ON SOME PROPERTIES OF HAMEL BASES AND THEIR APPLICATIONS TO MARCZEWSKI MEASURABLE FUNCTIONS

FRANÇOIS G. DORAIS, RAFAŁ FILIPÓW, AND TOMASZ NATKANIEC

Abstract. We introduce new properites of Hamel bases. We show that it is consistent with ZFC that such Hamel bases exist. Under the assumption that there exists a Hamel basis with one of these properties we construct a discontinuous and additive function that is Marczewski measurable. Moreover, we show that such a function can additionally have the intermediate value property (and even be an extendable function). Finally, we examine sums and limits of such functions.

1. Introduction

We consider $\mathbb{R}^n$, $n \geq 1$, as a linear space over the field $\mathbb{Q}$ of rationals. Every basis of $\mathbb{R}^n$ is called a Hamel basis. It is well known that Hamel bases cannot be Borel sets [26], however they can be quite regular. For example, there are Hamel bases that are both Lebesgue and Baire measurable (see e.g. [17]) and Marczewski measurable [19]. On the other hand, assuming the Continuum Hypothesis CH, it is not difficult to construct a Hamel basis H of $\mathbb{R}$ that is a union of ω_1 pairwise disjoint perfect sets. Ciesielski and Pawlikowski obtained the same result assuming the Covering Property Axiom CPA ([8], see also [7]). Recall that CPA contradicts CH. It holds, for example, in the iterated perfect set model, in which $\mathfrak{c} = \omega_2$. Such Hamel bases have been applied recently in the construction of a special type of additive functions [21] and in the construction of a function with Borel differences of unbounded Baire class ([8], [12]).

This article can be divided into two parts. In the first part (Section 3) we introduce new properties of Hamel bases. Then we prove some useful facts about these properties. And, finally, we show that some set-theoretic assumptions imply the existence of Hamel bases possessing these properties (Theorems 3.16 and 3.17). However, we do not know whether or not Hamel bases with these properties can be constructed in ZFC (Problem 7.1). Hamel bases with these properties have been independently constructed in [24], where they are used to construct Marczewski measurable periodic functions.

In the second part (Sections 4 and 5) we present some applications of Hamel bases with these properties to the theory of Marczewski measurable additive functions. It is well known that every continuous additive function is of the form $f(x) = ax$. Hamel bases can be used to construct discontinuous and additive functions.

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However, every Lebesgue or Baire measurable additive function is automatically continuous (see e.g. [17]).

In Section 4, we show some deep differences between Lebesgue measurability or Baire measurability, and Marczewski measurability by constructing a discontinuous and additive function that is Marczewski measurable (Theorem 4.2). (The existence of Marczewski measurable additive functions is obvious since every continuous function is Marczewski measurable.) We also notice that there are additive functions that are not Marczewski measurable (Example 4.1). (Of course, Marczewski measurable functions that are not additive exist as well.) Moreover, we prove that for all natural numbers n, k there is a bijection $f : \mathbb{R}^n \to \mathbb{R}^k$ such that both f and f^{-1} are additive and Marczewski measurable (Theorem 4.7). Note that this shows another deep difference between Lebesgue measurability or Baire measurability, and Marczewski measurability. Indeed, recall that every bijection $f : \mathbb{R}^n \to \mathbb{R}$, $n > 1$, such that both f and f^{-1} are additive functions is very irregular, namely the image $f[U]$ of any nonempty bounded open set $U \subset \mathbb{R}^n$ is saturated Lebesgue nonmeasurable and saturated without the Baire property [5].

Section 5 is devoted to Darboux-like functions that are additive and Marczewski measurable (in this section we consider only real-valued functions defined on $\mathbb{R}$). Darboux-like additive functions have been studied by many authors (e.g. [13], [14]). Recall that all inclusion relations among Darboux-like classes remain valid in the class of additive functions. Moreover, the class of extendable functions is the smallest Darboux-like class that meets the class of additive discontinuous functions ([6]). Darboux-like functions that are Marczewski measurable were also studied (e.g. [13], [14]). It is easy to see that there are Marczewski measurable functions that are not extendable. For a discontinuous Marczewski measurable function that is extendable one can take any discontinuous Darboux function of Baire class one (and recall that every Darboux function that is Baire class one is extendable). And extendable functions that are not Marczewski measurable are constructed in [15].

In the same section, we notice that there exists an additive discontinuous and extendable function that is not Marczewski measurable (Example 5.1) and that there exists an additive, discontinuous and Marczewski measurable function that is not extendable (Example 5.2). Finally, we prove that there exists an additive, discontinuous and extendable function that is Marczewski measurable (Theorem 5.3).

In Subsection 5.1, we study algebraic properties of the family of additive Marczewski measurable functions possessing Darboux-like properties. First, we examine sums of functions. It is known that every function $f : \mathbb{R} \to \mathbb{R}$ is the sum of two extendable functions (see e.g. [13]). Moreover, every additive function is the sum of two additive and extendable functions ([22]). (Of course, one cannot get non-additive functions as sums of additive and extendable functions since additive functions are closed under sums.) The proof of this fact shows that if f is Marczewski (Lebesgue, Baire) measurable then the summands are likewise measurable, hence every Marczewski measurable function is the sum of two Marczewski measurable extendable functions. (Again, one cannot get Marczewski non-measurable functions this way since Marczewski measurable functions are closed under sums.) We show that every additive and Marczewski measurable function is the sum of two additive, Marczewski measurable and extendable functions (Theorem 5.5). We also look at limits of sequences of functions. It is known that every function $f : \mathbb{R} \to \mathbb{R}$ is the pointwise (even discrete) limit of a sequence of extendable functions (see e.g.

[13]). Moreover, every additive function is the discrete (hence also pointwise) limit of a sequence of additive and extendable functions ([22]). (Of course, one cannot get non-additive functions this way since additive functions are closed under pointwise and discrete limits.) Here, as in the case of sums, the same proof shows that every Marczewski measurable function is the discrete (hence also pointwise) limit of a sequence of Marczewski measurable and extendable functions. (Again, one cannot get functions that are not Marcewski measurable this way since Marczewski measurable functions are closed under pointwise and discrete limits.) We show that every additive and Marczewski measurable function is the discrete (hence also pointwise) limit of a sequence of additive, Marczewski measurable and extendable functions (Theorem 5.6). We also look at the convergence of transfinite sequences of functions. It is known that under CH every function $f : \mathbb{R} \to \mathbb{R}$ is the transfinite limit of a sequence of Marczewski measurable functions. (Indeed, in [27], the author shows that under CH every function is a transfinite limit of a sequence of Borel measurable functions.) Moreover, every function is the transfinite limit of a sequence of extendable functions, and every additive function is the transfinite limit of a sequence of additive and extendable function ([22]). (One cannot get non-additive functions this way since additive functions are closed under transfinite limits.) We show that CH implies that every additive function is the transfinite limit of a sequence of additive Marczewski measurable and extendable functions (Theorem 5.7).

We would like to emphasize that all the above results are obtained under the assumption on the existence of a Hamel basis of $\mathbb{R}^n$ with some nice properties (or under CH in the last case). Thus, we only prove that these facts are consistent with ZFC. We do not know whether or not any of these facts are provable in ZFC (e.g. Problems 7.2 and 7.3).

2. Preliminaries

By $\mathbb{R}$, $\mathbb{Q}$ and ω we denote the set of all reals, rationals and natural numbers, respectively. Ordinal numbers will be identified with the set of their predecessors and cardinal numbers with their initial ordinals. The symbol $|A|$ stands for the cardinality of a set A. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$. By ZFC we mean Zermelo-Fraenkel set theory with the axiom of choice, by CPA we mean the Covering Property Axiom (see e.g. [7]), and by CH we mean the Continuum Hypothesis.

For sets $A, B \subset \mathbb{R}^n$ and $r \in \mathbb{R}$ we write $A + B = \{(x_1 + y_1, \ldots, x_n + y_n) : (x_1, \ldots, x_n) \in A, (y_1, \ldots, y_n) \in B\}$, $rA = \{(rx_1, \ldots, rx_n) : (x_1, \ldots, x_n) \in A\}$ and $A+r = \{(x_1+r, \ldots, x_n+r) : (x_1, \ldots, x_n) \in A\}$. We write $\sum_{i=1}^{n} A_i$ for $A_1+\cdots+A_n$, and $\sum_{i\in Z} A_i$ for $\{\sum_{i\in Z} a_i : a_i \in A_i, i \in Z\}$. If $Z = \emptyset$, then we put $\sum_{i\in Z} A_i = \{0\}$.

In this paper we always consider $\mathbb{R}^n$ as a linear space over $\mathbb{Q}$, except in Subsection 3.3 where it is explicitly indicated otherwise. A basis of $\mathbb{R}^n$ is called a *Hamel basis*. If we say that a set $A \subset \mathbb{R}^n$ is linearly independent or a function $f : \mathbb{R}^n \to \mathbb{R}^k$ is a linear function (linear isomorphism, etc.) we mean linear over $\mathbb{Q}$. For $A \subset \mathbb{R}^n$,

$$\mathrm{LIN}\,(A) = \bigcup_{k<\omega} \bigcup_{q_1,\ldots,q_k\in\mathbb{Q}} q_1 A + \cdots + q_k A$$

denotes the linear subspace of $\mathbb{R}^n$ spanned by A. Recall, a linear function $f : \mathbb{R}^n \to \mathbb{R}^k$ is also called an *additive function* and it is often defined as a function that satisfies *Cauchy's functional equation*

$$f(x+y) = f(x) + f(y)$$

for every $x, y \in \mathbb{R}^n$ (see e.g. [17]). It is known that every function defined on a Hamel basis has a unique extension to an additive function defined on the whole of $\mathbb{R}^n$.

Let X be a topological space. By a *perfect set* $P \subset X$ we mean a nonempty closed set without isolated points. Recall, that every perfect subset of $\mathbb{R}^n$ is of cardinality $\mathfrak{c}$. A set $A \subset X$ is $\mathfrak{c}$-*dense* if the intersection of A with every nonempty open set is of cardinality $\mathfrak{c}$. By $\mathcal{M}(X)$ we denote the family of all meager (first category) subsets of X. We set $\mathcal{M} = \mathcal{M}(\mathbb{R})$. By $\operatorname{cov}(\mathcal{M}(X))$ we denote the smallest cardinality of a family of meager subsets of X whose union covers X. It is known that $\operatorname{cov}(\mathcal{M}(X)) = \operatorname{cov}(\mathcal{M})$ for every Polish space (i.e. separable completely metrizable topological space) without isolated points X (see e.g. [1]). For definitions and properties of Borel and analytic sets see e.g. [16]. Let us recall here only a few properties of Borel and analytic sets that we will use frequently in this paper.

Theorem 2.1. *Let X and Y be Polish spaces.*

1. *Every Borel set in X is analytic.*
2. *Every uncountable analytic set in X contains a perfect subset.*
3. *Continuous functions $f : X \to Y$ are Borel measurable.*
4. *The image of an analytic set in X under a Borel measurable function $f : X \to Y$ is analytic.*
5. *For every Borel sets $B_1 \subset X$ and $B_2 \subset Y$ with the same cardinality there is a bijection $f : B_1 \to B_2$ such that both f and f^{-1} are Borel measurable. If B_1 and B_2 are uncountable then one can take f such that f and f^{-1} are discontinuous.*
6. *If X and Y do not have isolated points then there is a bijection $f : X \to Y$ such that both f and f^{-1} are Borel measurable and for every set $A \subset X$ we have that A is meager if and only if $f[A]$ is meager.*
7. *The set $B = a_1A_1 + \cdots + a_kA_k$ is analytic for every $a_i \in \mathbb{R}$ and analytic sets $A_i \subset \mathbb{R}^n$ $(i = 1, \ldots, k)$.*
8. *If $A \subset \mathbb{R}^n$ is nonmeager analytic set, then $A - A$ contains a nonempty open set.*

Proof. For (1), (2), (3), (4), (5) and (8) see e.g. [16]. For (6) see e.g. [4, Theorem 3.15]. Now we will show (7). The set B is the image of an analytic set $A \times \cdots \times A$ under the continuous function $f : (\mathbb{R}^n)^k \to \mathbb{R}^n$ given by $f(x_1, \ldots, x_k) = a_1x_1 + \cdots + a_kx_k$. Applying (3) and (4) we get that B is analytic. $\square$

The following theorems contains some topological facts concerning linearly independent sets.

