (in short *matrix ideal*),

$$\mathcal{I}(A) = \text{Exh}\left(\sup_{i\in\omega} \mu_{f_i}\right) = \left\{B \subseteq \mathbb{N} : \limsup_{i\to\infty} \sum_{k\in B} a_{i,k} = 0\right\},$$

is a nonpathological analytic P-ideal (see [1, the proof of Proposition 13]).

Matrix ideals can be used for checking pathologicity of ideals as shown in the following theorem.

Theorem 8.5 ([10, Theorems 5.7, 5.14 and Proposition 5.1])**.** *Every nonpathological ideal can be represented as the intersection of some matrix summability ideals i.e. if $\mathcal{I}$ is a nonpathological ideal, then there exists a family $\mathcal{M}$ of matrix summability ideals with*

$$\mathcal{I} = \bigcap \mathcal{M}.$$

In particular, every nonpathological ideal is contained in some matrix ideal.

Let us finish this section with examples of pathological ideals. In [9, Theorem 4.12], the authors constructed an F_σ P-ideal $\mathcal{I}$ which is not contained in any matrix summability ideal, and consequently (by Theorem 8.5) the ideal $\mathcal{I}$ is pathological. Another example of a pathological ideal can be found in [23, Section 3.2.1]. We would emphasize that both of these examples rely heavily on the construction of an F_σ ideal which is not contained in any summable ideal (given by K. Mazur [26, Theorem 1.9]). Using Theorem 11.2, one can see that the ideal constructed by Mazur is pathological, hence his ideal seems to be the first pathological ideal constructed ever.

9. Intersections of matrix ideals

The following theorem shows that item (2) of Theorem 8.4 characterizes ideals that can be represented as the intersections of matrix summability ideals in the realm of all ideals (not necessarily analytic P-ideals).

Theorem 9.1. *An ideal $\mathcal{I}$ is equal to the intersection of a family of matrix summability ideals if and only if $\mathcal{I} \restriction A \leq_K \mathcal{I}_d$ for every $A \notin \mathcal{I}$.*

Proof. '($\Rightarrow$)': Follows from [32, Corollary 3.4] and the proof of [32, Theorem 3.8] as noted by remarks under [32, Corollary 3.9].

'($\Leftarrow$)': Take $A \notin \mathcal{I}$. Let $f : \mathbb{N} \to A$ be such that for every $C \in \mathcal{I} \restriction A$ we have $f^{-1}[C] \in \mathcal{I}_d$.

We will finish the proof by constructing a regular matrix $B = (b_{i,k})$ such that $A \notin \mathcal{I}(B)$ and $\mathcal{I} \subseteq \mathcal{I}(B)$. For every $i, k \in \mathbb{N}$ put

$$b_{i,k} = \frac{|f^{-1}[\{k\}] \cap [1,i]|}{i}$$

Clearly, $b_{i,k} \geq 0$ for all $i, k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_{i,k} = 1$ for each $i \in \mathbb{N}$. Moreover, since for each $k \in \mathbb{N}$ we have $\{k\} \in \mathcal{I} \restriction A$, we get $f^{-1}[\{k\}] \in \mathcal{I}_d$. Therefore, for each $k \in \mathbb{N}$ we have

$$\lim_{i\to\infty} \frac{|f^{-1}[\{k\}] \cap [1,i]|}{i} = 0,$$

hence $\lim_{i\to\infty} b_{i,k} = 0$. Thus, the matrix B is regular.

Observe that

$$\sum_{k\in C} b_{i,k} = \frac{|f^{-1}[C] \cap [1,i]|}{i}$$

for every $i \in \mathbb{N}$ and $C \subseteq \mathbb{N}$.

Obviously, $f^{-1}[A] = \mathbb{N}$. It follows that

$$\sum_{k\in A} b_{i,k} = \frac{|f^{-1}[A] \cap [1,i]|}{i} = \frac{|[1,i]|}{i} = 1$$

for every $i \in \mathbb{N}$, thus $A \notin \mathcal{I}(B)$.

Moreover, since for every $C \in \mathcal{I} \restriction A$ we have $f^{-1}[C] \in \mathcal{I}_d$, it follows that for every such C we obtain

$$\lim_{i\to\infty} \frac{|f^{-1}[C] \cap [1,i]|}{i} = 0,$$

thus $\lim_{i\to\infty} \sum_{k\in C} b_{i,k} = 0$, hence $C \in \mathcal{I}(B)$. Therefore, $\mathcal{I} \restriction A \subseteq \mathcal{I}(B)$. Since $\mathcal{I} \subseteq \mathcal{I} \restriction A \cup \mathcal{P}(\mathbb{N} \setminus A)$ and $\mathbb{N} \setminus A \in \mathcal{I}(B)$, we obtain $\mathcal{I} \subseteq \mathcal{I}(B)$. □

In [22, Question 3.13], the authors asked whether the Solecki ideal $\mathcal{S}$ is pathological. Below we argue that the answer is positive, which gives another example of a pathological F_σ ideal (in [23, Section 3.3], the authors announced that the same result was independently obtained by Figueroa and Hrušák).

Corollary 9.2. *The Solecki ideal is pathological.*

Proof. It is known that the Solecki ideal $\mathcal{S}$ is a tall F_σ ideal (see e.g. [14, Section 3.6]). Suppose for the sake of contradiction that $\mathcal{S}$ is nonpathological. Then $\mathcal{S}$ would be the intersection of some family of matrix ideals by Theorem 8.5. Using Theorem 9.1, we obtain $\mathcal{S} \restriction A \leq_K \mathcal{I}_d$ for every $A \notin \mathcal{S}$. Thus $\mathcal{I}_d$ is pathological by Theorem 8.4, a contradiction. ☐

10. Generalized density ideals

If $(\mu_n)_{n\in\omega}$ is a sequence of measures on ω which are concentrated on finite pairwise disjoint intervals $I_n \subseteq \omega$ (i.e. $\{i \in \omega : \mu_n(\{i\}) > 0\} \subseteq I_n$ for each $n \in \omega$ and $I_n \cap I_m = \emptyset$ for $n \neq m$), then the *density ideal generated by* (μ_n),

$$\mathcal{I}_{\mu_n} = \operatorname{Exh}\left(\sup_{n\in\omega} \mu_n\right) = \left\{A \subseteq \omega : \limsup_{n\to\infty} \mu_n(A) = 0\right\},$$

is a nonpathological analytic P-ideal ([5]). If we replace measures μ_n with submeasures ϕ_n in the above definition, we obtain the *generalized density ideal generated by* (ϕ_n) which are also nonpathological analytic P-ideal ([6]).

In [4], the authors are interested in the following modification of density ideals. Let $\mathcal{F}$ be a family of finite subsets of ω and $f : \omega \to (0,\infty)$ be a sequence of positive reals. For each $F \in \mathcal{F}$, we define a measure on ω

$$\mu_{f,F}(A) = \sum_{i\in A\cap F} f(i),$$

and associated nonpathological analytic P-ideal given by

$$\mathcal{I}_{f,\mathcal{F}} = \operatorname{Exh}\left(\sup_{F\in\mathcal{F}} \mu_{f,F}\right).$$

The following theorem solves [4, Problem 4.3].

Theorem 10.1. *The ideal of all sets of exponential density zero,*

$$\mathcal{I}_\varepsilon = \left\{A \subseteq \omega : \limsup_{n\to\infty} \frac{\ln|A\cap n|}{\ln n} = 0\right\},$$

is a nonpathological analytic P-ideal that is not of the form $\mathcal{I}_{f,\mathcal{F}}$.

Proof. The ideal $\mathcal{I}_\varepsilon$ is a matrix summability ideal by [10, Theorem 5.20], so it is a nonpathological analytic P-ideal. Then it follows from [31, Proposition 4.6] that $\mathcal{I}_\varepsilon$ is a generalized density ideal, so there are pairwise disjoint intervals I_n and submeasures $\psi_n : \mathcal{P}(I_n) \to [0,\infty)$ such that

$$\mathcal{I}_\varepsilon = \left\{A \subseteq \omega : \limsup_{n\to\infty} \psi_n(A\cap I_n) = 0\right\}.$$

Assume that $\mathcal{I}_\varepsilon$ is of the form $\mathcal{I}_{f,\mathcal{F}}$ for some $\mathcal{F} \subseteq [\omega]^{<\omega}$ and $f : \omega \to (0,\infty)$. Let $\phi = \sup_{F\in\mathcal{F}} \mu_{f,F}$. Since $\mathcal{I}_\varepsilon$ is a generalized density ideal, we can notice that

$$\mathcal{I}_\varepsilon = \left\{A \subseteq \omega : \lim_{n\to\infty} \varphi(A\cap I_n) = 0\right\}.$$

We also know that $\lim_{n\to\infty}\varphi(\{n\})=0$, because $\mathcal{I}_\varepsilon$ is tall.

