On the difference property of the family of functions with the Baire property

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Abstract

We show that it is consistent with ZFC that the family of functions with the Baire property has the difference property. That is, every function for which $f(x+h)-f(x)$ has the Baire property for every $h \in \mathbb{R}$ is of the form $f = g + A$ where g has the Baire property and A is additive.

1 Introduction

For a function $f\colon \mathbb{R} \to \mathbb{R}$ and an $h \in \mathbb{R}$ we define the *difference function* $\Delta_h f\colon \mathbb{R} \to \mathbb{R}$ by $\Delta_h f(x) = f(x+h) - f(x)$. A function $A\colon \mathbb{R} \to \mathbb{R}$ is called *additive* if it satisfies Cauchy's functional equation $A(x+y) = A(x) + A(y)$ for all $x, y \in \mathbb{R}$.

The notion of difference property was introduced in the paper of de Bruijn [1] as follows. A class of functions $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ has the *difference property* if every function $f\colon \mathbb{R} \to \mathbb{R}$ such that $\Delta_h f \in \mathcal{F}$ for each $h \in \mathbb{R}$ is of the form $f = g + A$ where $g \in \mathcal{F}$ and A is additive.

We denote by $\mathcal{M}$ the family of meager sets. We will use two cardinal invariants $\mathrm{cov}(\mathcal{M}) = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{M} \wedge \bigcup \mathcal{F} = X\}$, and $\mathrm{non}^*(\mathcal{M}) = \min\{\kappa : \forall (A \notin \mathcal{M}) \exists (B \notin \mathcal{M}) B \subset A \wedge |B| \leq \kappa\}$.

A set A has the Baire property if it is the symmetric difference of an open set and a meager set. We say that a function $f\colon X \to Y$ has the Baire property if $f^{-1}(U)$ has the Baire property for every open set U.

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It is known that the family of Lebesgue measurable functions does not have the difference property under Continuum Hypothesis (see e.g. [6]). In the same way one can check that the family of functions with the Baire property does not have the difference property under Continuum Hypothesis.

On the other hand in [5, theorem 8] M. Laczkovich proved that it is consistent with *ZFC* that the family of Lebesgue measurable functions has the difference property, and in [6] he posed the problem (Problem 8.6): Is it consistent with *ZFC* that the family of functions with the Baire property has the difference property?

In this paper we prove two theorems.

Theorem 1.1 *If* $\mathrm{non}^*(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$ *then the family of functions with the Baire property has the difference property.*

And as a corollary we get a solution to the problem of Laczkovich.

Theorem 1.2 *It is consistent with ZFC that the family of functions with the Baire property has the difference property.*

Independently of our results theorem 1.2 was recently obtained in [7].

2 Preliminaries

Throughout this paper we will use the following notation. If $A, B \subset X$ we say that A, B are *equal modulo* $\mathcal{M}$, in symbols $A =_{\mathcal{M}} B$, if the symmetric difference $A \triangle B \in \mathcal{M}$. And if $f, g\colon X \to Y$ we write $f =_{\mathcal{M}} g$ if $\{x \in X : f(x) \neq g(x)\} \in \mathcal{M}$.

For a set $A \subset X \times Y$ and for $x \in X, y \in Y$ we define vertical section of A by $A_x = \{y \in Y : (x, y) \in A\}$ and horizontal section of A by $A^y = \{x \in X : (x, y) \in A\}$. And analogously for a function $f\colon X \times Y \to Z$ we write f_x (f^y respectively) for a function from Y into Z (from X into Z) such that $f_x(y) = f(x, y)$ ($f^y(x) = f(x, y)$).

When we say that a property $P(x)$ holds for almost all x (or for comeager many x) we mean that it holds for all x except a meager set.

For a function $f\colon \mathbb{R} \to \mathbb{R}$ we define a function $Df\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $Df(x, y) = f(x + y) - f(x) - f(y)$.

In the proofs we will use the following theorems.