Theorem 2.2. (1) *There is no analytic Hamel basis.*

2. *If an analytic set $A \subset \mathbb{R}^n$ is linearly independent then it is of Lebesgue measure zero and meager.*
3. *If $A_1, \ldots, A_k \subset \mathbb{R}^n$ are analytic sets such that $A = A_1 \cup \cdots \cup A_k$ is linearly independent, then $B = q_1A_1 + \cdots + q_kA_k$ is meager for every $q_1, \ldots, q_k \in \mathbb{Q}$.*

(4) *If* $B_1, \dots, B_k \subset \mathbb{R}^n$ *are Borel sets such that* $B = B_1 \cup \dots \cup B_k$ *is linearly independent, then* $q_1 B_1 + \dots + q_k B_k$ *is Borel measurable for every* $q_1, \dots, q_k \in \mathbb{Q}$.

Proof. For (1) and (2) see e.g. [17]. Now we will show (3). Suppose that B is nonmeager. By Theorem 2.1(7), B is analytic, so by Theorem 2.1(8) $B - B$ contains nonempty open set, hence $\mathrm{LIN}\,(B) = \mathbb{R}^n$. But $\mathrm{LIN}\,(A) = \mathrm{LIN}\,(B)$. Since A is linearly independent, A is a Hamel basis. But A is analytic, a contradiction with (1). The proof of (4) is written in Section 6. In fact, we do not use (4) in the constructions of Hamel bases and additive functions in this paper. However, we decided to include it because it is closely connected with the Hamel bases we consider in this paper. Moreover, it seems interesting in its own. Recall that the algebraic sums of Borel sets were already considered by Erdős and Stone [11] and Rogers [25] were the authors showed that there are two Borel sets $A, B \subset \mathbb{R}$ such that $A + B$ is not Borel (see also [3] for another example). Moreover, recently it was shown that there is an uncountable Borel set $B \subset \mathbb{R}$ such that $C + D$ is Borel for every Borel sets $C, D \subset B$ [18]. □

Let $\mathcal{R}$ be a family of relations on a set X, that is for each $R \in \mathcal{R}$ there is a positive integer $n = n(R)$ such that $R \subset X^n$. A subset Z of X is then said to be *$\mathcal{R}$-independent* if for all $R \in \mathcal{R}$ and distinct $z_1, \dots, z_n \in Z$, where $n = n(R)$, we have $(z_1, \dots, z_n) \notin R$. The following result is essentially due to Mycielski and Taylor; Mycielski [20] originally proved the result for countable families of closed nowhere dense relations and Taylor [29] then proved the result to any family of fewer than $\mathrm{cov}\,(\mathcal{M})$ closed nowhere dense binary relations. We present a different proof that unifies these two results.

Theorem 2.3. *If $\mathcal{R}$ is a family of fewer than* $\mathrm{cov}\,(\mathcal{M})$ *meager relations on a perfect Polish space X, then X contains a perfect $\mathcal{R}$-independent set.*

Proof. We refer the reader to [1] for all notions used in this proof and have not yet been defined.

Our proof relies on the following fact [1, Theorem 3.1.8]: *If $\mathcal{P}$ is a countable partial order and $\mathcal{D}$ is a family of dense subsets of $\mathcal{P}$ with* $|\mathcal{D}| < \mathrm{cov}\,(\mathcal{M})$*, then there is a filter G on $\mathcal{P}$ that meets every element of $\mathcal{D}$.*

For simplicity, we will assume that X is Baire space ω^ω (the general case can be easily done using a bijection $f : \omega^\omega \to X$ such that f and f^{-1} are Borel measurable and f preserves meager sets (Theorem 2.1(6))).

For $s \in \omega^{<\omega}$, write
$$[s] = \{x \in \omega^\omega : s \subset x\}.$$

It is not difficult to see that we can assume that

- all relations $R \in \mathcal{R}$ are closed nowhere dense (else replace each $R \in \mathcal{R}$ by countably many closed nowhere dense relations that cover R);
- the family $\mathcal{R}$ contains the diagonal $\{(x, x) : x \in \omega^\omega\}$;
- all relations $R \in \mathcal{R}$ are symmetric (else replace each $R \in \mathcal{R}$ by the relation $\bigcup_{\sigma \in \mathrm{Sym}(n)} \{(x_{\sigma(1)}, \dots, x_{\sigma(n)}) : (x_1, \dots, x_n) \in R\}$, where $n = n(R)$ and $\mathrm{Sym}(n)$ denotes the set of all permutations of $\{1, 2, \dots, n\}$).

Consider the partial order $\mathcal{P}$ whose conditions are pairs $p = (d_p, f_p)$ where $d_p \in \omega$ and $f_p : 2^{d_p} \to \omega^{<\omega}$ is such that $|f_p(s)| \geq d_p$ for all $s \in 2^{d_p}$; the ordering of $\mathcal{P}$ is defined by $p \leq q$ iff $d_p \leq d_q$ and $f_p(s \restriction d_p) \subset f_q(s)$ for all $s \in 2^{d_q}$.

For $k \in \omega$ and $R \in \mathcal{R}$, consider the set $\mathcal{D}_{k,R}$ of all conditions $p \in \mathcal{P}$ such that $k \leq d_p$ and

$$\prod_{s \in \Sigma} [f_p(s)] \cap R = \emptyset$$

for all $n(R)$-element subset Σ of 2^{d_p}. (Note that this condition is slightly ambiguous since no ordering of Σ is given, but since R is assumed to be symmetric any ordering will do.)

We claim $\mathcal{D}_{k,R}$ is always dense in $\mathcal{P}$. To see this fix a condition $p \in \mathcal{P}$. We may assume that $d_p \geq k$. Fix an enumeration $\Sigma_1, \dots, \Sigma_m$ of all $n(R)$-element subsets of 2^{d_p} and successively define conditions $p \leq p_1 \leq \cdots \leq p_m$ in such a way that $d_p = d_{p_1} = \cdots = d_{p_m}$ and $R \cap \prod_{s \in \Sigma_i} [f_{p_i}(s)] = \emptyset$ for every $i = 1, \dots, m$. This is always possible since R is closed nowhere dense. Then, p_m is the desired extension of p in $\mathcal{D}_{k,R}$.

By the fact quoted above, there is a filter G over $\mathcal{P}$ that meets all dense sets $\mathcal{D}_{k,R}$ for $k \in \omega$ and $R \in \mathcal{R}$. We claim that the set

$$Z = \{x \in \omega^\omega : (\forall p \in G)(\exists s \in 2^{d_p})(f_p(s) \subset x)\} = \bigcap_{p \in G} \bigcup_{s \in 2^{d_p}} [f_p(s)]$$

is as required.

Note that when R is the diagonal relation, then $p \in D_{k,R}$ if and only if $d_p \geq k$ and the clopen sets $[f_p(s)]$ are pairwise disjoint for $s \in 2^{d_p}$. Hence, Z is a perfect set (compare with Cantor schemes, see e.g. [16]).

Now, we show that Z is $\mathcal{R}$-independent. Take $R \in \mathcal{R}$ with $n = n(R)$, and distinct $z_1, \dots, z_n \in Z$. There is $k \in \omega$ such that $z_i \restriction k \neq z_j \restriction k$ for distinct $i, j = 1, \dots, n$. Let $p \in G \cap D_{k,R}$. For every $i = 1, \dots, n$ there are $s_i \in 2^{d_p}$ with $z_i \in [f_p(s_i)]$. Since $d_p \geq k$ and $z_i \restriction d_p \subset f_p(s_i)$, $s_1, \dots, s_n$ are pairwise distinct. Let $\Sigma = \{s_1, \dots, s_n\}$. Since $p \in D_{k,R}$, $\prod_{s \in \Sigma} [f_p(s)] \cap R = \emptyset$. Thus $(z_1, \dots, z_n) \notin R$. □

Lemma 2.4. *Let $A = \bigcup_{\alpha < \kappa} A_\alpha \subset \mathbb{R}^n$ be a linearly independent set that is the disjoint union of $\kappa < \operatorname{cov}(\mathcal{M})$ meager Borel sets A_α. Then there is a perfect set $P \subset \mathbb{R}^n \setminus A$ such that $P \cup A$ is linearly independent.*

Proof. Consider the set $\mathcal{R}$ of all relations $R = R_{p_1,\dots,p_m,\alpha_1,\dots,\alpha_k}$ defined by

$$R = \{(x_1, \dots, x_m) \in (\mathbb{R}^n)^m : p_1 x_1 + \cdots + p_m x_m \in \operatorname{LIN}(A_{\alpha_1} \cup \cdots \cup A_{\alpha_k})\}$$

where $p_1, \dots, p_m \in \mathbb{Q} \setminus \{0\}$ and $\alpha_1, \dots, \alpha_k < \kappa$.

We claim that these relations are all meager. First of all, let us note that the set $B = \operatorname{LIN}(A_{\alpha_1} \cup \cdots \cup A_{\alpha_k})$ is meager and analytic. Indeed, it follows from the fact that

$$B = \bigcup_{k < \omega} \bigcup_{q_1, \dots, q_k \in \mathbb{Q}} q_1 A_{\alpha_1} + \cdots + q_k A_{\alpha_k}$$

is the countable union of sets $q_1 A_{\alpha_1} + \cdots + q_k A_{\alpha_k}$ which are meager (by Theorem 2.2(3)) and analytic (by Theorem 2.1(7)).

Now we can show that R is meager. We have two cases. If $m = 1$ then $R = 1/p_1 \cdot B = B$ is meager. Now, we consider the case $m \geq 2$. Suppose that R is not meager. Since $R = f^{-1}[B]$ with $f : (\mathbb{R}^n)^m \to \mathbb{R}^n$, $f(x_1, \dots, x_m) = p_1 x_1 + \cdots + p_m x_m$ and f is continuous so R has the Baire property. Then by the Kuratowski-Ulam Theorem (see e.g. [16]) there is $(z_1, \dots, z_{m-1}) \in (\mathbb{R}^n)^{m-1}$ such that the set $C = \{x \in \mathbb{R}^n : p_1 z_1 + \cdots + p_{m-1} z_{m-1} + p_m x \in B\}$ is nonmeager and has the Baire property. Then

$B \supset p_m C + p_1 z_1 + \cdots + p_{m-1} z_{m-1}$ and the former set is meager whereas the latter set is nonmeager, a contradiction.

Since all relations in $\mathcal{R}$ are meager and $|\mathcal{R}| < \operatorname{cov}(\mathcal{M})$, Theorem 2.3 (with $X = \mathbb{R}^n$) implies that there is an $\mathcal{R}$-independent perfect set $P \subset \mathbb{R}^n$. It is not difficult to see that P is the required set. $\square$

Lemma 2.5. *Let $P \in \mathbb{R}^n$ be a perfect set, $A = \bigcup_{\alpha<\kappa} A_\alpha \subset \mathbb{R}^n$ be a linearly independent set that is the disjoint union of $\kappa < \operatorname{cov}(\mathcal{M})$ meager Borel sets A_α. Then either there is $x \in \mathbb{R}^n$ such that $|P \cap (x + \operatorname{LIN}(A))| = \mathfrak{c}$ or there is a perfect set $Q \subset P \setminus A$ such that $Q \cup A$ is linearly independent.*

Proof. Suppose that for all $x \in \mathbb{R}^n$ the set $P \cap (x + \operatorname{LIN}(A))$ is of size less than $\mathfrak{c}$. Consider the set $\mathcal{R}$ of all relations $R = R_{p_1,\ldots,p_m,\alpha_1,\ldots,\alpha_k}$ defined on P by

$$R = \{(x_1, \ldots, x_m) \in P^m : p_1 x_1 + \cdots + p_m x_m \in \operatorname{LIN}(A_{\alpha_1} \cup \cdots \cup A_{\alpha_k})\}$$

where $p_1, \ldots, p_m \in \mathbb{Q} \setminus \{0\}$ and $\alpha_1, \ldots, \alpha_k < \kappa$. We claim that these relations are all meager in P. Recall that the set $B = \operatorname{LIN}(A_{\alpha_1} \cup \cdots \cup A_{\alpha_k})$ is meager and analytic.