Now, note that there has to be $A\notin\mathcal{I}_\varepsilon$ such that $\limsup_{n\to\infty}\varphi(A\cap I_n)<\infty$. Otherwise, we would have $\mathcal{I}_\varepsilon=\{A\subseteq\omega:\varphi(A)<\infty\}$, which would make $\mathcal{I}_\varepsilon$ an F_σ ideal, a contradiction with the fact that no tall generalized density ideal is F_σ (see [7, Section 3]).

Therefore, there exists $A\notin\mathcal{I}_\varepsilon$, $\alpha>0$ and injective sequences $(n_k)\subseteq\omega$, $(F_{n_k})\subseteq\mathcal{F}$ such that $\lim_{k\to\infty}\varphi(A\cap F_{n_k}\cap I_{n_k})=\alpha$. Let $B_k=A\cap F_{n_k}\cap I_{n_k}$ and $B=\bigcup_{k\in\omega}B_k$. Clearly, $B\notin\mathcal{I}_\varepsilon$ as $\limsup_{n\to\infty}\varphi(B\cap I_n)=\alpha>0$. Since $\lim_{n\to\infty}\varphi(\{n\})=0$, we also get $\lim_{k\to\infty}|B_k|=\infty$. What is more, for every infinite $Z\subseteq\omega$ we have $\bigcup_{k\in Z}B_k\notin\mathcal{I}_\varepsilon$. Therefore, we may assume (by trimming down B_k if necessary) that $\liminf_{k\to\infty}\ln(|B_k|)/\ln(\max B_k)=\beta>0$.

Next, we divide every B_k into $\lfloor\sqrt{|B_k|}\rfloor$ disjoint, intertwined parts C^k_i (i.e. $||C^k_i\cap n|-|C^k_j\cap n||\leq 1$ for all i,j,n), each with cardinality around $\sqrt{|B_k|}$. Since $B_k\subseteq F_{n_k}$, for every $X\subseteq B_k$ we have $\varphi(X)=\mu_{f,F_{n_k}}(X)$. Therefore, for every k, there has to be at least one i_k with

$$\varphi(C^k_{i_k})\leq\frac{\varphi(B_k)}{\lfloor\sqrt{|B_k|}\rfloor},$$

thus for $C=\bigcup_{k\in\omega}C^k_{i_k}$ we obtain

$$\limsup_{n\to\infty}\varphi(C\cap I_n)=\limsup_{k\to\infty}\varphi(C^k_{i_k})\leq\limsup_{k\to\infty}\frac{\varphi(B_k)}{\lfloor\sqrt{|B_k|}\rfloor}=0,$$

because $\limsup_{k\to\infty}\varphi(B_k)\leq\alpha$ and $\lim_{k\to\infty}\lfloor\sqrt{|B_k|}\rfloor=\infty$.

On the other hand, since for every $k\in\omega$,

$$|C^k_{i_k}|\geq\left\lfloor\frac{|B_k|}{\lfloor\sqrt{|B_k|}\rfloor}\right\rfloor\geq\lfloor\sqrt{|B_k|}\rfloor$$

then for every $n=\max B_k$ we have

$$\frac{\ln(|C\cap n|)}{\ln n}\geq\frac{\ln|C^k_{i_k}|}{\ln n}\geq\frac{\ln(\lfloor\sqrt{|B_k|}\rfloor)}{\ln n}\approx\frac{\ln(|B_k|)}{2\ln n},$$

hence $\limsup_{n\to\infty}\ln(|C\cap n|)/\ln n\geq\beta/2$, thus $C\notin\mathcal{I}_\varepsilon$, which leads to a contradiction. □

11. F_σ IDEALS AND THE DEGREE OF PATHOLOGY

If ϕ is a lower semicontinuous submeasure, then $\widehat{\phi}_\sigma$ is a nonpathological lower semicontinuous submeasure (by Proposition 4.1(2)(7a)). If $P_\sigma(\phi)<\infty$ then

$$\widehat{\phi}_\sigma\leq\phi\leq P_\sigma(\phi)\cdot\widehat{\phi}_\sigma.$$

If ϕ is a lower semicontinuous submeasure with $P(\phi)<\infty$, then $P_\sigma(\phi)=P(\phi)$ (by Proposition 7.2(3)), so

$$\mathrm{Exh}(\phi)=\mathrm{Exh}\left(\widehat{\phi}_\sigma\right)\quad\text{and}\quad\mathrm{Fin}(\phi)=\mathrm{Fin}\left(\widehat{\phi}_\sigma\right),$$

and consequently, both $\mathrm{Exh}(\phi)$ and $\mathrm{Fin}(\phi)$ are nonpathological in this case (even though ϕ may be pathological).

In [23, Section 3.1], the authors constructed a lower semicontinuous submeasure ϕ such that $P(\phi)=\infty$ and $\mathrm{Fin}(\phi)$ is nonpathological. Then they asked a question [23, Question 3.6] whether one can find ϕ which additionally has $P(\phi\restriction\mathcal{P}(A))<\infty$

for each $A \in \mathrm{Fin}(\phi)$. The following proposition answers their question in the positive.

Proposition 11.1. *There exists a pathological lower semicontinuous submeasure ϕ on ω such that $P(\phi) = \infty$ and $\mathrm{Exh}(\phi) = \mathrm{Fin}(\phi) = \mathrm{Fin}$, thus $\mathrm{Fin}(\phi)$ is a nonpathological F_σ P-ideal. Moreover, $P(\phi \restriction \mathcal{P}(A)) < \infty$ for all $A \in \mathrm{Fin}(\phi)$.*

Proof. By [26, Lemma 1.8] for every $n > 0$ there exists a finite set K_n and a family $\mathcal{S}_n \subseteq \mathcal{P}(K_n)$ such that:

1. $\forall A_1, \ldots, A_n \in \mathcal{S}_n$ $(A_1 \cup \ldots \cup A_n \neq K_n)$;
2. if p is a probability distribution on K_n then there exists $A \in \mathcal{S}_n$ such that $p(A) \geq 1/2$.

Assume that $\{K_n : n \in \omega\}$ is a partition of $\mathbb{N}$ into intervals and define $\Phi_n : \mathcal{P}(K_n) \to [0, \infty)$ by

$$\Phi_n(A) = \min\left\{|\mathcal{S}| : \mathcal{S} \subseteq \mathcal{S}_n \text{ and } A \subseteq \bigcup \mathcal{S}\right\}$$

for any $A \subseteq K_n$. Notice that $\Phi_n(K_n) > n$.

Define $\phi : \mathcal{P}(K_n) \to [0, \infty)$ by $\phi(A) = \sum_{n=1}^{\infty} \phi_n(A \cap K_n)$. Clearly, if A is finite then $\Phi_n(A) = 0$ for all but finitely many n (thus $\phi(A) < \infty$) and if A is infinite then $\Phi_n(A) \geq 1$ for infinitely many n (thus $\phi(A) = \infty$). Therefore, $\mathrm{Exh}(\phi) = \mathrm{Fin}(\phi) = \mathrm{Fin}$.

To see that $P(\phi) = \infty$, first assume that there exists a measure $\mu : \mathcal{P}(K_n) \to [0, \infty)$ such that $\mu(K_n) \geq 2$. Then there exists $A \in \mathcal{S}_n$ such that $\mu(A) > 1$, because for the probability measure p given by $p(B) = \mu(B)/\mu(K_n)$ one has to find $A \in \mathcal{S}_n$ such that $p(A) > 1/2$, thus $\mu(A) > 1$. On the other hand, for every $A \in \mathcal{S}_n$ we have $\phi_n(A) = 1$, thus $\phi(A) = 1$, hence $\mu \not\leq \phi$. It follows that for every $\mu \leq \phi$ we have $\mu(K_n) < 2$. Therefore, for any measure $\mu \leq \phi$ we get

$$\frac{\phi(K_n)}{\mu(K_n)} > \frac{n}{2},$$

for all n, thus $P(\phi) = \infty$.

To prove the 'moreover' part, notice that for $A \in \mathrm{Fin}(\phi)$, there are only finitely many sets $B \subseteq A$ for which $(\phi \restriction \mathcal{P}(A))(B) > 0$. Since $(\phi \restriction \mathcal{P}(A))(A) = \phi(A) < \infty$ (thus $(\phi \restriction \mathcal{P}(A))(B) < \infty$ for every $B \subseteq A$), it is clear that $P_{\mathrm{Fin}}(\phi \restriction \mathcal{P}(A)) < \infty$. □

We know that nonpathological F_σ ideals can be represented as intersections of matrix ideals (see Theorem 8.5). We can possibly strengthen it a bit.