A recent result of I. Recław [8].

Theorem 2.1 *The family of functions with the Baire property has the double difference property. That means if a function Df has the Baire property then f is a sum of a function with the Baire property and an additive function.*

And one more theorem of Recław and Zakrzewski [9, theorem 2.14].

Theorem 2.2 *If* $\mathrm{non}^*(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$ *then for every $D \subset \mathbb{R} \times \mathbb{R}$ such that all its horizontal and vertical sections have the Baire property there exists a set $B \subset \mathbb{R} \times \mathbb{R}$ with the Baire property such that $D_x =_{\mathcal{M}} B_x$ for all $x \in \mathbb{R}$.*

Let us denote by $S_{\mathcal{M}}$ a sentence there is a set $A \subset \mathbb{R}\times\mathbb{R}$ such that $A \notin \mathcal{M}$ and $\mathbb{R} \times \mathbb{R} \setminus A \notin \mathcal{M}$ and $(A + (h, k)) \setminus A \in \mathcal{M}$ for all $(h, k) \in \mathbb{R} \times \mathbb{R}$. Then we have theorem of M. Laczkovich [5, theorem 2] which says, among other things that

Theorem 2.3 *If $S_{\mathcal{M}}$ is true then* $\mathrm{non}^*(\mathcal{M}) \geq \mathrm{cov}(\mathcal{M})$.

We will also need the Kuratowski-Ulam theorem (see e.g. [2]).

Theorem 2.4

1. *If $A \subset \mathbb{R} \times \mathbb{R}$ is a meager set, then almost all vertical sections of A are meager.*
2. *If $A \subset \mathbb{R} \times \mathbb{R}$ has the Baire property and if almost all vertical sections of A are meager, then the set A is meager.*

And an easy to prove fact.

Fact 2.5 *If a function $g: \mathbb{R} \to \mathbb{R}$ has the Baire property then the function $G: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $G(x, y) = g(x + y)$ has the Baire property (as a function of two variables) too.*

3 Theorem

For a function $F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and for $(h, k) \in \mathbb{R} \times \mathbb{R}$ let us define a function $\Delta_{(h,k)}F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by the formula $\Delta_{(h,k)}F(x, y) = F(x+h, y+k)-F(x, y)$.

Lemma 3.1 *If $\Delta_h f$ has the Baire property for every $h \in \mathbb{R}$, then the function $\Delta_{(h,k)}Df$ has the Baire property for every $(h, k) \in \mathbb{R} \times \mathbb{R}$.*

Proof. Take $(h,k) \in \mathbb{R}\times\mathbb{R}$. Then

$\Delta_{(h,k)}Df(x,y) = Df(x+h,y+k) - Df(x,y) = [f(x+y+h+k) - f(x+h) - f(y+k)] - [f(x+y) - f(x) - f(y)] = [f(x+y+h+k) - f(x+y)] - [f(x+h) - f(x)] - [f(y+k) - f(y)] = \Delta_{h+k}f(x+y) - \Delta_h f(x) - \Delta_k f(y)$.

The last two functions have the Baire property as functions of one variable so they have the Baire property as functions of two variables. And by the fact 2.5 the first function has the Baire property. So the function $\Delta_{(h,k)}Df$ has the Baire property. ∎

Lemma 3.2 *Let $F\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a function such that all its vertical and horizontal sections have the Baire property. If* $\mathrm{non}^*(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$ *then there is a function $G\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ with the Baire property such that $F_x =_{\mathcal{M}} G_x$ for all $x\in\mathbb{R}$.*

Proof. It is not difficult to check that it is sufficient to show the lemma for non negative function. So let function F be as in lemma and $F(x,y)\geq 0$. Now let us define a sequence of functions $F_n\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by the formula:

$$F_n(x,y) = \begin{cases} \frac{m}{2^n} & \text{if } \frac{m}{2^n} \leq F(x,y) < \frac{m+1}{2^n} \text{ for } m = 0,1,\ldots,n2^n-1, \\ n & F(x,y)\geq n. \end{cases}$$

Then we have that

$$F_n(x,y) = \Sigma_{m=0}^{n2^n-1}\frac{m}{2^n}\chi_{A_{m,n}}(x,y) + n\chi_{A_n}(x,y),$$

where $A_{m,n} = F^{-1}([\frac{m}{2^n},\frac{m+1}{2^n}))$ and $A_n = F^{-1}([n,+\infty))$. It is easy to see that the sequence $\{F_n\}_{n\in\omega}$ is point-wise convergent to the function F.