To show that R is meager consider two cases. If $m = 1$ then $R = P \cap 1/p_1 \cdot B = P \cap B$ is an analytic subset of the set $P \cap (0 + \operatorname{LIN}(A))$, thus it is countable hence meager. Now, we consider the case $m \geq 2$. Suppose that R is not meager. Since $R = f^{-1}[B]$ with $f : P^m \to \mathbb{R}^n$, $f(x_1, \ldots, x_m) = p_1 x_1 + \cdots + p_m x_m$ and f is continuous so R has the Baire property. Then by the Kuratowski-Ulam Theorem there is $(z_1, \ldots, z_{m-1}) \in (P)^{m-1}$ such that the set $C = \{x \in P : p_1 z_1 + \cdots + p_{m-1} z_{m-1} + p_m x \in B\}$ is nonmeager in P and has the Baire property, thus it is of size $\mathfrak{c}$. Set $x = -p_m^{-1}(p_1 z_1 + \cdots + p_{m-1} z_{m-1})$, then C is a subset of the set $P \cap (x + B) \subset P \cap (x + \operatorname{LIN}(A))$, a contradiction. Since all relations in $\mathcal{R}$ are meager and $|\mathcal{R}| < \operatorname{cov}(\mathcal{M})$, Theorem 2.3 (with $X = P$) implies that there is an $\mathcal{R}$-independent perfect set $Q \subset P$. The definition of the relations in $\mathcal{R}$ ensures that Q is the required set. $\square$

2.1. Marczewski measurability.

Let X be a topological space. A set $A \subset X$ is *Marczewski measurable* ($A \in (s)$ for short) if for every perfect set $P \subset X$ either $P \cap A$ or $P \setminus A$ contains a perfect set. If every perfect set $P \subset X$ contains a perfect subset that misses A, then A is called *Marczewski null* ($A \in (s_0)$ for short). It is known that (s) is a σ-algebra and (s_0) is a σ-ideal of (s) ([28]).

Let X, Y be topological spaces. A function $f : X \to Y$ is *Marczewski measurable* (or (s)-measurable for short) if it is measurable with respect to the σ-algebra (s) (i.e. if the preimage of any open set is Marczewski measurable). In [28], Marczewski proved that for Polish spaces X, Y a function $f : X \to Y$ is (s)-measurable if and only if every perfect set $P \subset X$ has a perfect subset Q such that $f \restriction Q$ is continuous. And as a corollary one can easily show that f is (s)-measurable if and only if every perfect set $P \subset X$ has a perfect subset Q such that $f \restriction Q$ is (s)-measurable.

We will also use the following theorem.

Theorem 2.6 ([28], [30]). *Let X, Y, Z and X_i, Y_i ($i = 1, \ldots, n$) be Polish spaces.*

1. *Every analytic set in X is (s)-measurable.*
2. *The composition of two (s)-measurable functions $f : X \to Y$ and $g : Y \to Z$ is (s)-measurable.*

(3) *If a function* $f_i : X_i \to Y_i$ $(i = 1, \ldots, n)$ *is* (s)*-measurable, then the function* $F : X_1 \times \cdots \times X_n \to Y_1 \times \cdots \times Y_n$ *given by* $F(x_1, \ldots, x_n) = (f_1(x_1), \ldots, f_n(x_n))$ *is* (s)*-measurable, as well.*
(4) *Let* $f : X \to Y$ *be a bijection such that* f *and* f^{-1} *are Borel measurable. If* $A \subset X$ *is* (s)*-measurable* $(A \in (s_0))$*, then the image* $f[A]$ *is also* (s)*-measurable* $(f[A] \in (s_0))$*.*
(5) *Every uncountable Borel set in* X *contains a Marczewski null set of cardinality* $\mathfrak{c}$*.*

2.2. **Darboux-like functions.** Let X, Y be topological spaces. A function $f : X \to Y$ is a *Darboux* function ($f \in \mathrm{D}$) if the image $f[C]$ is connected in Y whenever C is connected in X. In case $X = Y = \mathbb{R}$, Darboux functions coincide with functions having the intermediate value property. A function $f : X \to Y$ is a *connectivity* function ($f \in \mathrm{Conn}$) if the graph of f restricted to C is connected in $X \times Y$ whenever $C \subset X$ is connected. A function $f : X \to Y$ is called an *extendable* function ($f \in \mathrm{Ext}$) if there exists a connectivity function $g : X \times [0, 1] \to Y$ such that $f(x) = g(x, 0)$ for all $x \in X$. (For more on these and other Darboux-like classes of functions see e.g. [13] and [14].)

Relationships between different classes of Darboux-like functions in the case $X = Y = \mathbb{R}$ are described by Gibson's diagram ([13]). Note that every continuous function is extendable, every extendable function is connectivity, and every connectivity function is Darboux. And the family of extendable functions is the smallest class of Darboux-like functions that contains discontinuous functions.

In our constructions of extendable functions we will use the method of negligible sets ([2], see also [13, Section 7.2]). Recall that if $\mathcal{K}$ is a class of functions from $\mathbb{R}$ to $\mathbb{R}$ and $k \in \mathcal{K}$, then the set $M \subset \mathbb{R}$ is *k-negligible with respect to $\mathcal{K}$* (equivalently, M is *$\mathcal{K}$-negligible for k*), provided $f \in \mathcal{K}$ for every function $f : \mathbb{R} \to \mathbb{R}$ that agrees with k on $\mathbb{R} \setminus M$. (This is the same as saying that every function $f : \mathbb{R} \to \mathbb{R}$ obtained by arbitrarily redefining k on M is still a member of $\mathcal{K}$.)

Lemma 2.7 ([6, Proposition 4.3])**.** *For every* $\mathfrak{c}$*-dense meager* F_σ *set* $M \subset \mathbb{R}$ *there exists* $g \in \mathrm{Ext}$ *such that* $\mathbb{R} \setminus M$ *is* g*-negligible with respect to* Ext.

Notice that the function g in Lemma 2.7 can be chosen of Baire class 2 ($g \in B_2$). See the construction in [9]. (Actually, it is not written explicitly there that $g \in B_2$ but, as already noted in [23], it is straightforward to verify it.)

2.3. **Discrete and transfinite convergence.** A sequence $f_n : X \to Y$, $n < \omega$, of functions *converges discretely* to a function $f : X \to Y$ if for each $x \in X$ there is a positive integer n_x such that $f_n(x) = f(x)$ for $n \geq n_x$ ([10]).

We say that a function $f : X \to Y$ is a *limit of transfinite sequence* of functions $f_\alpha : X \to Y$, $\alpha < \omega_1$, if for each $x \in X$ there is $\alpha < \omega_1$ such that $f_\beta(x) = f(x)$ for all $\beta \geq \alpha$ ([27]).

3. Nice Hamel bases

Let $H \subset \mathbb{R}^n$ be a Hamel basis. We say that

(1) H *has property* (A) ($H \in$ (A) for short) if for every perfect set $P \subset \mathbb{R}^n$ there are $q_1, \ldots, q_k \in \mathbb{Q}$ such that the set

$$P \cap (q_1 H + \cdots + q_k H)$$

contains a perfect subset;

(2) *H has property* (B) ($H \in$ (B) for short) if for every perfect set $P \subset \mathbb{R}^n$ there are $q_1, \ldots, q_k \in \mathbb{Q}$ and a Borel set $B \subset H$ such that the set

$$P \cap (q_1 B + \cdots + q_k B)$$

contains a perfect subset;

(3) *H has property* (C) ($H \in$ (C) for short) if there exists a family $\mathcal{B}$ of pairwise disjoint Borel sets such that $H = \bigcup \mathcal{B}$ and for every perfect set $P \subset \mathbb{R}^n$ there are $q_1, \ldots, q_k \in \mathbb{Q}$ and $B_1, \ldots, B_k \in \mathcal{B}$ such that the set

$$P \cap (q_1 B_1 + \cdots + q_k B_k)$$

contains a perfect subset.

(4) *H has property* (D) ($H \in$ (D) for short) if it is the union of fewer than $\mathfrak{c}$ pairwise disjoint Borel sets.

A family $\mathcal{B}$ of pairwise disjoint Borel subsets of $\mathbb{R}^n$ is said to be a (C)*-family* if for every perfect set $P \subset \mathbb{R}^n$ there are $q_1, \ldots, q_k \in \mathbb{Q}$ and $B_1, \ldots, B_k \in \mathcal{B}$ such that the set

$$P \cap (q_1 B_1 + \cdots + q_k B_k)$$

contains a perfect subset. A (C)-family is said to be *independent* if $\bigcup \mathcal{B}$ is linearly independent. Thus, a Hamel basis $H \in$ (C) if and only if H is the union of an independent (C)-family.

3.1. **Basic properties.**

Proposition 3.1. *Let $H \subset \mathbb{R}^n$ be a Hamel basis. Then $H \in$ (D) $\Rightarrow H \in$ (C) $\Rightarrow$ $H \in$ (B) $\Rightarrow H \in$ (A).*

Proof. Implications $H \in$ (C) $\Rightarrow H \in$ (B) $\Rightarrow H \in$ (A) are easy to show. Below we prove that $H \in$ (D) $\Rightarrow H \in$ (C).

Let $\mathcal{B} = \{B_\alpha : \alpha < \kappa < \mathfrak{c}\}$ be a family of disjoint Borel sets such that $H = \bigcup \mathcal{B}$ is a Hamel basis of $\mathbb{R}^n$. We will show that $\mathcal{B}$ is a (C)-family. Let $P \subset \mathbb{R}^n$ be a perfect set. Since

$$\mathbb{R}^n = \bigcup_{n<\omega} \bigcup_{q_1,\ldots,q_n \in \mathbb{Q}} q_1 H + \cdots + q_n H,$$

there is $n < \omega$ and $q_1, \ldots, q_n \in \mathbb{Q}$ such that $P \cap (q_1 H + \cdots + q_n H)$ is uncountable. But

$$\begin{aligned} P \cap (q_1 H + \cdots + q_n H) &= P \cap \left(q_1 \bigcup_{\alpha<\kappa} B_\alpha + \cdots + q_n \bigcup_{\alpha<\kappa} B_\alpha \right) \\ &= \bigcup_{\alpha_1<\kappa} \cdots \bigcup_{\alpha_n<\kappa} P \cap (q_1 B_{\alpha_1} + \cdots + q_n B_{\alpha_n}), \end{aligned}$$

so there are $\alpha_1, \ldots, \alpha_n < \kappa$ such that $P \cap (q_1 B_{\alpha_1} + \cdots + q_n B_{\alpha_n})$ is uncountable. Since the latter set is analytic, it contains a perfect subset and we are done. □

Corollary 3.2. *A Hamel basis $H \subset \mathbb{R}^n$ has property* (D) *if and only if H is the union of an independent* (C)*-family of size less than $\mathfrak{c}$.*

Proof. The implication "$\Leftarrow$" is obvious. And the implication "$\Rightarrow$" follows from the proof of Proposition 3.1. □

For a given Hamel basis H of $\mathbb{R}$ and $n > 1$ define $K_n(H)$ as the set $\bigcup_{i \leq n} \prod_{j \leq n} K_{i,j}$, where $K_{i,j} = H$ if $i = j$ and $K_{i,j} = \{0\}$ otherwise. It is easy to see that $K_n(H)$ is a Hamel basis of $\mathbb{R}^n$.

Proposition 3.3. *A Hamel basis H of $\mathbb{R}$ has property* (A) *if and only if $K_n(H)$ has this property. The same holds for properties* (B), (C) *and* (D).

Proof. We will prove the proposition only for $n = 2$ and property (A). The proof for arbitrary n and properties (A), (B), (C) and (D) can be done the same way. Moreover, the implication "$\Leftarrow$" is easy, so we only prove "$\Rightarrow$".

Let $H \subset \mathbb{R}$ be a Hamel basis with property (A). Let $K = K_2(H) = H \times \{0\} \cup \{0\} \times H$. We will show that $K \in$ (A).

Let $P \subset \mathbb{R}^2$ be a perfect set. We have 3 cases:

1. there is $a \in \mathbb{R}$ such that $P_a = \{y : (a, y) \in P\}$ is uncountable,
2. there is $b \in \mathbb{R}$ such that $P^b = \{x : (x, b) \in P\}$ is uncountable,
3. sets P_a and P^b are countable for every $a, b \in \mathbb{R}$.