Theorem 11.2. *If $\mathcal{I}$ is a nonpathological F_σ ideal then it can be represented as the intersection of summable ideals. In particular, every nonpathological F_σ ideal is contained in some summbale ideal.*

Proof. Let ϕ be a nonpathological submeasure such that $\mathcal{I} = \mathrm{Fin}(\phi)$. Take $X \notin \mathcal{I}$. Then $\phi(X) = \infty$, thus, since ϕ is lower semicontinuous and nonpathological, there exist finite sets $A_1, A_2, \ldots \subseteq X$ and measures $\mu_1, \mu_2, \ldots$ with $\mu_n \leq \phi$ for every n such that $\mu_n(A_n) \geq 2^n$.

We will finish the proof by finding a summable ideal $\mathcal{I}_g$ such that $\mathcal{I} \subseteq \mathcal{I}_g$ and $X \notin \mathcal{I}_g$. Define $g : \mathbb{N} \to [0, \infty)$ by

$$g(i) = \sum_{n=1}^{\infty} \frac{\mu_n(\{i\})}{2^n}.$$

Then for every $A \subseteq \mathbb{N}$ we have

$$\sum_{i\in A} g(i) = \sum_{i\in A}\sum_{n=1}^{\infty} \frac{\mu_n(\{i\})}{2^n} = \sum_{n=1}^{\infty} \frac{\mu_n(A)}{2^n} \leq \sum_{n=1}^{\infty} \frac{\phi(A)}{2^n} = \phi(A).$$

It follows that if $\phi(A) < \infty$ then $\sum_{i\in A} g(i) < \infty$, thus $\mathcal{I} \subseteq \mathcal{I}_g$.

Now, observe that

$$\sum_{i\in X} g(i) = \sum_{n=1}^{\infty} \frac{\mu_n(X)}{2^n} \geq \sum_{n=1}^{\infty} \frac{\mu_n(A_n)}{2^n} \geq \sum_{n=1}^{\infty} 1 = \infty,$$

thus $X \notin \mathcal{I}_g$. □

Corollary 11.3. *If $\mathcal{I}$ is a nonpathological F_σ ideal then for every $X \notin \mathcal{I}$, $\mathcal{I} \restriction X$ can be extended to some summable ideal.*

Proposition 11.4. *Let $\mathcal{I}$ be an ideal. Then the following are equivalent.*

1. *$\mathcal{I}$ is an intersection of summable ideals.*
2. *for every $A \notin \mathcal{I}$ there exists some summable ideal $\mathcal{I}_f$ such that $\mathcal{I} \restriction A \leq_K \mathcal{I}_f$.*

Proof. The (1) ⇒ (2) implication is obvious.

To prove the (2) ⇒ (1) implication, take $A \notin \mathcal{I}$ and a summable ideal $\mathcal{I}_f$ such that $\mathcal{I} \restriction A \leq_K \mathcal{I}_f$. Let $g : \mathbb{N} \to A$ be a witness to that. Define the function $h : \mathbb{N} \to [0, \infty)$ by

$$h(n) = \sum_{i\in g^{-1}(\{n\})} f(i)$$

for every $n \in \mathbb{N}$. Clearly, $\sum_{n\in\mathbb{N}} h(n) = \infty$, thus $\mathcal{I}_h$ is a summable ideal such that $A \notin \mathcal{I}_h$.

We will finish the proof by showing that $\mathcal{I} \restriction A \subseteq \mathcal{I}_h$. Take $B \in \mathcal{I} \restriction A$. Then $g^{-1}(B) \in \mathcal{I}_f$, therefore $\sum_{i\in g^{-1}(B)} f(i) < \infty$, thus $\sum_{n\in B} h(n) < \infty$. It follows that $B \in \mathcal{I}_h$. □

Proposition 11.5. *If $\mathcal{I}$ is an F_σ ideal that can be represented as an intersection of summable ideals then for every $X \notin \mathcal{I}$ there exists a submeasure ϕ such that $\mathrm{Fin}(\phi) = \mathcal{I}$ and $\widehat{\phi}_\sigma(X) = \infty$.*

Proof. Let $X \notin \mathcal{I}$ and let ψ be such that $\mathrm{Fin}(\psi) = \mathcal{I}$. Since there exists a summable ideal $\mathcal{I}_f$ such that $\mathcal{I} \subseteq \mathcal{I}_f$ and $X \notin \mathcal{I}_f$, for a measure $\mu_f(A) = \sum_{n\in A} f(n)$ we have $\mu_f(X) = \infty$. Define $\phi = \max\{\psi, \mu_f\}$. Then ϕ is a lower semicontinuous submeasure as a maximum of a measure and a lower semicontinuous submeasure. Observe that $\widehat{\phi}_\sigma \geq \mu_f$, because μ_f is a σ-measure and $\mu_f \leq \phi$. Hence $\widehat{\phi}_\sigma(X) = \infty$.

To see that $\mathrm{Fin}(\phi) = \mathcal{I}$, first note that $\psi \leq \phi$, thus $\mathrm{Fin}(\phi) \subseteq \mathrm{Fin}(\psi) = \mathcal{I}$. On the other hand, if $A \in \mathcal{I} \subseteq \mathcal{I}_f$, then we have both $\psi(A) < \infty$ and $\mu_f(A) < \infty$. Therefore, $\phi(A) = \max\{\psi(A), \mu_f(A)\} < \infty$. □

Note that summable ideals are nonpathological F_σ ideals, so they can be represented as intersections of matrix ideals. Moreover, $\mathcal{I}_d$ is a matrix ideal that cannot be extended to any summable ideal, thus being represented as an intersection of summable ideals is strictly stronger than being represented as an intersection of matrix ideals. However, the following question is still open.

Question 11.6. Let $\mathcal{I}$ be an F_σ ideal. Then each of the following items implies the next.

(1) $\mathcal{I}$ is nonpathological.
(2) $\mathcal{I}$ can be represented as an intersection of summable ideals.
(3) $\mathcal{I}$ can be represented as an intersection of matrix ideals.

Can these implications be reversed?

We finish this section with an example which gives us answer to [23, Question 3.10]. First, we need to introduce some notions and notations necessary to the formulation of this question. For any finite set $K \subseteq \omega$ and a family $\mathcal{F} \subseteq \mathcal{P}(K)$ with $\bigcup \mathcal{F} = K$, we define the *covering number of $\mathcal{F}$ in K* by

$$\delta(K, \mathcal{F}) = \frac{\min\{|\{F \in \mathcal{F} : i \in F\}| : i \in K\}}{|\mathcal{F}|}.$$

Let ϕ be a lsc submeasure such that $\phi(\omega) = \infty$ and there is $M > 0$ such that

$$\mathcal{B}_M = \{A : \phi(A) < M\}$$

is a cover of ω. Let $(K_n : n < \omega)$ be a strictly increasing sequence of finite subsets of ω such that $\bigcup_{n\in\omega} K_n = \omega$ and let $(\mathcal{F}_n : n < \omega)$ be a sequence such that $\mathcal{F}_n \subseteq \mathcal{P}(K_n) \cap \mathcal{B}_M$ and $\bigcup \mathcal{F}_n = K_n$ for each n. Then we define

$$\delta(\phi, M, (K_n), (\mathcal{F}_n)) = \inf\{\delta(K_n, \mathcal{F}_n) : n \in \omega\}.$$

Lemma 11.7. *If* $\mathrm{Fin}(\phi)$ *is a tall ideal, then there exists* $M > 0$ *with*

$$\delta(\phi, M, (K_n), (\mathcal{F}_n) = 0,$$

where $K_n = \{i \in \omega : i < n\}$ *and* $\mathcal{F}_n = \{\{i\} : i < n\}$ *for each* $n \in \omega$.

Proof. Since $\mathrm{Fin}(\phi)$ is a tall ideal, there is $M > 0$ such that $\phi(\{n\}) \leq M$ for each $n \in \omega$. Then (K_n) is a strictly increasing sequence of finite subsets of ω with $\bigcup_{n\in\omega} K_n = \omega$, $\mathcal{F}_n \subseteq \mathcal{P}(K_n) \cap \mathcal{B}_M$ and $\bigcup \mathcal{F}_n = K_n$ for each $n < \omega$. Moreover, $\delta(K_n, \mathcal{F}_n) = 1/n$ for each n. Thus, $\delta(\phi, M, (K_n), (\mathcal{F}_n)) = 0$. □

In [23, Theorem 3.9], the authors proved that if $\delta(\phi, M, (K_n), (\mathcal{F}_n)) > 0$, then $\widehat{\phi}_\sigma(\omega) < \infty$. Then they asked [23, Question 3.10] whether $\widehat{\phi}_\sigma(\omega) < \infty$ implies $\delta(\phi, M, (K_n), (\mathcal{F}_n)) > 0$, and whether $\delta(\phi, M, (K_n), (\mathcal{F}_n)) = 0$ implies that $\mathrm{Fin}(\phi)$ is nonpathological. Below we show that the answers to both questions are negative.

