On the Difference Property of Borel Measurable and (s)-Measurable Functions

Rafał Filipów and Ireneusz Recław*
University of Gdańsk

Abstract

We prove that the class of (s)-measurable functions does not have the difference property. We show also under CH that there is a function with Borel differences but of unlimited Baire class. It solves a problem of M. Laczkovich.

1 Preliminaries

For a function $f : \mathbb{R} \to \mathbb{R}$ and an $h \in \mathbb{R}$ we define the **difference function** $\Delta_h f : \mathbb{R} \to \mathbb{R}$ by $\Delta_h f(x) = f(x+h) - f(x)$. A function $A : \mathbb{R} \to \mathbb{R}$ is called **additive** if it satisfies Cauchy's functional equation $A(x+y) = A(x) + A(y)$ for all $x, y \in \mathbb{R}$. We say that a given class of functions $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ has the **difference property** if every function $f : \mathbb{R} \to \mathbb{R}$ such that $\Delta_h f \in \mathcal{F}$ for each $h \in \mathbb{R}$ is of the form $f = g + A$ where $g \in \mathcal{F}$ and A is additive.

There are many results about difference property for specific classes of functions. Let us mention two of them. N. de Bruijn [1] proved that the class of continuous functions has the difference property and P. Erdős and M. Laczkovich (see [3]) proved that the difference property for Lebesgue measurable functions is independent from the axioms of Set Theory. In the second section we prove that the class of (s)-measurable functions does not have the difference property.

*The second author was partially supported by KBN Grant 2 PO3A 032 14.

1991 Mathematics Subject Classification. 26A21, 26A15.

Key words and phrases. Difference property, Borel functions, Marczewski measurable functions.

Instead of the difference property we can also consider more general problem: What can say about a function if all its differences are in a given class of functions? Considering this problem Laczkovich [2] asked a question: Assume that all differences of a function f are Borel measurable. Does there exist α such that all differences are of Baire class α? We prove in the third section that under CH the answer to it is NO.

2 The difference property for the family of (s)-measurable functions

By Perf we denote the family of all perfect sets on $\mathbb{R}$. We will assume that empty set does not belong to Perf. (s) denotes the class of (s)-measurable sets (Marczewski measurable sets). Recall that a set A is (s)-measurable iff every perfect set P has a perfect subset Q which is a subset of A or misses A. $A \in (s_0)$ iff every perfect set P has a perfect subset Q which misses A. It is known that (s_0) is a σ-ideal of (s). A function $f : \mathbb{R} \to \mathbb{R}$ is (s)-measurable if the preimage of any open subset is (s)-measurable. We will use the following characterization.

Theorem 2.1 (Marczewski [4]) *A function $f : \mathbb{R} \to \mathbb{R}$ is (s)-measurable iff every perfect $P \subseteq \mathbb{R}$ has a perfect subset Q such that $f|Q$ is continuous.*

Let $\mathbb{R} = \{h_\alpha : \alpha < \mathbf{c}\}$ and G_α denote the group generated by $\{h_\beta : \beta < \alpha\}$. Let $\mathcal{C}(X, Y)$ denote the family of all continuous function from X into Y.

Theorem 2.2 *The family of (s)-measurable functions does not have the difference property.*

Proof We will construct a function $f : \mathbb{R} \to \mathbb{R}$ such that

1. all its difference functions are (s)-measurable,

2. it is not a sum of an (s)-measurable function and an additive function.

Put

$$\mathcal{A} = \{(D, g) : D \in \text{Perf} \wedge g \in \mathcal{C}(D, \mathbb{R})\}.$$

Let $\{(D_\alpha, g_\alpha) : \alpha < c\}$ be an enumeration of the family $\mathcal{A}$. We will define two sequences $\{x_\alpha : \alpha < c\}$ and $\{y_\alpha : \alpha < c\}$ by transfinite induction. Let

$\alpha < c$, and suppose that x_β and y_β are defined for every $\beta < \alpha$. Let V_α denote the group generated by $h_\beta, x_\beta, y_\beta$ $(\beta < \alpha)$. Then $|V_\alpha| < c$, and we can choose an element $x_\alpha \in D_\alpha \setminus (\frac{1}{2} \cdot V_\alpha)$. We define $y_\alpha = 2x_\alpha$ if $g_\alpha(x_\alpha) = -2$ and $y_\alpha = x_\alpha$ otherwise. In this way we have defined x_α and y_α for every $\alpha < c$.