In the first case, there is a perfect set $Q \subset \mathbb{R}$ such that $\{a\} \times Q \subset P$ (since P_a is an uncountable Borel set). Then there are $q_1, \ldots, q_k \in \mathbb{Q}$ and a perfect set $R \subset \mathbb{R}$ such that $R \subset Q \cap (q_1 H + \cdots + q_k H)$. Let $p_1, \ldots, p_n \in \mathbb{Q}$ be such that $a \in p_1 H + \cdots + p_n H$. Then

$$\begin{gathered}\{a\} \times R \\ \subset \{a\} \times Q \cap \Big(q_1(\{0\} \times H) + \cdots + q_k(\{0\} \times H) + p_1(H \times \{0\}) + \cdots + p_n(H \times \{0\})\Big) \\ \subset P \cap (q_1 K + \cdots + q_k K + p_1 K + \cdots + p_n K),\end{gathered}$$

and case (1) is done. Case (2) can be done the same way. Now consider the case (3). In this case, the set $A = \{x : (x, y) \in P \text{ for some } y\}$ is Borel and there is a Borel function $f : A \to \mathbb{R}$ such that $(x, f(x)) \in P$ for every $x \in A$ (by Lusin-Novikov theorem see e.g. [16]).

The set A is uncountable (otherwise a set P_a would be uncountable for some a) and Borel so there is a perfect set $Q \subset A$.

The set $B = f[Q]$ is uncountable (otherwise Q would be contained in a countable union of level sets $f^{-1}(y)$ $(y \in B)$, so at least one $f^{-1}(y)$ should be uncountable and $f^{-1}(y) \subset P^y$, a contradiction) and analytic, so there is a perfect set $R \subset B$.

There are $p_1, \ldots, p_n \in \mathbb{Q}$ such that $R \cap (p_1 H + \ldots + p_n H)$ contains a perfect set R_1. Note that $f^{-1}[R_1] \cap Q$ is an uncountable Borel subset of Q, so it contains a perfect set Q_1. There are $q_1, \ldots, q_k \in \mathbb{Q}$ such that $Q_1 \cap (q_1 H + \ldots + q_k H)$ contains a perfect set Q_2. Then $f[Q_2] \subset R_1$ and

$$\begin{gathered}f \restriction Q_2 \\ \subset P \cap \Big(p_1(\{0\} \times H) + \cdots + p_n(\{0\} \times H) + q_1(H \times \{0\}) + \cdots + q_k(H \times \{0\})\Big) \\ \subset P \cap (p_1 K + \cdots + p_n K + q_1 K + \cdots + q_k K).\end{gathered}$$

Since $f \restriction Q_2$ is an uncountable Borel set, it contains a perfect subset and we are done. □

Remark. It follows from the proof of the above proposition that if $H \subset \mathbb{R}$ is a Hamel basis with property (C) and $\mathcal{C}$ is an independent (C)-family with $H = \bigcup \mathcal{C}$, then for each $n < \omega$ there is a Hamel basis $K \subset \mathbb{R}^n$ with property (C) and an independent $\mathcal{C}$-family $\mathcal{D}$ with $K = \bigcup \mathcal{D}$ such that $\mathcal{C}$ and $\mathcal{D}$ have the same number of uncountable elements.

The following example shows that not every Hamel basis has property (A).

Example 3.4. There exists a Hamel basis of $\mathbb{R}^n$ without property (A) (hence without properties (B), (C) and (D)).

Proof. First assume that $n = 1$. We will construct (by induction) a Hamel basis H of $\mathbb{R}$ such that no set $q_1H + \cdots + q_nH$ contains a perfect set. Let S be the family of all finite sequences of non-zero rationals and Perf be the family of all perfect sets in $\mathbb{R}$. Let $\mathbb{R} = \{r_\xi : \xi < \mathfrak{c}\}$ and $\mathrm{Perf} \times S = \{(P_\alpha, s_\alpha) : \alpha < \mathfrak{c}\}$. Let n_α denote the length of s_α. For every $\alpha < \mathfrak{c}$ choose x_α, h_α and p_α such that:

- $x_\alpha \in P_\alpha \setminus \mathrm{LIN}\,(H_\alpha)$, where $H_\alpha = \{h_\beta, x_\beta : \beta < \alpha\}$.
- h_α is equal to the first r_ξ that does not belong to the set $\mathrm{LIN}\,(H_\alpha \cup \{x_\alpha\})$.
- p_α is a non-zero rational such that $p_\alpha \cdot \sum_{i \in T} s_\alpha(i) \neq 1$ for each non-empty set $T \subset \{1, \cdots, n_\alpha\}$.

Set $H = \{h_\alpha : \alpha < \mathfrak{c}\} \cup \{p_\alpha x_\alpha : \alpha < \mathfrak{c}\}$. Observe that H is a Hamel basis. We will show that no set $q_1H + \cdots + q_nH$ contains a perfect set. Suppose opposite, then there exists $\alpha < \mathfrak{c}$ such that $P_\alpha \subset \sum_{i \leq n_\alpha} s_\alpha(i)H$. Then $x_\alpha \in \sum_{i \leq n_\alpha} s_\alpha(i)H$, i.e. $x_\alpha = \sum_{i \leq n_\alpha} s_\alpha(i)y_i$ for some $y_1, \cdots, y_{n_\alpha} \in H$. Since x_α is linearly independent with $H \setminus \{p_\alpha x_\alpha\}$, this means that there exists a non-empty set $T \subset \{1, \cdots, n_\alpha\}$ such that $x_\alpha = \sum_{i \in T} s_\alpha(i)p_\alpha x_\alpha$. But then $\sum_{i \in T} s_\alpha(i)p_\alpha = 1$, a contradiction.

Finally, assume that H is a Hamel basis of $\mathbb{R}$ without property (A). Proposition 3.3 yields that for every n the set $K_n(H)$ is a Hamel basis of $\mathbb{R}^n$ without property (A). □

Proposition 3.5. *Every Hamel basis of* $\mathbb{R}^n$ *with property* (B) *contains an uncountable Borel set.*

Proof. Suppose that there is a Hamel basis $H \in$ (B) without an uncountable Borel subset. Then, all sets of the form $q_1B + \cdots + q_kB$ with a Borel $B \subset H$ would be countable, so these sets could not contain a perfect subset, a contradiction. □

Proposition 3.6. *If there exists a Hamel basis of* $\mathbb{R}^n$ *with property* (A)*, then there is a Hamel basis of* $\mathbb{R}^n$ *with property* (A) *that does not have property* (B)*.*

Proof. Let H be a Hamel basis of $\mathbb{R}^n$ with property (A). If H does not contain a perfect set then we are done (by Proposition 3.5 and Theorem 2.1(2)). So suppose H contains a perfect set and let P_α ($\alpha < \mathfrak{c}$) enumerate all perfect sets contained in $-H \cup H$. We will define a coloring $s : H \to \{\pm 1\}$ by induction. At each stage $\alpha < \mathfrak{c}$, we will decide the color of at most one element of H. So at every stage, fewer than $\mathfrak{c}$ many elements of H have been colored.

At stage α of the construction, proceed as follows. If possible, find some $h \in H$ such that $s(h)$ has not yet been defined and such that $h \in P_\alpha$ or $-h \in P_\alpha$ but not both. If $h \notin P_\alpha$ then define $s(h) = 1$; if $-h \notin P_\alpha$ then define $s(h) = -1$. If, at the end of the entire construction, some elements of H have not yet been colored, give them all color 1.

Note that $G = \{s(h)h : h \in H\}$ is a Hamel basis with property (A) since $-H \cup H = -G \cup G$. However, G does not contain a perfect set. To see this, suppose that $P \subset G$ is perfect. Since $G \subset -H \cup H$, there is an $\alpha < \mathfrak{c}$ such that $P = P_\alpha$. Also, since $P \subset G$ we never have both $h \in P$ and $-h \in P$ for $h \in H$. It follows that at some stage α, we did find a $h \in H$ such that $s(h)$ had not yet been defined and either $h \in P_\alpha$ or $-h \in P_\alpha$ but not both. But then $-s(h)h \in P_\alpha \setminus G$, a contradiction. □

Corollary 3.7. *Assume* $\mathrm{cov}\,(\mathcal{M}) = \mathfrak{c}$ *or CPA. There exists a Hamel basis of* $\mathbb{R}^n$ *with property* (A) *and without property* (B)*.*

Proof. In Theorem 3.16 and 3.17 we will show that under $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$ and CPA, respectively, there is a Hamel basis with property (A). ☐

Proposition 3.8. *Assume CH. Let $H \subset \mathbb{R}^n$ be a Hamel basis. If $H \in$ (B), then $H \in$ (C).*

Proof. Let P_α $(\alpha < \mathfrak{c})$ be an enumeration of all perfect subsets of $\mathbb{R}^n$. For every $\alpha < \mathfrak{c}$ there are $q_1^\alpha, \dots, q_k^\alpha \in \mathbb{Q}$ and a Borel set $B_\alpha \subset H$ such that the set

$$P_\alpha \cap (q_1^\alpha B_\alpha + \dots + q_k^\alpha B_\alpha)$$

contains a perfect subset.

Let $C_\alpha = B_\alpha \setminus \bigcup_{\beta<\alpha} B_\beta$. Since $\mathfrak{c} = \omega_1$, the set C_α is Borel for every $\alpha < \mathfrak{c}$.

Let

$$\mathcal{B} = \{C_\alpha : \alpha < \mathfrak{c}\} \cup \left\{ \{h\} : h \in H \setminus \bigcup_{\alpha<\mathfrak{c}} C_\alpha \right\}.$$

It is easy to see that $\mathcal{B}$ is a disjoint family of Borel sets such that $H = \bigcup \mathcal{B}$. Now we will show that $\mathcal{B}$ is a (C)-family, and that will finish the proof.

Let $P \subset \mathbb{R}^n$ be a perfect set. Then there is α with $P = P_\alpha$, so

$$P \cap (q_1^\alpha B_\alpha + \dots + q_k^\alpha B_\alpha)$$

contains a perfect subset (hence is of cardinality $\mathfrak{c}$). Since

$$\begin{aligned} P \cap (q_1^\alpha B_\alpha + \dots + q_k^\alpha B_\alpha) &\subset P \cap \left(q_1^\alpha \bigcup_{\beta\le\alpha} C_\beta + \dots + q_k^\alpha \bigcup_{\beta\le\alpha} C_\beta \right) \\ &= \bigcup_{\beta_1\le\alpha} \cdots \bigcup_{\beta_k\le\alpha} P \cap (q_1^\alpha C_{\beta_1} + \dots + q_k^\alpha C_{\beta_k}), \end{aligned}$$

and the latter set is a union of fewer than $\mathfrak{c}$ sets, it follows that there are $\beta_1, \dots, \beta_k$ such that $P \cap (q_1^\alpha C_{\beta_1} + \dots + q_k^\alpha C_{\beta_k})$ is of cardinality $\mathfrak{c}$. But the latter set is analytic, so it contains a perfect subset, and we are done. ☐

Proposition 3.9. *Assume $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$. Property (C) does not imply property (D).*

Proof. In Theorem 3.16 we will show that under $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$ there is a Hamel basis with property (C) and there is no Hamel basis with property (D). ☐

3.2. **Properties of (C)-families.**

Proposition 3.10. *Let $\mathcal{B}$ be an independent (C)-family. If a Hamel basis H contains $\bigcup \mathcal{B}$ then $H \in$ (C).*

Proof. Let $H \setminus \bigcup \mathcal{B} = \{h_\alpha : \alpha < \kappa \le \mathfrak{c}\}$. Let $\mathcal{C} = \mathcal{B} \cup \{\{h_\alpha\} : \alpha < \kappa\}$. Then $\mathcal{C}$ is a (C)-family and $H = \bigcup \mathcal{C}$. ☐

Proposition 3.11. *Let $\mathcal{B}$ be an independent (C)-family of subsets of $\mathbb{R}^n$. If H is a Hamel basis such that $\bigcup \mathcal{B} \subset H$ then $H \setminus \bigcup \mathcal{B}$ is a Marczewski null set.*

Proof. Let $K = H \setminus \bigcup \mathcal{B}$. Note that $\operatorname{LIN}(\bigcup \mathcal{B}) \cap \operatorname{LIN}(K) = \{0\}$. Let $P \subset \mathbb{R}^n$ be a perfect set. There are $q_1, \dots, q_k \in \mathbb{Q}$, $B_1, \dots, B_k \in \mathcal{B}$ and a perfect set $Q \subset P \cap (q_1 B_1 + \dots + q_k B_k)$. We have two cases. If $|Q \cap K| \le 1$ then we can, of course, find a perfect set $Q' \subset Q \setminus K$. Then $Q' \subset P \setminus K$ and we are done. Otherwise, there are two distinct elements $x, y \in Q \cap K$. And then $0 \ne x - y \in \operatorname{LIN}(\bigcup \mathcal{B}) \cap \operatorname{LIN}(K)$, a contradiction. ☐

Proposition 3.12. *Any independent* (C)*-family contains at least* $\operatorname{cov}(\mathcal{M})$ *uncountable elements.*

Proof. Let $\mathcal{B}$ be an independent (C)-family having less than $\operatorname{cov}(\mathcal{M})$ uncountable elements. Let $\mathcal{C} = \{B \in \mathcal{B} : B \text{ is uncountable}\}$. Since $|\mathcal{C}| < \operatorname{cov}(\mathcal{M})$, it follows from Lemma 2.4 that there is a perfect set $Q \subset \mathbb{R}^n \setminus \bigcup \mathcal{C}$ such that $\mathrm{LIN}(Q) \cap \mathrm{LIN}(\bigcup \mathcal{C}) = \{0\}$.