To answer the first question, we take a submeasure ψ_3 which is defined in Proposition 6.4(4). We have already proved there that ψ_3 is pathological and $\widehat{(\psi_3)}_\sigma(\omega) < \infty$. Once we prove that $\mathrm{Fin}(\psi_3)$ is tall, Lemma 11.7 will show that $\delta(\phi, M, (K_n), (\mathcal{F}_n)) = 0$ for an appropriate M, (K_n) and $(\mathcal{F}_n)$, so the answer to the first question will be obtained.

Tallness of $\mathrm{Fin}(\psi_3)$ follows from the fact that for every $n \in \omega$ and for every $i \in I_n$ we have $\psi_3(\{i\}) \leq (n+1)/2^{n+1}$. To prove this fact, suppose for the sake of contradiction that there is $n \in \omega$ and there is $i \in I_n$ such that $\psi_3(\{i\}) > (n+1)/2^{n+1}$. Then we take the measure $\mu = \phi_n(\{i\})\delta_i$ and observe that $\mu \leq \phi_n$, so $\mu(\mathcal{I}_n) \leq 1/2^{n+1}$. On the other hand, $\mu(I_n) \geq \mu(\{i\}) = \phi_n(\{i\}) = \psi_3(\{i\})/(n+1) > 1/2^{n+1}$, a contradiction.

To answer the second question, we take a pathological F_σ P-ideal $\mathcal{I}$ constructed in [9, Theorem 4.12]. Since $\mathcal{I}$ is tall, Lemma 11.7 show that $\delta(\phi, M, (K_n), (\mathcal{F}_n)) = 0$ for an appropriate M, (K_n) and $(\mathcal{F}_n)$, so the answer to the second question is obtained.

12. Van der Waerden ideal

We start this section by recalling two versions of van der Waerden Theorem (see e.g. [11, Section 2]).

Theorem 12.1. *Let $A \subseteq \mathbb{N}$ be a set containing arithmetic progressions of any given length. For any partition of A into finitely many subsets, at least one of them contains arithmetic progressions of any given finite length.*

Theorem 12.2. *For any $n \in \mathbb{N}$ there exists a number $W_n \in \mathbb{N}$ such that for any set $A \subseteq \mathbb{N}$ containing an arithmetic progression of length W_n if $A = B \cup C$, then at least one of the sets B, C contains an arithmetic progression of length n.*

Due to the first of these theorems, the family $\mathcal{W}$ which consists of those subsets of $\mathbb{N}$ that do not contain arithmetic progressions of arbitrary finite length is an ideal called *van der Waerden ideal.* It is known that $\mathcal{W}$ is an F_σ ideal ([8, Example 4.12]) and that it can be extended to a summable ideal ([18, Theorem 3.1]), thus by the homogeneity of $\mathcal{W}$ ([19, Example 2.6]) it is equal to the intersection of a family of summable ideals. To the best of our knowledge, the possible pathology of $\mathcal{W}$ (or lack of thereof) has not been researched before.

To prove the nonpathology of $\mathcal{W}$, we will also need a finitary version of Szemerédi's Theorem ([29], see also [11, Section 1.4]).

Theorem 12.3 ([29])**.** *Let $n \in \mathbb{N}$ and let $0 < \delta \leq 1$. Then there exists a number $N(n, \delta)$ such that for any arithmetic progression A of length $k \geq N(n, \delta)$ every set $B \subseteq A$ of size at least δk contains an arithmetic progression of length n.*

Theorem 12.4. *$\mathcal{W}$ is a nonpathological F_σ ideal.*

Proof. We will obtain the theorem by finding an lsc submeasure ϕ such that $\mathrm{Fin}(\phi) = \mathcal{W}$ and $P_{\mathrm{Fin}}(\phi) \leq 2$.

To begin, we will define inductively the sequence (V_n). Let $V_1 = 1$, $V_2 = 2$ and assume that we have defined $V_1, \dots, V_{n-1}$ for some $n \geq 3$.

Then for every $i < n$ we define as N_i the smallest natural number such that for every arithmetic progression A of length at least N_i and every set $B \subseteq A$ such that

$$\frac{|B|}{|A|} \geq \frac{i}{n}$$

the set B has to contain an arithmetic progression of length V_i. Such N_i exists by Theorem 12.3 (we put $N_i = N(V_i, i/n)$).

We pick as V_n the smallest natural number such that $V_n \geq N_i$ for every $i < n$ and $V_n \geq W_{V_{n-1}}$, where W_j is defined as in Theorem 12.2.

We can now proceed to defining the function $\phi : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ by

$$\phi(A) = \sup\{n \in \mathbb{N} : A \text{ contains an arithmetic progression of length } V_n\}$$

for every $A \subseteq \mathbb{N}$.

Clearly, $\phi(\emptyset) = 0$, $\phi(\mathbb{N}) = \infty$ and $\phi(A) \leq \phi(B)$ whenever $A \subseteq B$.

We will prove that $\phi(A \cup B) \leq \phi(A) + \phi(B)$. We may assume that A, B are nonempty and disjoint. We have two cases.

If $\phi(A \cup B) = \infty$ then $A \cup B$ contains an arithmetic progression of arbitrary length, thus by Theorem 12.1 at least one of the two sets A, B also needs to contain an arithmetic progression of any given length, thus at least one of $\phi(A)$, $\phi(B)$ has to be infinite, hence $\phi(A) + \phi(B) = \infty$.

If $\phi(A \cup B) = n$ for some $n \in \mathbb{N}$ then $A \cup B$ contains an arithmetic progression of length V_n. Since $V_n \geq W_{V_{n-1}}$, at least one of the two sets A, B contains an arithmetic progression of length V_{n-1} – we may assume that it is A. Since B is nonempty, $\phi(B) \geq 1$. Hence, $\phi(A) + \phi(B) \geq (n-1) + 1 = n$.

Therefore, ϕ is a submeasure. It is easy to see that it is lsc and that $\mathrm{Fin}(\phi) = \mathcal{W}$.

To finish the proof, we need to show that $P_{\mathrm{Fin}}(\phi) \leq 2$. Take a finite, nonempty set $A \subseteq \mathbb{N}$. Then $\phi(A) = n$, hence it contains an arithmetic progression B of length V_n.

If $n \leq 2$, then the measure $\mu(C) = |C \cap B|$ fulfills $\mu \leq \phi$ and $\mu(B) = \mu(A) = \phi(A)$, so we are done.

If $n > 2$, we define the measure $\nu : \mathcal{P}(A) \to [0, \infty)$ by

$$\nu(C) = \frac{n \cdot |C \cap B|}{V_n}$$

for every $C \subseteq A$. Observe that $\nu(A) = \nu(B) = n$. We may also notice that for any $i \leq n$ we have that $\phi(C) \geq i$ whenever $\nu(C) \geq i$. Indeed, if $\nu(C) \geq i$ then

$$\frac{|C \cap B|}{|B|} \geq \frac{i}{n}.$$

Since B is an arithmetic progression of length V_n, by the fact that $V_n \geq N(V_i, i/n)$ we find that C contains an arithmetic progression of length V_i.

It follows that for every $C \subseteq A$ we have

$$\frac{\nu(C)}{\phi(C)} \leq \frac{i+1}{i} \leq 2.$$

Therefore, if we take the measure $\mu : \mathcal{P}(A) \to [0, \infty)$ given by $\mu(C) = \nu(C)/2$ we obtain $\mu \leq \phi$. Since $\mu(A) = n/2$, we get

$$\frac{\phi(A)}{\mu(A)} = 2,$$

thus $P_{\mathrm{Fin}}(\phi) \leq 2$. □

Let us recall one more F_σ ideal which is also defined with the aid of Ramsey theory. Namely, using Folkman's theorem (see e.g. [11, Section 3.4]) we obtain the *Folkman ideal* ([19])

$$\mathcal{F} = \{A \subseteq \mathbb{N} : \exists n \, \forall D \subseteq \mathbb{N} \; (|D| = n \implies FS(D) \not\subseteq A)\},$$

where $FS(D)$ is the set of all sums of distinct elements of D. We do not know whether $\mathcal{F}$ is a nonpathological ideal.

13. JOSEFSON-NISSENZWEIG PROPERTY

Let X be a Tychonoff space. By $C(X)$ we denote the set of all continuous real-valued functions on X, and by $C^*(X)$ we denote the subspace of $C(X)$ consisting of all bounded functions.

By a *measure on* X we mean a σ-additive measure defined on the σ-algebra of all Borel subsets of X which is regular, signed and has bounded total variation. We say that a measure μ on X is *finitely supported* if $\mu = \sum_{i=1}^{k} a_i \delta_{x_i}$ for some $x_i \in X$

and $a_i \in \mathbb{R}$, where δ_x is the probability measure concentrated on x. In the case μ is finitely supported, its variation is given by

$$||\mu|| = \sum_{i=1}^{k} |a_i|$$

and we will write

$$\mu(f) = \int_X f\, d\mu = \sum_{i=1}^{k} a_i f(x_i)$$

for every $f \in C(X)$.