We have that all (vertical and horizontal) sections of sets $A_{n,m}, A_n$ have the Baire property. Indeed, $(A_{m,n})_x = (F_x)^{-1}([\frac{m}{2^n},\frac{m+1}{2^n}))$ and a function F_x has the Baire property for all $x\in\mathbb{R}$. Analogously, we show it for horizontal sections and for sets A_n.

Now, applying theorem 2.2 we find sets $B_{m,n}, B_n$ with the Baire property such that $(B_{m,n})_x =_{\mathcal{M}} (A_{m,n})_x$ and $(B_n)_x =_{\mathcal{M}} (A_n)_x$ for all $x\in\mathbb{R}$.

Let $C = \bigcup_n\bigcup\{B_{m,n}\cap B_{k,n} : m\neq k, m,k = 0,1,\ldots,n2^n-1\}\cup\bigcup_n\bigcup\{B_n\cap B_{m,n} : m = 0,1,\ldots,n2^n-1\}$. Anyone can check that the sets $B_{m,n}\cap B_{k,n}$ and $B_n\cap B_{m,n}$ are meager. Indeed, suppose that there are $n,m,k, m\neq k$ such that $B_{m,n}\cap B_{k,n}\notin\mathcal{M}$ (case of sets B_n and $B_{m,n}$ can be done in the same way). Since the sets $B_{m,n}, B_{k,n}$ have the Baire property so by theorem 2.4.2 there is a set $X\notin\mathcal{M}$ such that $(B_{m,n}\cap B_{k,n})_x\notin\mathcal{M}$ for every $x\in X$.

On the other hand $(B_{m,n})_x =_{\mathcal{M}} (A_{m,n})_x$ and $(B_{k,n})_x =_{\mathcal{M}} (A_{k,n})_x$ for every $x \in \mathbb{R}$ hence $(A_{m,n} \cap A_{k,n}) \notin \mathcal{M}$ for every $x \in X$. Since $X \neq \emptyset$ we get that $A_{m,n} \cap A_{k,n} \neq \emptyset$ a contradiction (since by definition of sets $A_{m,n}$ we have that $A_{m,n} \cap A_{k,n} = \emptyset$).

Finally we get that the set C is meager.

Now define the following sets $D_{m,n} = B_{m,n} \setminus C$ and $D_n = B_n \setminus C$. These sets have the Baire property and for almost all $x \in \mathbb{R}$ we have $(D_{m,n})_x =_{\mathcal{M}} (A_{m,n})_x$ and $(D_n)_x =_{\mathcal{M}} (A_n)_x$. Indeed, it follows from the fact that almost all vertical sections of the set C are meager (by theorem 2.4.1).

Now define a sequence of functions $G_n\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by a formula $G_n(x,y) = \Sigma_{m=0}^{n2^n-1} \frac{m}{2^n} \chi_{D_{m,n}}(x,y) + n\chi_{D_n}(x,y)$.