Since $\mathcal{B}$ has property (C), there are $q_i \in \mathbb{Q}$ and $B_i \in \mathcal{B}$ $(i = 1, \ldots, k)$ such that $Q \cap (q_1 B_1 + \cdots + q_k B_k)$ contains a perfect subset. We can assume that $B_1, \ldots, B_l \in \mathcal{C}$ and $B_{l+1}, \ldots, B_k \in \mathcal{B} \setminus \mathcal{C}$. Set $S = q_{l+1} B_{l+1} + \cdots + q_k B_k$ and note that this set is countable. Then

$$Q \cap (q_1 B_1 + \cdots + q_k B_k) = \bigcup_{z \in S} Q \cap (q_1 B_1 + \cdots + q_l B_l + z)$$

so there is a $z \in S$ such that $Q \cap (q_1 B_1 + \cdots + q_l B_l + z)$ is uncountable. Take two distinct elements $x, y \in Q \cap (q_1 B_1 + \cdots + q_l B_l + z)$. Then $0 \neq x - y \in \mathrm{LIN}(Q) \cap \mathrm{LIN}(\bigcup \mathcal{C})$, a contradiction. ☐

Since $\operatorname{cov}(\mathcal{M}) \geq \omega_1$, we get the following corollary.

Corollary 3.13. *Any independent* (C)*-family contains uncountably many uncountable elements.*

3.3. $\mathfrak{c}$-dense Hamel bases.

Note that in this subsection, we consider $\mathbb{R}^n$ also as a linear space over $\mathbb{R}$.

The following two facts are used in the proofs of the lemmas from this subsection.

Fact 1. *Let* $\{b_1, \ldots, b_n\} \subset \mathbb{R}^n$ *be a basis of* $\mathbb{R}^n$ *as a linear space over* $\mathbb{R}$*. Then the set* $\{q_1 b_1 + \cdots + q_n b_n : q_1, \ldots, q_n \in \mathbb{Q}\}$ *is dense in* $\mathbb{R}^n$*.*

Proof. Follows from the facts that the function $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x_1, \cdots, x_n) = x_1 b_1 + \cdots x_n b_n$ is continuous surjection and $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. ☐

Fact 2. *Let* $H \subset \mathbb{R}^n$ *be a Hamel basis. Then there are* $b_1, \ldots, b_n \in H$ *such that* $\{b_1, \ldots, b_n\}$ *is a basis of* $\mathbb{R}^n$ *as a linear space over* $\mathbb{R}$*.*

Proof. Note that $H \setminus \mathrm{LIN}_{\mathbb{R}}(A) \neq \emptyset$ for every set $A \subset \mathbb{R}^n$ of cardinality less than n. Indeed, suppose that $H \subset \mathrm{LIN}_{\mathbb{R}}(A)$ for some set A of cardinality less than n. Since $\mathrm{LIN}_{\mathbb{Q}}(A) \subset \mathrm{LIN}_{\mathbb{R}}(A)$, $\mathbb{R}^n = \mathrm{LIN}_{\mathbb{Q}}(H) \subset \mathrm{LIN}_{\mathbb{R}}(\mathrm{LIN}_{\mathbb{R}}(A)) = \mathrm{LIN}_{\mathbb{R}}(A)$, and so A contains a basis of $\mathbb{R}^n$ as a linear space over $\mathbb{R}$, a contradiction. Now, we can inductively take n points from H that are linearly independent over $\mathbb{R}$. And these points form a basis of $\mathbb{R}^n$ as a linear space over $\mathbb{R}$. ☐

Lemma 3.14. *If there exists a Hamel basis of* $\mathbb{R}^n$ *with property* (C)*, then there exists a Hamel basis of* $\mathbb{R}^n$ *with property* (C) *that contains uncountably many pairwise disjoint* $\mathfrak{c}$*-dense Borel sets.*

Proof. Let H be a Hamel basis of $\mathbb{R}^n$ with property (C). Let $\mathcal{B}$ be an independent (C)-family such that $H = \bigcup \mathcal{B}$. By Fact 2, there exist $c_1, \ldots, c_n \in H$ such that $\{c_1, \ldots, c_n\}$ is a basis of $\mathbb{R}^n$ as a linear space over $\mathbb{R}$. Let $C_i \in \mathcal{B}$ $(i = 1, \ldots, n)$ be such that $c_i \in C_i$. Set $\mathcal{C} = \{C_1, \ldots, C_n\}$, $\mathcal{B}' = \{B \in \mathcal{B} : B \text{ is uncountable}\}$, $\mathcal{B}_0 = \mathcal{B}' \setminus \mathcal{C}$ and $\mathcal{B}'' = \mathcal{B} \setminus (\mathcal{B}' \cup \mathcal{C})$.

By Corollary 3.13, $\mathcal{B}_0$ is uncountable.

For every $B \in \mathcal{B}_0$ there are pairwise disjoint uncountable Borel sets D_j^B $(j < \omega)$ such that $B = \bigcup_j D_j^B$.

By Fact 1, for every $B \in \mathcal{B}_0$ there are $q_{j,i}^B \in \mathbb{Q}$ such that

$$C_B = \bigcup_{j<\omega} \left(D_j^B + \sum_{i=1}^{n} q_{j,i}^B c_i \right)$$

is a $\mathfrak{c}$-dense Borel set.

Let $\mathcal{D} = \{C_B : B \in \mathcal{B}_0\} \cup \mathcal{C} \cup \mathcal{B}''$.

We will show that $\mathcal{D}$ is an independent (C)-family and $K = \bigcup \mathcal{D}$ is a Hamel basis.

First of all, it is not difficult to check that $\bigcup \mathcal{D}$ is linearly independent and spans $\mathbb{R}^n$ so K is a Hamel basis. Now we check that $\mathcal{D}$ is a (C)-family.

Let $P \subset \mathbb{R}$ be a perfect set. There are $q_1, \dots, q_k \in \mathbb{Q}$ and $B_1, \dots, B_k \in \mathcal{B}$ such that $P \cap (q_1 B_1 + \cdots + q_k B_k)$ contains a perfect subset. We can assume that $B_i \in \mathcal{B}_0$ for $i \le l$, $B_i \in \mathcal{C}$ for $l < i \le m$, and $B_i \in \mathcal{B}''$ for $i > m$. Set $S = q_{l+1} B_{l+1} + \cdots + q_m B_m$ and $T = q_{m+1} B_{m+1} + \cdots + q_k B_k$. Since

$$\begin{aligned} P \cap (q_1 B_1 + \cdots + q_k B_k) &= P \cap \left(q_1 \bigcup_j D_j^{B_1} + \cdots + q_l \bigcup_j D_j^{B_l} + S + T \right) \\ &= \bigcup_{j_1<\omega} \cdots \bigcup_{j_l<\omega} P \cap \left(q_1 D_{j_1}^{B_1} + \cdots + q_l D_{j_l}^{B_l} + S + T \right), \end{aligned}$$

it follows that there are $j_1, \dots, j_l < \omega$ such that $P \cap \left(q_1 D_{j_1}^{B_1} + \cdots + q_l D_{j_l}^{B_l} + S + T \right)$ is uncountable (hence contains a perfect subset). Since

$$\begin{aligned} &P \cap \left(q_1 D_{j_1}^{B_1} + \cdots + q_l D_{j_l}^{B_l} + S + T \right) \\ &= P \cap \Bigg[q_1 \left(D_{j_1}^{B_1} + \sum_{i=1}^{n} q_{j_1,i}^{B_1} c_i \right) + \cdots + q_l \left(D_{j_l}^{B_l} + \sum_{i=1}^{n} q_{j_l,i}^{B_l} c_i \right) \\ &\qquad -q_1 \left(\sum_{i=1}^{n} q_{j_1,i}^{B_1} c_i \right) - \cdots - q_l \left(\sum_{i=1}^{n} q_{j_l,i}^{B_l} c_i \right) + S + T \Bigg] \\ &= P \cap \Bigg[q_1 \left(D_{j_1}^{B_1} + \sum_{i=1}^{n} q_{j_1,i}^{B_1} c_i \right) + \cdots + q_l \left(D_{j_l}^{B_l} + \sum_{i=1}^{n} q_{j_l,i}^{B_l} c_i \right) \\ &\qquad - \left(\sum_{i=1}^{l} q_i q_{j_i,1}^{B_i} \right) c_1 - \cdots - \left(\sum_{i=1}^{l} q_i q_{j_i,n}^{B_i} \right) c_n + S + T \Bigg] \\ &\subset P \cap \Bigg[q_1 C_{B_1} + \cdots + q_l C_{B_l} - \left(\sum_{i=1}^{l} q_i q_{j_i,1}^{B_i} \right) C_1 - \cdots - \left(\sum_{i=1}^{l} q_i q_{j_i,n}^{B_i} \right) C_n + S + T \Bigg], \end{aligned}$$

it follows that the latter set contains a perfect subset. Thus $\mathcal{D}$ is a (C)-family. □

The following fact is used in the proof of the next lemma. Probably this is a part of mathematical folklore, but we decided to include it for logical completeness.

Fact 3. *Every uncountable Borel set in* $\mathbb{R}^n$ *is a disjoint union of* ω_1 *uncountable Borel sets.*

Proof. Let $B \subset \mathbb{R}^n$ be an uncountable Borel set. Let $A \subset \mathbb{R}$ be an analytic set that is not Borel (see e.g. [16, Corollary 26.2]). Let $A_\alpha, C_\alpha \subset \mathbb{R}^n$ be pairwise disjoint Borel sets such that $A = \bigcup_{\alpha<\omega_1} A_\alpha$ and $C = \mathbb{R}^n \setminus A = \bigcup_{\alpha<\omega_1} C_\alpha$ (see e.g. [16, Theorem 25.16]). Since A is not Borel, the sets A_α and C_α are nonempty for ω_1 many α. Thus, we can assume that all of them are nonempty.

Let $f : B \to \mathbb{R}^n \times \mathbb{R}$ be a Borel measurable bijection (Theorem 2.1(5)). Then $\{f^{-1}(A_\alpha \times \mathbb{R}) : \alpha < \omega_1\} \cup \{f^{-1}(C_\alpha \times \mathbb{R}) : \alpha < \omega_1\}$ is the required partition of B into ω_1 uncountable Borel sets. □

Lemma 3.15. *If there exists a Hamel basis of* $\mathbb{R}^n$ *with property* (B)*, then there exists a Hamel basis of* $\mathbb{R}^n$ *with property* (B) *that contains uncountably many pairwise disjoint* $\mathfrak{c}$*-dense Borel sets.*

Proof. Let H be a Hamel basis of $\mathbb{R}^n$ with property (B). We have two cases. If the CH holds then, by Proposition 3.8, H has property (C) and so, by Lemma 3.14, there is a Hamel basis $K \in$ (C) that contains uncountably many pairwise disjoint $\mathfrak{c}$-dense Borel sets. Then $K \in$ (B) and we are done. Now we assume that CH fails ($\omega_1 < \mathfrak{c}$).