Definition 13.1 ([20])**.** A Tychonoff space X has the *Josefson-Nissenzweig property* (the *bounded Josefson-Nissenzweig property*, resp.), or shortly the JNP (the BJNP, resp.), if X admits a sequence (μ_n) of finitely supported measures on X such that $||\mu_n|| = 1$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu_n(f) = 0$ for every $f \in C(X)$ (resp. $f \in C^*(X)$).

Clearly, if Y is a subspace of X and Y has the JNP (BJNP, resp.) then X has the JNP (BJNP, resp.) too.

For an ideal $\mathcal{I}$ on $\mathbb{N}$, consider the space $X(\mathcal{I}) = \mathbb{N} \cup \{\infty\}$ with the topology such that every point in $\mathbb{N}$ is isolated while every open neighborhood of ∞ is of the form $\{\infty\} \cup (\mathbb{N} \setminus A)$ for some $A \in \mathcal{I}$.

The JNP and BJNP for the space $X(\mathcal{I})$ were recently investigated in [20].

Theorem 13.2 ([20, Theorem A])**.** *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following conditions are equivalent.*

1. *$X(\mathcal{I})$ has the JNP.*
2. *$X(\mathcal{I})$ contains a non-trivial convergent sequence.*
3. $\mathcal{I} \approx_K \mathrm{Fin}$.

Theorem 13.3 ([20, Theorem C])**.** *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following conditions are equivalent.*

1. *$X(\mathcal{I})$ has the BJNP.*
2. *There is a matrix ideal $\mathcal{J}$ such that $\mathcal{I} \subseteq \mathcal{J}$.*

Note that originally in [20] the space $X(\mathcal{I})$ had the BJNP if and only if $\mathcal{I}$ could be extended to a density ideal, but extension to a density ideal was proved to be equivalent to extension to a matrix ideal in [32, Section 3].

We say that a family $\mathcal{A}$ of subsets of $\mathbb{N}$ is $\mathcal{I}$-almost disjoint if $\mathcal{A} \subseteq \mathcal{I}^+$ and $A \cap B \in \mathcal{I}$ for all distinct $A, B \in \mathcal{A}$.

Let $\mathcal{I}$ be an ideal and $\mathcal{A}$ be an nonempty $\mathcal{I}$-almost disjoint family of subsets of $\mathbb{N}$. Consider the space $\Psi_{\mathcal{I}}(\mathcal{A}) = \mathbb{N} \cup \mathcal{A}$ with the topology such that every point in $\mathbb{N}$ is isolated while every open neighborhood of $A \in \mathcal{A}$ is of the form $\{A\} \cup A \setminus B$, where $B \in \mathcal{I}$. If $\mathcal{I} = \mathrm{Fin}$, the space $\Psi_{\mathrm{Fin}}(\mathcal{A}) = \Psi(\mathcal{A})$ is known as *Mrówka-Isbell space* (see e.g. [13]).

We can apply Theorem 13.3 to spaces $\Psi_{\mathcal{I}}(\mathcal{A})$.

Corollary 13.4. *An ideal $\mathcal{I}$ on $\mathbb{N}$ is an intersection of matrix ideals $\iff \Psi_{\mathcal{I}}(\mathcal{A})$ has the BJNP for every $\mathcal{I}$-almost disjoint family $\mathcal{A}$.*

Proof. ($\Longleftarrow$): Let $A \notin \mathcal{I}$ and $\mathcal{A} = \{A\} \subseteq \mathcal{I}^+$. Then $\Psi_{\mathcal{I}}(\mathcal{A})$ is equal to $X(\mathcal{J}_A)$ with $\mathcal{J}_A = \{B \subseteq \omega : A \cap B \in \mathcal{I}\}$. Thus $X(\mathcal{J}_A)$ has the BJNP, hence by Theorem 13.3

the ideal $\mathcal{J}_A$ can be extended to a matrix ideal. Therefore, $\mathcal{I}$ is an intersection of matrix ideals.

($\Longrightarrow$): Let $\mathcal{A} \subseteq \mathcal{I}^+$ be an $\mathcal{I}$-almost disjoint family and take $A \in \mathcal{A}$. Put $\mathcal{B} = \{A\} \subseteq \mathcal{I}^+$. Since $\mathcal{I}$ is an intersection of matrix ideals, $\mathcal{I} \restriction A$ can be extended to a matrix ideal, so $X(\mathcal{I} \restriction A)$, has the BJNP, which means that $\Psi_{\mathcal{I}}(\mathcal{B})$ has the BJNP as well. Moreover, $\Psi_{\mathcal{I}}(\mathcal{B})$ is a subspace of $\Psi_{\mathcal{I}}(\mathcal{A})$, hence $\Psi_{\mathcal{I}}(\mathcal{A})$ has the BJNP. □

Theorem 13.5. *Let $(Y_j)_{j\in J}$ be pairwise disjoint Tychonoff spaces. The following conditions are equivalent.*

1. *$X = \bigsqcup\{Y_j : j \in J\}$ has the JNP (BJNP, resp.).*
2. *Y_j has the JNP (BJNP, resp.) for some $j \in J$.*

Proof. (2) $\Longrightarrow$ (1): Obvious, because every Y_j is a subspace of X.

(1) $\Longrightarrow$ (2): Let $\{\{x_i^n : i \leq k_n\} : n \in \mathbb{N}\}$ and $\{\{a_i^n : i \leq k_n\} : n \in \mathbb{N}\}$ be the witness for the JNP (BJNP, resp.), i.e. for every $f \in C(X)$ ($f \in C^*(X)$, resp.) we have

$$\lim_{n\to\infty} \sum_{i\leq k_n} a_i^n f(x_i^n) = 0 \quad \text{and} \quad \sum_{i\leq k_n} |a_i^n| = 1 \text{ for every } n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, denote the set $\{x_i^n : i \leq k_n\}$ by X_n, the set $\{x_i^n : a_i^n > 0\}$ by X_n^+ and the set $\{x_i^n : a_i^n < 0\}$ by X_n^-.

For each $n \in \mathbb{N}$ define the measure $\delta_n : \mathcal{P}(X) \to [0,1]$ by

$$\delta_n(A) = \sum_{\{i\leq k_n : x_i^n \in A\}} |a_i^n|$$

for each $A \subseteq X$. We have two cases: either there exists $j_0 \in J$ such that $\delta_n(Y_{j_0})$ does not tend to zero or $\lim_{n\to\infty} \delta_n(Y_j) = 0$ for all $j \in J$.

In the first case, we will show that Y_{j_0} has the JNP (BJNP, resp.). Assume otherwise and pick $\alpha > 0$ and a sequence $n_1 < n_2 < \ldots$ such that

$$\lim_{l\to\infty} \delta_{n_l}(Y_{j_0}) = \alpha \quad \text{and} \quad \delta_{n_l}(Y_{j_0}) > 0 \text{ for each } l \in \mathbb{N}.$$

Since Y_{j_0} does not have the JNP (BJNP, resp.), there exists a function $g \in C(Y_{j_0})$ ($g \in C^*(Y_{j_0})$) such that

$$\limsup_{l\to\infty} \left| \sum_{\{i\leq k_{n_l} : x_i^{n_l} \in Y_{j_0}\}} \frac{a_i^{n_l} g(x_i^{n_l})}{\delta_{n_l}(Y_{j_0})} \right| > 0.$$

Define the function $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} g(x) & \text{if } x \in Y_{j_0}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $f \in C(X)$ ($f \in C^*(X)$). However,

$$\left| \sum_{i\leq k_{n_l}} a_i^{n_l} f(x_i^{n_l}) \right| = \left| \delta_{n_l}(Y_{j_0}) \sum_{\{i\leq k_{n_l} : x_i^{n_l} \in Y_{j_0}\}} \frac{a_i^{n_l} g(x_i^{n_l})}{\delta_{n_l}(Y_{j_0})} \right|,$$

which does not tend to zero, a contradiction.

In the second case, observe that for every $n \in \mathbb{N}$ we have $\delta_n(X_n^+) \geq 1/2$ or $\delta_n(X_n^-) \geq 1/2$. Put

$$Z_n = \begin{cases} X_n^+ & \text{if } \delta_n(X_n^+) \geq 1/2, \\ X_n^- & \text{otherwise.} \end{cases}$$

We will obtain the thesis by showing that the second case contradicts the assumption that X has the JNP (BJNP, resp.). We will do so by finding a function $f \in C^*(X)$ such that for infinitely many $n \in \mathbb{N}$ we have

$$\left| \sum_{i \leq k_n} a_i^n f(x_i^n) \right| \geq \frac{1}{4}$$

We will now inductively construct sequences of natural numbers (n_m), sets (D_m) with $\lim_{n\to\infty} \delta_n(D_m) = 0$ and functions (f_m) with $\operatorname{dom}(f_m) \subseteq X$ and $\operatorname{ran}(f_m) \subseteq [0,1]$.