Now we show that the set $Z = \{(x,y) \in \mathbb{R}\times\mathbb{R} : \{G_n(x,y)\}$ is convergent$\}$ is comeager. Since the functions G_n have the Baire property, the set Z has the Baire property too.
Take $x \in \{z \in \mathbb{R} : C_z \in \mathcal{M}\}$. We show that Z_x is comeager. Let $H_x^n = \{y \in \mathbb{R} : G_n(x,y) = F_n(x,y)\}$. For all n sets H_x^n are comeager. Indeed, $H_x^n = \{y \in \mathbb{R} : G_n(x,y) = F_n(x,y)\} = \{y \in \mathbb{R} : \Sigma_{m=0}^{n2^n-1} \frac{m}{2^n} \chi_{D_{m,n}}(x,y) + n\chi_{D_n}(x,y) = \Sigma_{m=0}^{n2^n-1} \frac{m}{2^n} \chi_{A_{m,n}}(x,y) + n\chi_{A_n}(x,y)\} = \{y \in \mathbb{R} : \Sigma_{m=0}^{n2^n-1} \frac{m}{2^n} \chi_{(D_{m,n})_x}(y) + n\chi_{(D_n)_x}(y) = \Sigma_{m=0}^{n2^n-1} \frac{m}{2^n} \chi_{(A_{m,n})_x}(y) + n\chi_{(A_n)_x}(y)\} \supset \bigcap_m [\mathbb{R} \setminus ((D_{m,n})_x \triangle (A_{m,n})_x)] \cap (\mathbb{R} \setminus ((A_n)_x \triangle (D_n)_x))$. But we know that when $x \in \{z \in \mathbb{R} : C_z \in \mathcal{M}\}$ then $(D_{m,n})_x =_{\mathcal{M}} (A_{m,n})_x$ and $(D_n)_x =_{\mathcal{M}} (A_n)_x$ so the sets $((D_{m,n})_x) \triangle (A_{m,n})_x$ and $(A_n)_x \triangle (D_n)_x$ are meager. Hence the set H_x^n is comeager for all n so a set $H = \bigcap_n H_x^n$ is comeager too. Since for all $y \in H$ we have that $G_n(x,y) = F_n(x,y)$ and the sequence $\{F_n(x,y)\}$ is convergent so the sequence $\{G_n(x,y)\}$ is convergent for all $y \in H$ as well. Finally we get that $H \subset Z_x$ so the section Z_x is comeager. Since the set $\{z \in \mathbb{R} : C_z \in \mathcal{M}\}$ is comeager (by theorem 2.4.1) we get that comeager many vertical sections of set Z are comeager so by theorem 2.4.2 the set Z is comeager.

Let us define the function $G\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by the formula

$$G(x,y) = \begin{cases} \lim_{n\to\infty} G_n(x,y) & \text{if } (x,y) \in Z, \\ 0 & \text{if } (x,y) \notin Z. \end{cases}$$

Anyone can see that the function G has the Baire property.

Now we show that the function G is "almost good" i.e. for comeager many $x \in \mathbb{R}$ we have $G_x =_{\mathcal{M}} F_x$. Take $x \in \{z \in \mathbb{R} : C_z \in \mathcal{M}\}$. Then for all $y \in H$ we have $G_n(x,y) = F_n(x,y)$ so $G_x(y) = F_x(y)$. Now it is sufficient to change the function G in the following way

$$\overline{G}(x,y) = \begin{cases} G(x,y) & \text{if } x \in \{z \in \mathbb{R} : C_z \in \mathcal{M}\} \text{ and } y \in \mathbb{R}, \\ F(x,y) & \text{otherwise.} \end{cases}$$

Since the set $\{z \in \mathbb{R} : C_z \in \mathcal{M}\} \times \mathbb{R}$ is comeager we have that the function $\overline{G}$ has the Baire property and for all $x \in \mathbb{R}$ we have $(\overline{G})_x =_{\mathcal{M}} (F)_x$ as well. That finishes the proof of lemma. ■

Lemma 3.3 *Let f be such that $\Delta_h f$ has the Baire property for every $h \in \mathbb{R}$. If* $\text{non}^*(\mathcal{M}) < \text{cov}(\mathcal{M})$ *then $\Delta_{(h,k)}(Df - G) =_{\mathcal{M}} 0$ for all $(h,k) \in \mathbb{R} \times \mathbb{R}$, where G is a function like in lemma 3.2 for a function $F = Df$.*