By Fact 2, there exist $c_1, \dots, c_n \in H$ such that $\{c_1, \dots, c_n\}$ is a basis of $\mathbb{R}^n$ as a linear space over $\mathbb{R}$. By Proposition 3.5 there is an uncountable Borel set $B \subset H$. We can assume that $B \cap \{c_1, \dots, c_n\} = \emptyset$.

Let $B_{\alpha,j}$ ($\alpha < \omega_1, j < \omega$) be pairwise disjoint uncountable Borel sets such that $B = \bigcup_{\alpha<\omega_1} \bigcup_{j<\omega} B_{\alpha,j}$ (they exist by Fact 3). By Fact 1, for every $B_{\alpha,j}$ there are $q_i^{\alpha,j} \in \mathbb{Q}$ such that

$$C_\alpha = \bigcup_{j<\omega} \left(B_{\alpha,j} + \sum_{i=1}^{n} q_i^{\alpha,j} c_i \right)$$

is a $\mathfrak{c}$-dense Borel set. Let $C_{\alpha,j} = B_{\alpha,j} + \sum_{i=1}^n q_i^{\alpha,j} c_i$, and $C = \bigcup_{\alpha<\omega_1} C_\alpha$.

We will show that $K = (H \setminus B) \cup C$ is a Hamel basis with property (B), and that will finish the proof since C_α are pairwise disjoint $\mathfrak{c}$-dense Borel sets.

First of all, it is not difficult to check that K is a Hamel basis. Now we check that K has property (B).

Let $P \subset \mathbb{R}$ be a perfect set. There are $q_1, \dots, q_l \in \mathbb{Q}$ and a Borel set $A \subset H$ such that $P \cap (q_1 A + \dots + q_l A)$ contains a perfect set.

Since

$$P \cap (q_1 A + \dots + q_l A) \subset P \cap \Big(q_1(A \setminus B \cup B) + \dots + q_l(A \setminus B \cup B) \Big)$$

$$= \bigcup_{W \subset \{1,\dots,l\}} P \cap \left(\sum_{i\in W} q_i(A \setminus B) + \sum_{i\in\{1,\dots,l\}\setminus W} q_i B \right),$$

there is a $W \subset \{1, \dots, l\}$ such that $P \cap \Big(\sum_{i\in W} q_i(A \setminus B) + \sum_{i\in\{1,\dots,l\}\setminus W} q_i B\Big)$ is uncountable, hence this last intersection contains a perfect subset (because it is an uncountable analytic set). Set $Z = \{1, \dots, l\} \setminus W$, $S = \sum_{i\in W} q_i(A \setminus B)$ and $\Phi = (\omega_1 \times \omega)^Z$ (i.e. Φ is the family of all functions $\phi : Z \to \omega_1 \times \omega$). If $Z = \emptyset$ then $P \cap \sum_{i\in W} q_i(A \setminus B)$ contains a perfect set and this finishes the proof because $A \setminus B$ is a Borel subset of K. Now we will assume that $Z \neq \emptyset$. Note that $|\Phi| = \omega_1 < \mathfrak{c}$.

Since

$$P \cap \left(\sum_{i\in W} q_i (A \setminus B) + \sum_{i\in Z} q_i B \right) = P \cap \left(S + \sum_{i \in Z} q_i \bigcup_{\alpha<\omega_1} \bigcup_{n<\omega} B_{\alpha,n} \right)$$
$$= \bigcup_{\phi\in\Phi} P \cap \left(S + \sum_{i\in Z} q_i B_{\phi(i)} \right),$$

it follows that there is some $\phi \in \Phi$ such that $P \cap \left(S + \sum_{i\in Z} q_i B_{\phi(i)}\right)$ is uncountable (hence contains a perfect subset). Since

$$P \cap \left(S + \sum_{i\in Z} q_i B_{\phi(i)} \right) = P \cap \left(S + \sum_{i\in Z} q_i \left(C_{\phi(i)} - \sum_{j=1}^{n} q_j^{\phi(i)} c_j \right) \right)$$
$$= P \cap \left(S + \sum_{i\in Z} q_i C_{\phi(i)} - \sum_{i\in Z} \left(q_i \sum_{j=1}^{n} q_j^{\phi(i)} c_j \right) \right)$$
$$= P \cap \left(S + \sum_{i\in Z} q_i C_{\phi(i)} - \sum_{i\in Z} q_i q_1^{\phi(i)} c_1 - \cdots - \sum_{i\in Z} q_i q_n^{\phi(i)} c_n \right),$$

it follows that if we put $p_j = -\sum_{i\in Z} q_i q_j^{\phi(i)}$ $(j = 1, \ldots, n)$ and

$$D = (A \setminus B) \cup \bigcup_{i\in Z} C_{\phi(i)} \cup \{c_1, \ldots, c_n\}$$

then $p_1, \ldots, p_n \in \mathbb{Q}$, D is a Borel set, $D \subset K$ and

$$P \cap \left(S + \sum_{i\in Z} q_i C_{\phi(i)} + p_1 c_1 + \cdots + p_n c_n \right) \subset P \cap \left(\sum_{i\in W} q_i D + \sum_{i\in Z} q_i D + \sum_{j=1}^{n} p_j D \right).$$

So the latter set contains a perfect subset and thus $K \in$ (B). ☐

3.4. **Existence of nice Hamel bases.**

Theorem 3.16. *Assume* $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$. *There exists a Hamel basis of* $\mathbb{R}^n$ *with property* (C) *(hence also with properties* (A) *and* (B)*) and there is no Hamel basis with property* (D).

Proof. First, we show that there is a Hamel basis with property (C).

Let $\{P_\alpha : \alpha < \mathfrak{c}\}$ be the family of all perfect subsets of $\mathbb{R}^n$. We will construct sets Q_α $(\alpha < \mathfrak{c})$ such that for every $\alpha < \mathfrak{c}$:

1. $Q_\alpha \cap Q_\beta = \emptyset$ for every $\beta < \alpha$,
2. Q_α is either a perfect set, a singleton set, or the empty set,
3. $\bigcup_{\beta\leq\alpha} Q_\beta$ is linearly independent,
4. there are $q_1, \ldots, q_k \in \mathbb{Q}$ and $\beta_1, \ldots, \beta_k \leq \alpha$ such that $P_\alpha \cap (q_1 Q_{\beta_1} + \cdots + q_k Q_{\beta_k})$ contains a perfect subset.

Then $\mathcal{B} = \{Q_\alpha : \alpha < \mathfrak{c}\}$ is an independent (C)-family and by Proposition 3.10 any Hamel basis containing $\bigcup \mathcal{B}$ has property (C).

Assume that Q_β for $\beta < \alpha$ are already constructed. By Theorem 2.2(2), all sets Q_α are meager. Let $A = \bigcup_{\beta<\alpha} Q_\beta$. By Lemma 2.5, we have two cases.

Case 1. There is some $x \in \mathbb{R}^n$ with $|P \cap (x + \mathrm{LIN}(A))| = \mathfrak{c}$. If $x \in \mathrm{LIN}(A)$, then let $Q_\alpha = \emptyset$; otherwise, let $Q_\alpha = \{x\}$. It is easy to see that (1), (2) and (3) are satisfied. For (4), we consider two subcases. On the one hand, if $Q_\alpha = \emptyset$, then $x \in \mathrm{LIN}\,(A)$, so

$$P \cap (x + \mathrm{LIN}\,(A)) = P \cap \mathrm{LIN}\,(A) = P \cap \bigcup_{k<\omega} \bigcup_{q_1,\dots,q_k \in \mathbb{Q}} \bigcup_{\beta_1,\dots,\beta_k<\alpha} q_1 Q_{\beta_1} + \dots + q_k Q_{\beta_k}$$

hence there are $k < \omega$, $q_1, \dots, q_k \in \mathbb{Q}$ and $\beta_1, \dots, \beta_k < \alpha$ such that $|P \cap (q_1 Q_{\beta_1} + \dots + q_k Q_{\beta_k})| = \mathfrak{c}$. Since $q_1 Q_{\beta_1} + \dots q_k Q_{\beta_k}$ is analytic by Theorem 2.1(7), $P \cap (q_1 Q_{\beta_1} + \dots + q_k Q_{\beta_k})$ contains a perfect subset. On the other hand, if $Q_\alpha = \{x\}$, then

$$P \cap (x + \mathrm{LIN}\,(A)) = P \cap \left(x + \bigcup_{k<\omega} \bigcup_{q_1,\dots,q_k \in \mathbb{Q}} \bigcup_{\beta_1,\dots,\beta_k<\alpha} q_1 Q_{\beta_1} + \dots + q_k Q_{\beta_k} \right)$$

hence there are $k < \omega$, $q_1, \dots, q_k \in \mathbb{Q}$ and $\beta_1, \dots, \beta_k < \alpha$ such that $|P \cap (x + q_1 Q_{\beta_1} + \dots + q_k Q_{\beta_k})| = \mathfrak{c}$. Since $x + q_1 Q_{\beta_1} + \dots q_k Q_{\beta_k}$ is analytic, $P \cap (x + q_1 Q_{\beta_1} + \dots + q_k Q_{\beta_k})$ contains a perfect subset and

$$P \cap (x + q_1 Q_{\beta_1} + \dots + q_k Q_{\beta_k}) = P \cap (Q_\alpha + q_1 Q_{\beta_1} + \dots + q_k Q_{\beta_k})\,.$$

Case 2. There is a perfect set $Q \subset P_\alpha \setminus A$ such that $Q \cup A$ is linearly independent. Then we take $Q_\alpha = Q$. In this case, all properties (1), (2), (3) and (4) are easy to verify.

Now, we show that there is no Hamel basis with property (D). Suppose that $\mathcal{B}$ is a disjoint family of Borel sets such that $|\mathcal{B}| < \mathfrak{c}$ and $\bigcup \mathcal{B}$ is a Hamel basis. Then

$$\mathbb{R}^n = \bigcup_{k<\omega} \bigcup_{q_1,\dots,q_k \in \mathbb{Q}} \bigcup_{B_1,\dots,B_k \in \mathcal{B}} q_1 B_1 + \dots + q_k B_k.$$

Since each one of the sets $q_1 B_1 + \dots + q_k B_k$ is meager (Theorem 2.2(3)), this contradicts the hypothesis that $\mathfrak{c} = \mathrm{cov}\,(\mathcal{M})$. ☐

Theorem 3.17 (Ciesielski-Pawlikowski [8])**.** *Assume CPA. There exists a Hamel basis of* $\mathbb{R}^n$ *with property* (D) *(hence also with the properties* (A)*,* (B) *and* (C)*).*

Proof. In [8], the authors proved that under CPA there exists a Hamel basis H of $\mathbb{R}$ that is the union of ω_1 pairwise disjoint perfect sets.

Since under CPA we have $\omega_1 < \mathfrak{c}$, the Hamel basis H has property (D). And by Proposition 3.3 there is a Hamel basis of $\mathbb{R}^n$ with property (D) for every n. ☐

4. Marczewski measurable functions

Example 4.1. There exists an additive function $f : \mathbb{R}^n \to \mathbb{R}^k$ that is not (s)-measurable.

Proof. Let $P \subset \mathbb{R}^n$ be a linearly independent perfect set. Let H be a Hamel basis with $P \subset H$. Let $B \subset P$ be a Bernstein set in P (i.e. B and $P \setminus B$ meet all perfect subsets of P). Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be an additive function such that $f(x) = (1, \dots, 1)$ for $x \in B$ and $f(x) = 0$ for $x \in P \setminus B$. Since B and $P \setminus B$ are dense in every perfect subset of P, it follows that f is not (s)-measurable. ☐

Theorem 4.2. *If there is a Hamel basis of* $\mathbb{R}^n$ *with property* (A)*, then there exists an additive function* $f : \mathbb{R}^n \to \mathbb{R}^k$ *that is discontinuous and* (s)*-measurable.*

Proof. Let $H \subset \mathbb{R}^n$ be a Hamel basis with property (A). Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be an additive function such that $f(x) = (1, \dots, 1)$ for every $x \in H$. Obviously, the function f is discontinuous. We claim that f is (s)-measurable.

Let $P \subset \mathbb{R}^n$ be a perfect set. There are $q_1, \dots, q_m \in \mathbb{Q}$ and a perfect set $Q \subset P \cap (q_1 H + \dots + q_m H)$.