First, let $n_1 = 1$ and $D_1 = \bigcup\{Y_j : Y_j \cap X_1 \neq \emptyset\}$. Because each Y_j is Tychonoff and X_1 is finite, there exists a continuous function $f_1 : D_1 \to [0,1]$ such that $f_1(x) = 1$ for $x \in Z_1$ and $f_1(x) = 0$ for $x \in X_1 \setminus Z_1$. Since X_1 is finite, D_1 is made of finitely many Y_j, thus $\lim_{n\to\infty} \delta_n(D_1) = 0$ as $\delta_n(Y_j)$ tends to zero for every $j \in J$.

Suppose we have constructed n_m, f_m, Z_m for $m < l$ for some $l \in \mathbb{N}$. Then we put as n_l such $n \in \mathbb{N}$ that $\delta_n(D_{l-1}) < 1/8$. Let $D_l = \bigcup\{Y_j : \bigcup_{i \leq n_l} X_i \cap Y_j \neq \emptyset\}$. Because f_{l-1} is continuous on D_{l-1}, which is a closed subset of D_l, X_{n_l} is finite and each Y_j is Tychonoff, there exists a continuous function $f_l : D_l \to [0,1]$ such that

- $f_l \restriction D_{l-1} = f_{l-1}$;
- $f_l(x) = 1$ for $x \in Z_{n_l} \setminus D_{l-1}$;
- $f_l(x) = 0$ for $x \in (X_{n_l} \setminus Z_{n_l}) \setminus D_{l-1}$.

Once again, D_l is made of finitely many Y_j, thus $\lim_{n\to\infty} \delta_n(D_l) = 0$.

We can now define the function $f : X \to [0,1]$ in such way that

- $f \restriction D_l = f_l$ for every $l \in \mathbb{N}$;
- $f(x) = 0$ for $x \in X \setminus \bigcup_{l\in\mathbb{N}} D_l$.

It is easy to see that $f \in C^*(X)$ as each f_l is continuous while $\bigcup_{l\in\mathbb{N}} D_l$ and each D_l are all clopen subsets of X.

On the other hand, for all $l \geq 2$ we obtain

$$\begin{aligned} \left| \sum_{i \leq k_{n_l}} a_i^{n_l} f\left(x_i^{n_l}\right) \right| &\geq \left| \sum_{\left\{i \leq k_{n_l} : x_i^{n_l} \in D_l \setminus D_{l-1}\right\}} a_i^{n_l} f\left(x_i^{n_l}\right) \right| - \left| \sum_{\left\{i \leq k_{n_l} : x_i^{n_l} \in D_{l-1}\right\}} a_i^{n_l} f\left(x_i^{n_l}\right) \right| \\ &\geq \left| \sum_{\left\{i \leq k_{n_l} : x_i^{n_l} \in Z_{n_l} \setminus D_{l-1}\right\}} a_i^{n_l} \right| - \sum_{\left\{i \leq k_{n_l} : x_i^{n_l} \in D_{l-1}\right\}} |a_i^{n_l}| \\ &= \delta_{n_l}(Z_{n_l} \setminus D_{l-1}) - \delta_{n_l}(D_{l-1}) \geq \left(\frac{1}{2} - \frac{1}{8}\right) - \frac{1}{8} = \frac{1}{4}. \end{aligned}$$

□

Corollary 13.6. *Let $\mathcal{A}$ be a countable $\mathcal{I}$-almost disjoint family. Then the following conditions are equivalent.*

1. *$\Psi_{\mathcal{I}}(\mathcal{A})$ has the BJNP.*
2. *There exists $A \in \mathcal{A}$ such that $X(\mathcal{I} \restriction A)$ has the BJNP.*
3. *There exists $A \in \mathcal{A}$ such that $\mathcal{I} \restriction A$ extends to a matrix ideal.*

Proof. (2) $\iff$ (3): follows from Theorem 13.3.

(1) $\iff$ (2): follows from Theorem 13.5 as for every countable $\mathcal{I}$-almost disjoint family $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ there exists a countable pairwise disjoint family $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ such that $\Psi_{\mathcal{I}}(\mathcal{A})$ is homeomorphic to $\bigsqcup_{n\in\mathbb{N}} X(\mathcal{I} \restriction B_n)$ while $X(\mathcal{I} \restriction B_n)$ is homeomorphic to $X(\mathcal{I} \restriction A_n)$ for every $n \in \mathbb{N}$. □

14. The intersection of matrix ideals can be non-Borel

In this section, we will prove the consistency of existence of a non-Borel ideal that is an intersection of matrix ideals (Corollary 14.4), which consistently answers [9, Question 3]. The consistency is expressed with the aid of the *tower number* $\mathfrak{t}$ which is the smallest length of a tower of infinite subsets of ω, and in the proof we will also use the *almost disjointness number* $\mathfrak{a}$ which is the smallest cardinality of any MAD family on ω (i.e. an infinite maximal almost disjoint family of infinite subsets of ω). For more on these cardinals, see e.g. [3].

For a MAD family $\mathcal{A}$, we write $\mathcal{I}(\mathcal{A})$ to denote the ideal generated by $\mathcal{A}$:

$$B \in \mathcal{I}(\mathcal{A}) \iff B \setminus (A_1 \cup \cdots \cup A_n) \text{ is finite for some } A_1, \ldots, A_n \in \mathcal{A}.$$

In [16, Theorem 3.9], the authors proved that assuming $\mathfrak{t} = \mathfrak{c}$, there exists a MAD family $\mathcal{A}$ such that $\mathcal{I}(A)$ is K-homogeneous. We will use a slightly modified version of the above result – we will additionally require that the constructed almost disjoint family consists of sets from a tall ideal (see Lemma 14.1 and Theorem 14.3). The proof will be almost the same as in the original result, but we will provide it for the sake of completeness.

Throughout this section, we will assume that every family $\mathcal{F} \subseteq \omega^\omega$ consists of injections. For functions $f, g : X \to Y$ we write $f =^* g$ if $f(x) = g(x)$ for all but finitely many $x \in X$. Moreover, we write $A \subseteq^* B$ if $B \setminus A$ is finite.

Let $\mathcal{A}$ be an almost disjoint family and let $\mathcal{F} \subseteq \omega^\omega$. We say that $\mathcal{A}$ *respects* $\mathcal{F}$ if $f^{-1}[A] \in \mathcal{I}(\mathcal{A})$ for all $f \in \mathcal{F}$ and all $A \in \mathcal{A}$.

Lemma 14.1. *Let $\mathcal{I}$ be a tall ideal. Let $\mathcal{A} \subseteq \mathcal{I}$ be an almost disjoint family that respects $\mathcal{F} \subseteq \omega^\omega$. Assume also that $|\mathcal{A}| < \mathfrak{t}$, $|\mathcal{F}| < \mathfrak{t}$ and $X \in \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A})$. Then there exists an almost disjoint family $\mathcal{B} \subseteq \mathcal{I}$ such that $\mathcal{A} \subseteq \mathcal{B}$, $|\mathcal{B}| < \mathfrak{t}$, $\mathcal{B}$ respects $\mathcal{F}$ and $\mathcal{B} \cap [X]^\omega \neq \emptyset$.*

Proof. If $\mathcal{A} \cap [X]^\omega \neq \emptyset$ then we can put $\mathcal{B} = \mathcal{A}$ and we are done. Hence, we can assume that $\mathcal{A} \cap [X]^\omega = \emptyset$.

First, notice that since $X \notin \mathcal{I}(\mathcal{A})$ and $|\mathcal{A}| < \mathfrak{t} \leq \mathfrak{a}$, the set $\{A \cap X : A \in \mathcal{A} \wedge |A \cap X| = \omega\}$ cannot be a MAD family on X, so there exists a set $Y \in [X]^\omega$ such that $Y \cap A \in \mathrm{Fin}$ for every $A \in \mathcal{A}$. In particular, $Y \notin \mathcal{I}(\mathcal{A})$. It follows that we can assume without loss of generality that $A \cap X \in \mathrm{Fin}$ for every $A \in \mathcal{A}$.

Second, we can assume without loss of generality that the identity function $f(n) = n$ belongs to $\mathcal{F}$.