We put $A = \bigcup_{\alpha<c}(G_\alpha + y_\alpha)$ and $f = \chi_A$.

We prove that the function f satisfies the requirements. First we note that $|(A + h) \setminus A| < c$ for every h. Indeed, let $h = h_\beta$. Then $G_\alpha + h_\beta = G_\alpha$ for every $\alpha > \beta$. This implies $(A + h) \setminus A \subset \bigcup_{\alpha \leq \beta}(G_\alpha + y_\alpha)$, which proves $|(A + h) \setminus A| < c$. Since $A \setminus (A + h) = [(A - h) \setminus A] + h$, it follows that $|(A + h) \bigtriangleup A| < c$ for every h. Since $\{x \in \mathbb{R} : \Delta_h f(x) \neq 0\} = (A - h) \bigtriangleup A$, and since every set $B \subset \mathbb{R}$ with $|B| < c$ belongs to (s), we get that $\Delta_h f$ is $(s)-$measurable for every h.

Suppose that $f = g+A$, where g is $(s)-$measurable and A is additive. By Theorem 2.1, there is a $P_1 \in \mathrm{Perf}$ such that g is continuous on P_1. Applying Theorem 2.1 again we find a $P_2 \subset 2 \cdot P_1$ such that g is continuous on P_2. Then $f(2x)-2f(x) = g(2x)-2g(x)$ is continuous on $\frac{1}{2} \cdot P_2$. There is an $\alpha < c$ such that $\frac{1}{2} \cdot P_2 = D_\alpha$ and $f(2x) - 2f(x) = g_\alpha(x)$ for every $x \in D_\alpha$. Now we distinguish between two cases.

If $g_\alpha(x_\alpha) \neq -2$, then $y_\alpha = x_\alpha \in A$. We claim that $2x_\alpha \notin A$. Indeed, if $2x_\alpha \in A$ then $2x_\alpha \in G_\beta + y_\beta$ for some $\beta < c$. This implies $x_\alpha \in \frac{1}{2} \cdot V_\alpha$ if $\beta < \alpha$; $x_\alpha \in G_\alpha \subset V_\alpha$ if $\beta = \alpha$; and $y_\beta \in G_\beta + 2x_\alpha \subset V_\beta$ if $\beta > \alpha$. Since each of these statements is impossible, we obtain $2x_\alpha \notin A$. Thus $f(2x_\alpha) - 2f(x_\alpha) = -2 \neq g_\alpha(x_\alpha)$, a contradiction.

If $g_\alpha(x_\alpha) = -2$, then $y_\alpha = 2x_\alpha \in A$. We can prove $x_\alpha \notin A$ using an argument similar to the one above. Thus $f(2x_\alpha) - 2f(x_\alpha) = 1 \neq -2 = g_\alpha(x_\alpha)$, also impossible.

This completes the proof. ■

3 A solution to a problem of Laczkovich

Theorem 3.1 *Assume CH. Then there is a function $f : \mathbb{R} \to \mathbb{R}$ such that $\Delta_h f$ is Borel for each $h \in \mathbb{R}$ and for each $\alpha < \omega_1$ there is h such that $\Delta_h f$ is not of Baire class α.*

The following lemma is due to J. Mycielski [5].

Lemma 3.1 *For each meager sets C, E with $0 \notin C$ there is a perfect set D linearly independent over rationals such that $D \cap E = \emptyset$ and $(D - D) \cap C = \emptyset$.*

Remark 3.1 *If $(D - D) \cap (C - C) = \{0\}$ then for each t, $|(C + t) \cap D| \leq 1$. If D linearly independent over rationals then for each $t \neq 0$, $|(D + t) \cap D| \leq 1$.*

Theorem 3.2 *Assume CH. There is a set $A \subset \mathbb{R}$ such that for $t \in \mathbb{R}$ $(A + t) \setminus A$ is Borel and for each $\alpha < \omega_1$ there is $t \in \mathbb{R}$ with $(A + t) \setminus A \notin \Sigma^0_\alpha$.*

Proof Let $\mathbb{R} = \{h_\alpha : \alpha < \omega_1\}$. Let G_α denote the group generated by $\{h_\beta : \beta < \alpha\}$. We will define perfect sets D_α and Borel sets $B_\alpha \subset D_\alpha$ for each $\alpha < \omega_1$. Then $A = \bigcup_{\alpha < \omega_1} (B_\alpha + G_\alpha)$.