Then $f(x) = (q_1 + \dots + q_m)(1, \dots, 1)$ for every $x \in Q$. Thus $f \restriction Q$ is constant (hence continuous). □

Theorem 4.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be an additive function and $B_i \subset \mathbb{R}^n$ ($i = 1, \dots, m$) be Borel sets. If $f \restriction B_i$ is (s)-measurable for every $i = 1, \dots, m$, then $f \restriction q_1 B_1 + \dots + q_m B_m$ is (s)-measurable for every $q_1, \dots, q_m \in \mathbb{Q}$.*

Proof. Let $A = q_1 B_1 + \dots + q_m B_m$. Note that A is analytic, hence (s)-measurable. Let the function $\ell : B_1 \times \dots \times B_m \to \mathbb{R}^n$ be given by

$$\ell(x_1, \dots, x_m) = q_1 x_1 + \dots + q_m x_m.$$

This function is continuous (hence Borel measurable) thus, by the Jankov-von Neumann Uniformization Theorem (see e.g. [16, Theorem 18.1 and Exercise 18.3]) there is a function $s : A \to B_1 \times \dots \times B_m$ that is measurable with respect to the smallest σ-algebra generated by analytic sets and such that $\ell(s(x)) = x$ for every $x \in A$. The function s is also (s)-measurable because the σ-algebra of (s)-measurable sets contains all analytic sets (Theorem 2.6(1)).

Let the function $F : B_1 \times \dots \times B_m \to (\mathbb{R}^k)^m$ be given by $F(x_1, \dots, x_m) = (f(x_1), \dots, f(x_m))$. By Theorem 2.6(3), F is (s)-measurable.

Let the function $\phi : (\mathbb{R}^k)^m \to \mathbb{R}^k$ be given by $\phi(x_1, \dots, x_m) = q_1 x_1 + \dots + q_m x_m$. Then, of course, ϕ is continuous.

Since $f(x) = \phi(F(s(x)))$ for every $x \in A$, it follows from Theorem 2.6(2) that $f \restriction A$ is (s)-measurable. □

Corollary 4.4. *Let $H \subset \mathbb{R}^n$ be a Hamel basis with property* (B)*. If $f : \mathbb{R}^n \to \mathbb{R}^k$ is an additive function such that $f \restriction B$ is (s)-measurable for every Borel set $B \subset H$, then f is (s)-measurable.*

Proof. Let $P \subset \mathbb{R}^n$ be a perfect set. There are $q_1, \dots, q_m \in \mathbb{Q}$, a Borel set $B \subset H$ and a perfect set $Q \subset P \cap (q_1 B + \dots + q_m B)$.

By Theorem 4.3, $f \restriction q_1 B + \dots + q_m B$ is (s)-measurable, so $f \restriction Q$ is also (s)-measurable. And that finishes the proof. □

Corollary 4.5. *Let $\mathcal{B}$ be a* (C)*-family of subsets of $\mathbb{R}^n$. If $f : \mathbb{R}^n \to \mathbb{R}^k$ is an additive function such that $f \restriction B$ is (s)-measurable for each $B \in \mathcal{B}$ then f is (s)-measurable.*

Proof. Let $P \subset \mathbb{R}^n$ be a perfect set. There are $B_1, \dots, B_m \in \mathcal{B}$, $q_1, \dots, q_m \in \mathbb{Q}$ and a perfect set $Q \subset P \cap (q_1 B_1 + \dots + q_m B_m)$.

By Theorem 4.3, $f \restriction q_1 B_1 + \dots + q_m B_m$ is (s)-measurable, so $f \restriction Q$ is also (s)-measurable. And that finishes the proof. □

Theorem 4.6. *If there exists a Hamel basis of $\mathbb{R}^n$ with property* (B)*, then there exists a linear isomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that both f and f^{-1} are discontinuous and (s)-measurable.*

Proof. Let $H \subset \mathbb{R}^n$ be a Hamel basis of $\mathbb{R}^n$ with property (B). Take two distinct $h_0, h_1 \in H$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an additive function such that

$$f(h) = \begin{cases} h & \text{if } h \in H \setminus \{h_0, h_1\}, \\ h_i & \text{if } h = h_{1-i}, i = 0, 1. \end{cases}$$

Then, of course, f is a linear isomorphism such that f and f^{-1} are discontinuous. We will show that f and f^{-1} are (s)-measurable. Since $f^{-1} = f$, it is enough to show that f is (s)-measurable.

Let $P \subset \mathbb{R}^n$ be a perfect set. There are $q_1, \ldots, q_k \in \mathbb{Q}$, a Borel set $B \subset H$ and a perfect set Q such that $Q \subset P \cap (q_1 B + \cdots + q_k B)$.

Since $f \restriction B$ is Borel measurable, it follows from Theorem 4.3 that $f \restriction q_1 B + \cdots + q_k B$ is (s)-measurable, and so $f \restriction Q$ is (s)-measurable as well. That finishes the proof. ☐

Theorem 4.7. *If there exists a Hamel basis $H \subset \mathbb{R}$ with property* (C)*, then there exists a linear isomorphism $f : \mathbb{R}^n \to \mathbb{R}^k$ such that both f and f^{-1} are discontinuous and (s)-measurable.*

Proof. By Proposition 3.3, the Hamel bases $H_1 = K_n(H) \subset \mathbb{R}^n$ and $H_2 = K_k(H) \subset \mathbb{R}^k$ have property (C). Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be independent (C)-families such that $H_1 = \bigcup \mathcal{B}_1$ and $H_2 = \bigcup \mathcal{B}_2$.

Let $\mathcal{C}_i = \{B \in \mathcal{B}_i : B \text{ is uncountable}\}$ $(i = 1, 2)$. By the remark following Proposition 3.3, we can assume that $|\mathcal{C}_1| = |\mathcal{C}_2|$. Let $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ be a bijection.

For each $B \in \mathcal{C}_1$ let $g_B : B \to \phi(B)$ be a bijection such that both g_B and g_B^{-1} are discontinuous Borel measurable functions (Theorem 2.1(5)).

Let $B_1 \in \mathcal{C}_1$ ($\mathcal{C}_1$ is nonempty by Corollary 3.13) and $B_2 = \phi(B_1)$.

By Theorem 2.6(5) there is a Marczewski null set $A_1 \subset B_1$ of cardinality $\mathfrak{c}$.

Let $K_i = H_i \setminus \bigcup \mathcal{B}_i$ $(i = 1, 2)$. By Proposition 3.11, $K_i \in (s_0)$ $(i = 1, 2)$.

Let $h : B_1 \cup K_1 \to B_2 \cup K_2$ be a bijection such that $h(x) = g_{B_1}(x)$ for every $x \in B_1 \setminus A_1$ (there is one since $|g_{B_1}[A_1]| = \mathfrak{c}$, so the sets $A_1 \cup K_1$ and $g_{B_1}[A_1] \cup K_2$ are of size $\mathfrak{c}$).

Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a linear isomorphism such that $f \restriction B = g_B$ for each $B \in \mathcal{C}_1 \setminus \{B_1\}$ and $f \restriction (B_1 \cup K_1) = h$.

We will show that $f \restriction B$ and $f^{-1} \restriction C$ are (s)-measurable for every $B \in \mathcal{B}_1$ and $C \in \mathcal{B}_2$. Then, by Corollary 4.5, both f and f^{-1} are (s)-measurable, and that finishes the proof.

If $B \in \mathcal{B}_1 \setminus \mathcal{C}_1$ and $C \in \mathcal{B}_2 \setminus \mathcal{C}_2$, then B and C are countable, so $f \restriction B$ and $f^{-1} \restriction C$ are Borel measurable.

If $B \in \mathcal{C}_1 \setminus \{B_1\}$ and $C \in \mathcal{C}_2 \setminus \{B_2\}$, then $f \restriction B = g_B$ and $f^{-1} \restriction C = g^{-1}_{\phi^{-1}(C)}$ are Borel measurable.

Finally, we have to show that $f \restriction B_1$ and $f^{-1} \restriction B_2$ are (s)-measurable. Let $P \subset B_1$ be a perfect set. Since $A_1 \in (s_0)$, there is a perfect set $Q \subset B_1 \setminus A_1$. Then $f \restriction Q = h \restriction Q = g_{B_1} \restriction Q$ is Borel measurable. Thus, $f \restriction B_1$ is (s)-measurable. By Theorem 2.6(4), $g_{B_1}[A_1] \in (s_0)$, so the same argument as before shows that $f^{-1} \restriction B_2$ is (s)-measurable. ☐

5. DARBOUX-LIKE FUNCTIONS

Example 5.1. There exists an additive and extendable function $f : \mathbb{R} \to \mathbb{R}$ that is not (s)-measurable.

Proof. Let $H \subset \mathbb{R}$ be a Hamel basis that includes a $\mathfrak{c}$-dense F_σ meager set E. (See e.g. [6, Corollary 4.4].) We may assume that $H \setminus E$ contains a perfect set F. In fact, for any perfect set $F \subset E$, the set $E \setminus F$ has the same property as E. Let $g : \mathbb{R} \to \mathbb{R}$ be an extendable function such that $\mathbb{R} \setminus E$ is g-negligible with respect to Ext. Fix $f_0 : F \to \mathbb{R}$ that is not (s)-measurable. Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function such that $f(x) = g(x)$ for $x \in E$ and $f(x) = f_0(x)$ for $x \in F$. Then f is extendable and not (s)-measurable. □

Example 5.2. If there exists a Hamel basis with property (A), then there exists an additive and (s)-measurable function that is not Darboux (hence not extendable).

Proof. Take the additive, discontinuous and (s)-measurable function constructed in the proof of Theorem 4.2 (for $n = k = 1$). Since the range of this function is contained in $\mathbb{Q}$, it is not Darboux. □

Theorem 5.3. *If there is a Hamel basis of* $\mathbb{R}$ *with property* (B)*, then there exists an additive discontinuous function* $f : \mathbb{R} \to \mathbb{R}$ *that is* (s)*-measurable and extendable.*

Proof. Let $H \subset \mathbb{R}$ be a Hamel basis with property (B). By Lemma 3.15 we can assume that there is $\mathfrak{c}$-dense Borel set $B \subset H$.

Let $G \subset B$ be a $\mathfrak{c}$-dense meager F_σ set.

By Lemma 2.7, there exists a Borel measurable and extendable function $g : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{R} \setminus G$ is Ext-negligible for g.

Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function such that $f(x) = g(x)$ for every $x \in G$ and $f(x) = 0$ for every $x \in H \setminus G$.

Then f is a discontinuous, additive and extendable function.

Since $f \restriction B$ is Borel measurable for every Borel set $B \subset H$, it follows from Corollary 4.4 that f is (s)-measurable. □

Recall that for $n \geq 2$ every additive and extendable function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous ([6, Theorem 4.8]). Hence, the analogon of Theorem 5.3 cannot be proved. However, it is possible to replace extendability by a weaker Darboux property as the following example shows.

Example 5.4. If there is a Hamel basis of $\mathbb{R}$ with property (B), then there exist additive discontinuous and Darboux functions $F, G : \mathbb{R}^n \to \mathbb{R}^k$ such that F is (s)-measurable but G is not (s)-measurable.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be an additive discontinuous and Darboux function. It is easy to see that the function $F : \mathbb{R}^n \to \mathbb{R}^k$ defined by $F(x_1, \ldots, x_n) = (f(x_1), 0, \ldots, 0)$ is additive discontinuous and Darboux too. Moreover, F is (is not) (s)-measurable iff f has the same property (see Theorem 5.3 and Example 5.1). □

5.1. **Algebraic properties.**

Theorem 5.5. *Assume there is a Hamel basis with property* (B)*. Then every additive* (s)*-measurable function* $f : \mathbb{R} \to \mathbb{R}$ *is the sum of two additive* (s)*-measurable and extendable functions.*

Proof. Fix an additive and (s)-measurable function $f : \mathbb{R} \to \mathbb{R}$. Let $H \subset \mathbb{R}$ be a Hamel basis with property (B). By Lemma 3.15 we can assume that there are two disjoint $\mathfrak{c}$-dense Borel sets $B_0, B_1 \subset H$.

Let $G_i \subset B_i$ $(i = 0, 1)$ be a $\mathfrak{c}$-dense meager F_σ set.

By Lemma 2.7, there exists a Borel measurable and extendable function $g_i : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{R} \setminus G_i$ is Ext-negligible for g_i $(i = 0, 1)$.