Let $\mathcal{F}'$ be the closure of $\mathcal{F}$ under compositions. Observe that $\mathcal{A}$ respects $\mathcal{F}'$. Indeed, for any $f, g \in \mathcal{F}$ and $A \in \mathcal{A}$ we have

$$(f \circ g)^{-1}[A] = g^{-1}\left[f^{-1}[A]\right] \subseteq \bigcup_{i=1}^{n} g^{-1}[A_i]$$

for some $A_i \in \mathcal{A}$ as $f^{-1}[A] \in \mathcal{I}(\mathcal{A})$. Since $g^{-1}[A_i] \in \mathcal{I}(\mathcal{A})$, we obtain $(f \circ g)^{-1}[A] \in \mathcal{I}(\mathcal{A})$.

Put $\{f_\alpha : \alpha < \kappa\}$ as an enumeration of elements of $\mathcal{F}'$ such that f_0 is the identity function. Note that $\mathcal{F}'$ consists of injections and that $|\mathcal{F}'| \le |[\mathcal{F}]^{<\omega}| < \mathfrak{t}$.

We will now construct inductively a sequence $(T_\alpha)_{\alpha<\kappa}$ such that

1. $T_\alpha \in [X]^\omega$ for every $\alpha < \kappa$;
2. if $\alpha > \beta$ then $T_\alpha \subseteq^* T_\beta$;
3. for each $\alpha < \kappa$, we have $f_\alpha^{-1}[T_\alpha] \in \mathcal{I}(\mathcal{A})$ or
$$f_\alpha^{-1}[T_\alpha] \in \mathcal{I} \text{ and } f_\alpha^{-1}[T_\alpha] \cap A \in \mathrm{Fin} \text{ for all } A \in \mathcal{A};$$
4. for every $\beta, \gamma \le \alpha < \kappa$, if $f_\beta^{-1}[T_\alpha] \notin \mathcal{I}(\mathcal{A})$ and $f_\gamma^{-1}[T_\alpha] \notin \mathcal{I}(\mathcal{A})$ then
$$f_\beta^{-1}[T_\alpha] \cap f_\gamma^{-1}[T_\alpha] \in \mathrm{Fin} \quad \text{or} \quad f_\beta^{-1} \restriction T_\alpha =^* f_\gamma^{-1} \restriction T_\alpha.$$

Fix $\alpha < \kappa$, and assume we have constructed T_β for $\beta < \alpha$. Since $|\alpha| < \mathfrak{t}$, there exists a set $S \in [X]^\omega$ such that $S \subseteq^* T_\beta$ for all $\beta < \alpha$.

If there exists an infinite $C \subseteq S$ such that $f_\alpha^{-1}[C] \in \mathcal{I}(\mathcal{A})$ then put $S_0 = C$.

If there is no such C then $f_\alpha^{-1}[S] \cap A \in \mathrm{Fin}$ for all $A \in \mathcal{A}$ as otherwise, should there be $A_0 \in \mathcal{A}$ such that $f_\alpha^{-1}[S] \cap A_0 \notin \mathrm{Fin}$ then $C = f_\alpha\left[f_\alpha^{-1}[S] \cap A_0\right]$ would be an infinite subset of S with $f_\alpha^{-1}[C] \subseteq A_0 \in \mathcal{I}(\mathcal{A})$.

In this case, notice that we have $f_\alpha^{-1}[S] \notin \mathcal{I}(\mathcal{A})$, thus $f_\alpha^{-1}[S] \notin \mathrm{Fin}$. Since $\mathcal{I}$ is tall, there exists an infinite $D \subseteq f_\alpha^{-1}[S]$ such that $D \in \mathcal{I}$. We put $S_0 = f_\alpha[D]$. Notice that S_0 is an infinite subset of S such that $f_\alpha^{-1}[S_0] \in \mathcal{I}$.

Next, let $(\alpha+1) \times (\alpha+1) = \{(\beta_\xi, \gamma_\xi) : \xi < \lambda\}$ with $\lambda = |(\alpha+1) \times (\alpha+1)| < \mathfrak{t}$. We will now construct inductively a sequence $(S_\xi)_{\xi<\lambda}$ such that $S_\xi \subseteq^* S_\eta$ for $\eta < \xi$ and if $f_{\beta_\xi}^{-1}[S_\xi] \notin \mathcal{I}(\mathcal{A})$ and $f_{\gamma_\xi}^{-1}[S_\xi] \notin \mathcal{I}(\mathcal{A})$ then

$$f_{\beta_\xi}^{-1}[S_{\xi+1}] \cap f_{\gamma_\xi}^{-1}[S_{\xi+1}] \in \mathrm{Fin} \quad \text{or} \quad f_{\beta_\xi}^{-1} \restriction S_{\xi+1} =^* f_{\gamma_\xi}^{-1} \restriction S_{\xi+1}.$$

We have already defined S_0.

For a given $\xi < \lambda$, suppose we have defined S_ξ. We have two cases.

In the first case, there exists $C \in [S_\xi]^\omega$ such that $f_{\beta_\xi}^{-1}[C] \cap f_{\gamma_\xi}^{-1}[C] \in \mathrm{Fin}$. Then we put $S_{\xi+1} = C$.

In the second case, for all $C \in [S_\xi]^\omega$ we have $f_{\beta_\xi}^{-1}[C] \cap f_{\gamma_\xi}^{-1}[C] \notin \mathrm{Fin}$. Since $f_{\beta_\xi}^{-1}$ and $f_{\gamma_\xi}^{-1}$ are injections, we can realize that we have $f_{\beta_\xi}^{-1} \restriction S_\xi =^* f_{\gamma_\xi}^{-1} \restriction S_\xi$. We put $S_{\xi+1} = S_\xi$.

For a limit $\eta < \lambda$, suppose we have defined all S_ξ for $\xi < \eta$. Then we find such S_η that $S_\eta \subseteq^* S_\xi$ for all $\xi < \eta$ – we can find such, because $\eta < \lambda < \mathfrak{t}$.

Finally, we put as T_α an infinite subset of S_0 such that $T_\alpha \subseteq^* S_\xi$ for all $\xi < \lambda$.

Now that we have constructed the sequence (T_α), we may find the set $T \in [X]^\omega$ such that $T \subseteq^* T_\alpha$ for every $\alpha < \kappa$. Since $\mathcal{I}$ is tall, we may also demand that $T \in \mathcal{I}$.

As a next step we define the family

$$\mathcal{B} = \mathcal{A} \cup \{T\} \cup \left\{ f_\alpha^{-1}[T] : \alpha < \kappa,\, f_\alpha^{-1}[T] \notin \mathcal{I}(\mathcal{A}),\, \forall \beta < \alpha \, (f_\alpha^{-1}[T] \neq^* f_\beta^{-1}[T]) \right\}.$$

It is easy to see that $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{B} \subseteq \mathcal{I}$ and $|\mathcal{B}| \le |\mathcal{A}| + 1 + \kappa < \mathfrak{t}$. Moreover, $T \in \mathcal{B} \cap [X]^\omega$.

Next, notice that $T \subseteq X$, thus $T \cap A \in \mathrm{Fin}$ for every $A \in \mathcal{A}$. Moreover, notice that if $f_\alpha^{-1}[T] \in \mathcal{B}$, $f_\beta^{-1}[T] \in \mathcal{B}$ are two distinct sets for some $\alpha > \beta$ then $f_\alpha^{-1}[T] \cap f_\beta^{-1}[T] \in \mathrm{Fin}$ as $f_\alpha^{-1}[T_\alpha] \cap f_\beta^{-1}[T_\alpha] \in \mathrm{Fin}$ and $T \subseteq^* T_\alpha$. Additionally, if $f_\alpha^{-1}[T] \in \mathcal{B}$ then $f_\alpha^{-1}[T] \cap A \in \mathrm{Fin}$ for every $A \in \mathcal{A}$ as $f_\alpha^{-1}[T_\alpha] \cap A \in \mathrm{Fin}$ and $T \subseteq^* T_\alpha$. Moreover, $f_0^{-1}[T] = T \notin \mathcal{I}(\mathcal{A})$ as T is an infinite subset of X and $A \cap X \in \mathrm{Fin}$ for every $A \in \mathcal{A}$. Therefore, $\mathcal{B}$ is an almost disjoint family.

To finish the proof, we need to show that $\mathcal{B}$ respects $\mathcal{F}$. Pick any $f \in \mathcal{F}$ and $B \in \mathcal{B}$. Obviously, $f = f_\alpha$ for some $\alpha < \kappa$.

If $B \in \mathcal{A}$ then $f^{-1}[B] \in \mathcal{I}(\mathcal{A}) \subseteq \mathcal{I}(\mathcal{B})$, because $\mathcal{A}$ respects $\mathcal{F}$.

If $B = T$ then either $f^{-1}[T] \in \mathcal{I}(\mathcal{A}) \subseteq \mathcal{I}(\mathcal{B})$ or $f^{-1}[T] \in \mathcal{B} \subseteq \mathcal{I}(\mathcal{B})$ or there exists $\beta < \alpha$ such that $f^{-1}[T] =^* f_\beta^{-1}[T] \in \mathcal{B}$, thus $f^{-1}[T] \in \mathcal{I}(\mathcal{B})$.