We define by transfinite induction sets D_α, B_α. Let D_α be (by Lemma 3.1) a perfect set linearly independent over rationals such that

1. $D_\alpha \cap \bigcup_{\beta < \alpha} (D_\beta + G_\alpha) = \emptyset$
2. $(D_\alpha - D_\alpha) \cap (\bigcup_{\beta < \alpha} (D_\beta - D_\beta) \cup \mathbb{Q}) = \{0\}$.

   Let $B_\alpha \subset D_\alpha$ be an arbitrary Borel set not from Σ^0_α.

Let us consider $(A + t) \setminus A$. Let $t = h_\alpha$ for some α.

Claim 3.1 $(A + h_\alpha) \setminus A = \left[\bigcup_{\beta \leq \alpha} (B_\beta + G_\beta + h_\alpha) \right] \setminus \bigcup_{\beta \leq \alpha} (B_\beta + G_\beta)$

Proof ($\Rightarrow$). For $\beta > \alpha$ we have $B_\beta + G_\beta + h_\alpha = B_\beta + G_\beta$. So if $x \in (A + h_\alpha) \setminus A$ then $x \in \bigcup_{\beta \leq \alpha} (B_\beta + G_\beta + h_\alpha)$.

($\Leftarrow$). If there is $\beta \leq a$ with $x \in (B_\beta + G_\beta + h_\alpha) \setminus A$ then $x \in (A + h_\alpha) \setminus A$. So it is enough to show that $(B_\beta + G_\beta + h_\alpha) \setminus \bigcup_{\gamma \leq \alpha} (B_\gamma + G_\gamma) \subset (B_\beta + G_\beta + h_\alpha) \setminus A$. If not, then there should exist $x \in \left[(B_\beta + G_\beta + h_\alpha) \cap A \right] \setminus \bigcup_{\gamma \leq \alpha} (B_\gamma + G_\gamma)$. If $x \in A \setminus \bigcup_{\gamma \leq \alpha} (B_\gamma + G_\gamma)$ then there is $\delta > \alpha$ with $x \in D_\delta + G_\delta$ but from the construction $D_\delta \cap \bigcup_{\gamma < \delta} (B_\gamma + G_\delta) = \emptyset$ so also $(D_\delta + G_\delta) \cap \bigcup_{\gamma < \delta} (B_\gamma + G_\delta) = \emptyset$ but $B_\beta + G_\beta + h_\alpha \subset B_\beta + G_\delta$, a contradiction. ∎

So $(A + t) \setminus A$ is Borel for each t.

Now we show that for each $\alpha < \omega_1$ there is $t \in \mathbb{R}$ such that $(A + t) \setminus A \notin \Sigma^0_\alpha$. It is easy to see that for each $\alpha < \omega_1$ there is $\gamma \geq \alpha$ such that $h_\gamma \notin G_\gamma$. We have $[(A + h_\gamma) \setminus A] \cap (D_\gamma + h_\gamma) = \left[\left(\bigcup_{\beta \leq \gamma} (B_\beta + G_\beta + h_\gamma) \right) \setminus \bigcup_{\beta \leq \gamma} (B_\beta + G_\beta) \right] \cap (D_\gamma + h_\gamma)$. So by Remark $((A + h_\gamma) \setminus A) \cap (D_\gamma + h_\gamma) = (B_\gamma + h_\gamma) \triangle Z$ where Z is countable.

So for $t = h_\gamma$ the set $(A + t) \setminus A$ is not in Σ^0_α. ■

Proof of Theorem 3.1 $f = \chi_A$ is the required function. ■

The authors would like to thank the referee for giving simpler proof of Theorem 2.2.

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Department of Mathematics, University of Gdańsk
Wita Stwosza 57, 80-952 Gdańsk, Poland
(E-mail: R.Filipow@impan.gda.pl, reclaw@ksinet.univ.gda.pl)