Let $h_0, h_1 : \mathbb{R} \to \mathbb{R}$ be additive functions such that $(i = 0, 1)$

$$h_i(x) = \begin{cases} g_i(x) & \text{for } x \in B_i, \\ f(x) - g_{1-i}(x) & \text{for } x \in B_{1-i}, \\ i \cdot f(x) & \text{for } x \in H \setminus (B_0 \cup B_1). \end{cases}$$

By Lemma 2.7, the functions h_0, h_1 are extendable. Since $h_i \restriction B$ $(i = 0, 1)$ is (s)-measurable for every Borel set $B \subset H$, it follows from Corollary 4.4 that h_i is (s)-measurable. Finally, $f(x) = h_0(x) + h_1(x)$ for every $x \in \mathbb{R}$, and we are done. $\square$

Theorem 5.6. *Assume there is a Hamel basis with property* (B). *Then every additive* (s)*-measurable function* $f : \mathbb{R} \to \mathbb{R}$ *is the limit of a discretely convergent sequence of additive* (s)*-measurable and extendable functions.*

Proof. Fix an additive and (s)-measurable function $f : \mathbb{R} \to \mathbb{R}$. Let $H \subset \mathbb{R}$ be a Hamel basis with property (B). By Lemma 3.15 we can assume that there are pairwise disjoint $\mathfrak{c}$-dense Borel sets $B_n \subset H$ $(n < \omega)$.

Fix $n < \omega$. Let $G_n \subset B_n$ be a $\mathfrak{c}$-dense meager F_σ set. By Lemma 2.7, there exists a Borel measurable and extendable function $g_n : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{R} \setminus G_n$ is Ext-negligible for g_n.

Let $f_n : \mathbb{R} \to \mathbb{R}$ be an additive function such that

$$f_n(x) = \begin{cases} g_n(x) & \text{for } x \in B_n, \\ f(x) & \text{for } x \in H \setminus B_n. \end{cases}$$

By Lemma 2.7, the function f_n is extendable. Since $f_n \restriction B$ is (s)-measurable for every Borel set $B \subset H$, it follows from Corollary 4.4 that f_n is (s)-measurable.

We verify that the sequence $(f_n)_n$ discretely converges to f. Fix $x \in \mathbb{R}$. Then there is $n_0 < \omega$ with $x \in \mathrm{LIN}\left(\bigcup_{i<n_0} B_i \cup (H \setminus \bigcup_{i<\omega} B_i)\right)$, so $f_n(x) = f(x)$ for $n \geq n_0$. $\square$

Theorem 5.7. *Assume CH. Every additive function* $f : \mathbb{R} \to \mathbb{R}$ *is the limit of a transfinite sequence* $(f_\alpha)_{\alpha<\omega_1}$ *of additive,* (s)*-measurable and extendable functions.*

Proof. Fix an additive and (s)-measurable function $f : \mathbb{R} \to \mathbb{R}$. Since CH implies $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$, it follows from Theorem 3.16 that there is a Hamel basis $H \subset \mathbb{R}$ with property (B). By Lemma 3.15 we can assume that there are pairwise disjoint $\mathfrak{c}$-dense Borel sets $B_\alpha \subset H$ $(\alpha < \omega_1)$.

Let $H = \{h_\alpha : \alpha < \mathfrak{c}\}$. For each $\alpha < \mathfrak{c}$ set $H_\alpha = \{h_\beta : \beta \leq \alpha\}$.

Since $|H_\alpha| < \mathfrak{c} = \omega_1$ for every $\alpha < \mathfrak{c}$, there are uncountably many $\beta < \omega_1$ such that $B_\beta \cap H_\alpha = \emptyset$. Thus we can assume that already the sequence $(B_\alpha)_{\alpha<\omega_1}$ is such that $B_\alpha \cap H_\alpha = \emptyset$ for every $\alpha < \omega_1$.

Fix $\alpha < \omega_1$. Let $G_\alpha \subset B_\alpha$ be a $\mathfrak{c}$-dense meager F_σ set. By Lemma 2.7, there exists a Borel measurable and extendable function $g_\alpha : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{R} \setminus G_\alpha$ is Ext-negligible for g_α.

Let $f_\alpha : \mathbb{R} \to \mathbb{R}$ be an additive function such that

$$f_\alpha(x) = \begin{cases} g_\alpha(x) & \text{for } x \in B_\alpha, \\ f(x) & \text{for } x \in H_\alpha, \\ 0 & \text{for } x \in H \setminus (H_\alpha \cup B_\alpha). \end{cases}$$

By Lemma 2.7 the function f_α is extendable.

We show that $f_\alpha \restriction B$ is (s)-measurable for every Borel set $B \subset H$. Then, by Corollary 4.4 the function f_α is (s)-measurable.

Take a Borel set $B \subset H$ and perfect set $P \subset B$. If $B_\alpha \cap P$ is uncountable, then there is a perfect subset $Q \subset B_\alpha \cap P$. Then $f_\alpha \restriction Q = g_\alpha \restriction Q$ is Borel measurable. Suppose $B_\alpha \cap P$ is countable. Since $|H_\alpha| < \mathfrak{c}$, there is a perfect set $Q \subset P \setminus (H_\alpha \cup B_\alpha)$. Then $f_\alpha \restriction Q = 0$ is continuous. Thus $f_\alpha \restriction B$ is (s)-measurable.

Finally, we will show that f is the transfinite limit of the sequence $(f_\alpha)_{\alpha<\omega_1}$. Fix $x \in \mathbb{R}$. There is $\alpha < \mathfrak{c} = \omega_1$ with $x \in \mathrm{LIN}\,(H_\alpha)$. Then $f_\beta(x) = f(x)$ for $\beta > \alpha$. □

6. Appendix

Lemma 6.1. *If $B \subset \mathbb{R}^n$ is a linearly independent Borel set, then $q_1 B + \cdots + q_k B$ is Borel measurable for all $q_1, \ldots, q_k \in \mathbb{Q}$.*

Proof. For $k = 1$ the lemma is, obviously, true. Suppose that the lemma is true for every $k \leq l$. We will show that it is true for $k = l+1$. Let $q_1, \ldots, q_{l+1} \in \mathbb{Q}$. We can assume that $q_1, \ldots, q_{l+1} \neq 0$ (otherwise we are done by the inductive hypothesis). Let the function $f : B^{l+1} \to \mathbb{R}^n$ be given by $f(x_1, \ldots, x_{l+1}) = q_1 x_1 + \ldots q_{l+1} x_{l+1}$.

Let
$$\nabla = \left\{(x_1, \ldots, x_{l+1}) \in B^{l+1} : x_i \neq x_j \text{ for every } i \neq j\right\}$$
and
$$\Delta_{i,j} = \left\{(x_1, \ldots, x_{l+1}) \in B^{l+1} : x_i = x_j\right\}$$
for every $1 \leq i < j \leq l+1$. Then ∇ and $\Delta_{i,j}$ are Borel and $B^{l+1} = \nabla \cup \bigcup_{1\leq i<j\leq l+1} \Delta_{i,j}$.

Since B is linearly independent and $q_1, \ldots, q_{l+1} \neq 0$, the preimage of every singleton under $f \restriction \nabla$ is finite (it follows from the fact that a linear combination of pairwise distinct vectors from a linearly independent set with nonzero coefficients is unique up to the order of vectors). Thus, by the Lusin-Novikov Theorem (see e.g. [16, Theorem 18.10 and Exercise 18.14]), the set $f\,[\nabla]$ is Borel.

Since
$$f\,[\Delta_{i,j}] = \sum_{\substack{m=1\\ i\neq m\neq j}}^{l+1} q_m B + (q_i + q_j)B,$$
$f\,[\Delta_{i,j}]$ is Borel (by the inductive hypothesis).

Then, the equality
$$q_1 B + \cdots + q_l B + q_{l+1} B = f\left[B^{l+1}\right]$$
$$= f\left[\nabla \cup \bigcup_{1\leq i<j\leq l+1} \Delta_{i,j}\right] = f\,[\nabla] \cup \bigcup_{1\leq i<j\leq l+1} f\,[\Delta_{i,j}]$$
completes the proof of the lemma. □

Lemma 6.2. *If $A_1, \ldots, A_m \subset \mathbb{R}^n$ are Borel sets such that $\mathrm{LIN}\,(A_j) \cap \mathrm{LIN}\left(\bigcup_{i\neq j} A_i\right) = \{0\}$ for every $j \leq m$, then $A_1 + \cdots + A_m$ is Borel.*

Proof. Let the function $f : (R^n)^m \to \mathbb{R}^n$ be given by $f(x_1, \ldots, x_m) = x_1 + \cdots + x_m$. Note that f is continuous and $A = A_1 \times \cdots \times A_m$ is Borel. Since it is not difficult to see that $f \restriction A$ is one-to-one, it follows from the Lusin-Souslin Theorem (see e.g. [16, Theorem 15.1]) that the image $f[A]$ is Borel. Then, the equality $f[A] = A_1 + \cdots + A_m$ completes the proof of the lemma. □

Theorem 6.3 (Theorem 2.2(4))**.** *If $B_1, \ldots, B_k \subset \mathbb{R}^n$ are Borel sets such that $B = B_1 \cup \cdots \cup B_k$ is linearly independent, then $q_1 B_1 + \cdots + q_k B_k$ is Borel measurable for every $q_1, \ldots, q_k \in \mathbb{Q}$.*

Proof. Let $B_1, \ldots, B_k \subset \mathbb{R}^n$ be Borel sets such that $B = B_1 \cup \cdots \cup B_k$ is linearly independent, and $q_1, \ldots, q_k \in \mathbb{Q}$.

Claim. We can assume that either $B_i = B_j$, or $B_i \cap B_j = \emptyset$ for $i, j \leq k$

Proof of Claim. For a set A, let $A^0 = A$ and $A^1 = \mathbb{R}^n \setminus A$. For $i \leq k$ let S_i be the set of all functions $s : \{1, \ldots, k\} \to \{0, 1\}$ such that $s(i) = 0$, and let $S = \bigcup_{i \leq k} S_i$. For every $s \in S$, we define

$$A_s = B_1^{s(1)} \cap \cdots \cap B_k^{s(k)}.$$

Note that A_s are pairwise disjoint Borel sets and $B_l = \bigcup_{s \in S_l} A_s$ for every $l \leq k$. Since $\bigcup_{s \in S} A_s = B_1 \cup \cdots \cup B_k$, the former union is linearly independent. Then, the equality

$$q_1 B_1 + \cdots + q_k B_k = q_1 \bigcup_{s \in S_1} A_s + \cdots + q_k \bigcup_{s \in S_k} A_s = \bigcup_{s_1 \in S_1} \cdots \bigcup_{s_k \in S_k} q_1 A_{s_1} + \cdots + q_k A_{s_k}$$

completes the proof of Claim. □

Let $\sim$ be the equivalence relation on the set $N = \{1, \ldots, k\}$ defined by the condition $i \sim j$ if and only if $B_i = B_j$. Let $N_\sim = \{[i_1]_\sim, \ldots, [i_m]_\sim\}$. By Lemma 6.1, for each $j \leq m$ the set $A_j = \sum_{i \in [i_j]_\sim} q_i B_i$ is Borel. Since $B_1, \ldots, B_k$ are pairwise disjoint and $B_1 \cup \cdots \cup B_k$ is linearly independent, it follows that $\mathrm{LIN}(A_j) \cap \mathrm{LIN}\left(\bigcup_{i \neq j} A_i\right) = \{0\}$ for every $j \leq m$. Then, Lemma 6.2 and the equality $q_1 B_1 + \cdots + q_k B_k = A_1 + \cdots + A_m$ finish the proof. □

7. Open problems

Problem 7.1. Does there exist in ZFC a Hamel basis with property (A), (B) or (C)?

Problem 7.2. Does there exist in ZFC a discontinuous, additive and (s)-measurable function?

Problem 7.3. Does there exist in ZFC a discontinuous, additive, (s)-measurable and extendable function?

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Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043
E-mail address: `dorais@umich.edu`
URL: `http://www-personal.umich.edu/~dorais/`

Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland
E-mail address: `rfilipow@mat.ug.edu.pl`
URL: `http://rfilipow.mat.ug.edu.pl/`

Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland
E-mail address: `mattn@mat.ug.edu.pl`
URL: `http://mattn.mat.ug.edu.pl/`