If $B = f_\beta^{-1}[T]$ for some $\beta < \kappa$, then there exists some $\gamma < \kappa$ such that $f_\gamma^{-1} = f_\alpha^{-1} \circ f_\beta^{-1}$. It follows that $f^{-1}[B] = f_\gamma^{-1}[T]$, which belongs to $\mathcal{I}(\mathcal{B})$ by the reasoning presented in the previous case. ☐

Lemma 14.2 ([16, Lemma 3.11])**.** *Let $\mathcal{A}$ be an almost disjoint family that respects $\mathcal{F} \subseteq \omega^\omega$. Assume also that $|\mathcal{A}| < \mathfrak{a}$, and $X \notin \mathcal{I}(\mathcal{A})$. Then there exists an injection $f : \omega \to X$ such that $\mathcal{A}$ respects $\mathcal{F} \cup \{f\}$.*

Theorem 14.3. *Assume $\mathfrak{t} = \mathfrak{c}$. If $\mathcal{I}$ is a tall ideal then there exists a MAD family $\mathcal{A} \subseteq \mathcal{I}$ such that $\mathcal{I}(\mathcal{A})$ is K-homogeneous.*

Proof. First, let $\{X_\alpha : \alpha < \mathfrak{c}\}$ be the enumeration of elements of $[\omega]^\omega$. We will inductively construct sequences $(\mathcal{A}_\alpha)_{\alpha<\mathfrak{c}}$ and $(\mathcal{F}_\alpha)_{\alpha<\mathfrak{c}}$ such that for every $\alpha < \mathfrak{c}$

1. $\mathcal{A}_\alpha$ is an AD-family such that $\mathcal{A}_\alpha \subseteq \mathcal{I}$;
2. $\mathcal{F}_\alpha \subseteq \omega^\omega$ consists of injections;
3. if $\beta < \alpha$ then $\mathcal{A}_\beta \subseteq \mathcal{A}_\alpha$ and $\mathcal{F}_\beta \subseteq \mathcal{F}_\alpha$;
4. $\mathcal{A}_0$ is a partition of ω into infinitely many infinite sets belonging to $\mathcal{I}$;
5. $\mathcal{F}_0 = \emptyset$;
6. $|\mathcal{A}_\alpha| < \mathfrak{c}$ and $|\mathcal{F}_\alpha| < \mathfrak{c}$;
7. $\mathcal{A}_\alpha$ respects $\mathcal{F}_\alpha$;
8. there exists $A \in \mathcal{A}_{\alpha+1}$ such that $A \cap X_\alpha \notin \mathrm{Fin}$;
9. if $X_\alpha \notin \mathcal{I}(\mathcal{A}_{\alpha+1})$ then there is $f \in \mathcal{F}_{\alpha+1}$ with $\mathrm{ran}(f) \subseteq X_\alpha$.

Fix $0 < \alpha < \mathfrak{c}$ and assume we have constructed $\mathcal{A}_\beta$ and $\mathcal{F}_\beta$ for all $\beta < \alpha$.

If α is a limit ordinal then we put $\mathcal{A}_\alpha = \bigcup_{\beta<\alpha} \mathcal{A}_\beta$ and $\mathcal{F}_\alpha = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$. Since $\mathfrak{t}$ is a regular cardinal (see e.g. [3, Proposition 6.4]), we obtain $|\mathcal{A}_\alpha| < \mathfrak{c}$ and $|\mathcal{F}_\alpha| < \mathfrak{c}$.

Suppose now that $\alpha = \beta + 1$. If $X_\beta \in \mathcal{I}(\mathcal{A}_\beta)$, we put $\mathcal{A}_\alpha = \mathcal{A}_\beta$ and $\mathcal{F}_\alpha = \mathcal{F}_\beta$. If $X_\beta \notin \mathcal{I}(\mathcal{A}_\beta)$ then by Lemma 14.1 we can find an AD-family $\mathcal{A}_\alpha \subseteq \mathcal{I}$ such that $\mathcal{A}_\beta \subseteq \mathcal{A}_\alpha$, $|\mathcal{A}_\alpha| < \mathfrak{c}$, $\mathcal{A}_\alpha$ respects $\mathcal{F}_\beta$ and there exists an infinite $A \in \mathcal{A}_\alpha$ such that $A \subseteq X_\beta$. In both cases, the families $\mathcal{A}_\alpha$ and $\mathcal{F}_\beta$ satisfy items (1)-(8). Now, we will define $\mathcal{F}_\alpha$ to satisfy item (9).

If $X_\beta \in \mathcal{I}(\mathcal{A}_\alpha)$, we put $\mathcal{F}_\alpha = \mathcal{F}_\beta$. In the other case, by Lemma 14.2 there exists an injection $f \in \omega^\omega$ with $\mathrm{ran}(f) \subseteq X_\beta$ such that $\mathcal{A}_\alpha$ respects $\mathcal{F}_\beta \cup \{f\}$. Then we put $\mathcal{F}_\alpha = \mathcal{F}_\beta \cup \{f\}$.

Now that we have constructed sequences $(\mathcal{A}_\alpha)_{\alpha<\mathfrak{c}}$ and $(\mathcal{F}_\alpha)_{\alpha<\mathfrak{c}}$, let

$$\mathcal{A} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{A}_\alpha \quad \text{and} \quad \mathcal{F} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha.$$

Observe that $\mathcal{A}$ respects $\mathcal{F}$. Indeed, take any $A \in \mathcal{A}$ and $f \in \mathcal{F}$. The sequences $(\mathcal{A}_\alpha)$ and $(\mathcal{F}_\alpha)$ are increasing, thus we can find $\alpha < \mathfrak{c}$ such that $A \in \mathcal{A}_\alpha$ and $f \in \mathcal{F}_\alpha$. Since $\mathcal{A}_\alpha$ respects $\mathcal{F}_\alpha$, we obtain $f^{-1}[A] \in \mathcal{I}(\mathcal{A}_\alpha) \subseteq \mathcal{I}(\mathcal{A})$.

By items (1), (4) and (8) of the construction, $\mathcal{A}$ is a MAD family such that $\mathcal{A} \subseteq \mathcal{I}$.

To finish the proof, we need to show that $\mathcal{I}(\mathcal{A})$ is K-homogeneous. Take $X \notin \mathcal{I}(\mathcal{A})$. Then there exists $\alpha < \mathfrak{c}$ such that $X = X_\alpha$ and clearly $X_\alpha \notin \mathcal{I}(\mathcal{A}_{\alpha+1})$. By item (9) of the construction there exists $f \in \mathcal{F}_{\alpha+1} \subseteq \mathcal{F}$ with $\operatorname{ran}(f) \subseteq X_\alpha$. Since $\mathcal{A}$ respects $\mathcal{F}$, we know that for every $A \in \mathcal{A}$ we have $f^{-1}[A] \in \mathcal{I}(\mathcal{A})$. Therefore, this f is a witness for $\mathcal{I}(\mathcal{A}) \restriction X \leq_K \mathcal{I}(\mathcal{A})$. ☐

Corollary 14.4. *Assume* $\mathfrak{t} = \mathfrak{c}$. *There exists a non-Borel ideal that is the intersection of some matrix ideals.*

Proof. Take $\mathcal{I}(\mathcal{A})$ from Theorem 14.3 with $\mathcal{I} = \mathcal{I}_d$. By [25, Proposition 4.6], $\mathcal{I}(\mathcal{A})$ is a non-Borel ideal. Since $\mathcal{A} \subseteq \mathcal{I}_d$, we have $\mathcal{I}(\mathcal{A}) \subseteq \mathcal{I}_d$, thus $\mathcal{I}(\mathcal{A}) \leq_K \mathcal{I}_d$. Moreover, by K-homogeneity of $\mathcal{I}(\mathcal{A})$, for every $A \notin \mathcal{I}(\mathcal{A})$ we obtain

$$\mathcal{I}(\mathcal{A}) \restriction A \leq_K \mathcal{I}(\mathcal{A}) \leq_K \mathcal{I}_d,$$

thus, by Theorem 9.1, $\mathcal{I}(\mathcal{A})$ is an intersection of matrix ideals. ☐

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(R. Filipów) INSTITUTE OF MATHEMATICS, FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS, UNIVERSITY OF GDAŃSK, UL. WITA STWOSZA 57, 80-308 GDAŃSK, POLAND
Email address: `Rafal.Filipow@ug.edu.pl`
URL: `http://mat.ug.edu.pl/~rfilipow`

(J. Tryba) INSTITUTE OF MATHEMATICS, FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS, UNIVERSITY OF GDAŃSK, UL. WITA STWOSZA 57, 80-308 GDAŃSK, POLAND
Email address: `Jacek.Tryba@ug.edu.pl`