Proof. Fix $(h,k) \in \mathbb{R} \times \mathbb{R}$. For every $x \in \mathbb{R}$ denote by $B(x)$ a comeager set such that $G_x(y) = (Df)_x(y)$ for all $y \in B(x)$. We have to show that a set $A = (\Delta_{(h,k)}(Df - G))^{-1}(0)$ is comeager. At the first observe that all vertical section of the set A are comeager. Indeed, $A_x = \{y \in \mathbb{R} \colon \Delta_{(h,k)}(Df - G)(x,y) = 0\} = \{y \in \mathbb{R} \colon \Delta_{(h,k)}(Df)(x,y) = \Delta_{(h,k)}(G)(x,y)\} = \{y \in \mathbb{R} \colon Df(x+h, y+k) - Df(x,y) = G(x+h, y+k) - G(x,y)\}$, but we know that $(Df)_{x+h}(y+k) = G_{x+h}(y+k)$ for all $y \in B(x+h) - k$ and $(Df)_x(y) = G_x(y)$ for all $y \in B(x)$. Finally we get that $(B(x+h) - k) \cap B(x) \subset A_x$ so the set A_x is comeager for all $x \in \mathbb{R}$.

The function $\Delta_{(h,k)}(Df - G)$ has the Baire property by lemma 3.1 and the fact that the function G has the Baire property. So the set A has the Baire property too. Now by theorem 2.4.2 the set A is comeager. ■

Proof of theorem 1.1 Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function such that all its difference functions have the Baire property. We will show that the function Df has the Baire property and it will finish the proof because we have the theorem 2.1 which says that the family of functions with the Baire property has the double difference property.

Suppose that the function Df does not have the Baire property. Then there is $r \in \mathbb{R}$ such that a set $A = (Df - G)^{-1}((-\infty, r))$ does not have the Baire property, where G is as in lemma 3.2 for a function $F = Df$.

For every $(h,k) \in \mathbb{R} \times \mathbb{R}$ the set $(A + (h,k)) \setminus A$ is meager. Indeed, $(A + (h,k)) \setminus A = \{(x,y) \colon (Df - G)(x,y) < r\} + (h,k) \setminus \{(x,y) \colon (Df - G)(x,y) < r\} = \{(x+h, y+k) \colon (Df - G)(x,y) < r\} \setminus \{(x,y) \colon (Df - G)(x,y) < r\} = \{(x,y) \colon (Df - G)(x-h, y-k) < r\} \setminus \{(x,y) \colon (Df - G)(x,y) < r\} = \{(x,y) \colon (Df - G)(x-h, y-k) < r \text{ and } (Df - G)(x,y) \geq r\} \subset$

$\{(x,y)\colon \Delta_{(-h,-k)}(Df-G)(x,y)\neq 0\}$ and the last set is meager by lemma 3.3 so the set $(A+(h,k))\setminus A$ is meager too.

Then we see that the sentence $S_{\mathcal{M}}$ is true (the set A constructed above is good one). But by theorem 2.3 it leads to contradiction with our assumption that $\mathrm{non}^*(\mathcal{M})<\mathrm{cov}(\mathcal{M})$. The contradiction we got by supposition that the function Df does not have the Baire property. So that finishes the proof. ■

Proof of theorem 1.2 It is known that a model for $\mathrm{cov}(\mathcal{M})=2^\omega=\omega_2$ is the model obtained by adding ω_2 many Cohen reals to a model of GCH. On the other hand it was shown that in this model we also have $\mathrm{non}^*(\mathcal{M})=\omega_1$ (see e.g. [4, proof of theorem 3] or [3, theorem 2.6]). Hence in this model the inequality $\mathrm{non}^*(\mathcal{M})<\mathrm{cov}(\mathcal{M})$ is true. By the theorem 1.1 in that model the family of functions with the Baire property has the difference property. ■

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