$\mathcal{I}$-SELECTION PRINCIPLES FOR SEQUENCES OF FUNCTIONS

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Abstract. We generalize three classical selection principles (Arzela-Ascoli theorem, Mazurkiewicz's theorem and Helly's theorem) on the ideal convergence. In particular, we show that for every analytic P-ideal $\mathcal{I}$ with the BW property (and every F_σ ideal $\mathcal{I}$) the following selection theorems hold:

- If $\langle f_n \rangle_n$ is a sequence of uniformly bounded equicontinuous functions on $[0,1]$ then there exists $A \notin \mathcal{I}$ such that $\langle f_n \rangle_{n \in A}$ is uniformly convergent;
- if $\langle f_n \rangle_n$ is a sequence of uniformly bounded continuous functions then there exists a perfect set P and a set $A \notin \mathcal{I}$ such that $\langle f_n \upharpoonright P \rangle_{n \in A}$ is pointwise convergent;
- if $\langle f_n \rangle_n$ is a sequence of uniformly bounded monotone functions then there exists a set $A \notin \mathcal{I}$ such that $\langle f_n \rangle_{n \in A}$ is pointwise convergent.

1. Introduction

For every theorem that deals with convergence of sequence there is natural question whether this theorem can be generalized in some sense to ideal convergence[1], and for what class of ideals on ω such a generalization is true. Many such generalizations have been made for classical theorems of convergence of functions. In [1], Balcerzak, Dems and Komisarski proved an ideal version of Egorov's theorem for the ideal $\mathcal{I}_d$ of statistical density zero sets. Recently, Mrożek [22] extended their result to all analytic P-ideals. Other ideals were considered by Kadets and Leonov in [12]. Moreover, in [1] authors showed that ideal version of Riesz theorem holds for every P-ideal. Ideal convergence of continuous functions (and other Baire classes) was already investigated by Katětov [13]. He proved that for every $\alpha \in \omega_1$ there is an ideal $\mathcal{I}_\alpha$ such that the family of all $\mathcal{I}_\alpha$-limits of continuous functions coincides with the family of all functions of Baire α-class. If we do not assume any regularity property on an ideal then the situation can be even worse: there is an ideal $\mathcal{I}$ and a sequence of continuous functions $\langle f_n \rangle_{n \in \omega}$ such that $\mathcal{I}$-limit of $\langle f_n \rangle$ is Lebesgue non-measurable (see e.g. [17]). However, if we assume that $\mathcal{I}$ is an analytic P-ideal then ideal limits of continuous functions behave like ordinary limits. In [17] it is shown that $\mathcal{I}_d$-limits of sequences of continuous functions are of

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[1]Filter convergence was introduced by Cartan [4] in 1937 and became an important tool in general topology and functional analysis since 1940, when Bourbaki's book [3] appeared. Nowadays many authors prefer to use an equivalent notion of ideal convergence ([17], [1]). In this paper we use this latter language.

the first Baire class. Finally, in [5] and, independently, in [18] it was generalized to all analytic P-ideals and all Baire classes.

In [8] we examined the ideal version of the Bolzano-Weierstrass theorem: if it is true that for every bounded sequence $\langle x_n\rangle_{n\in\omega}$ of reals there exists $A \notin \mathcal{I}$ such that the subsequence $\langle x_n\rangle_{n\in A}$ is $\mathcal{I}$-convergent. It turned out that the answer depends on an ideal $\mathcal{I}$. We say that an ideal has the BW property if it satisfies this version of the Bolzano-Weierstrass theorem. For discussion, characterizations and applications of ideals with the BW property see [8].

It is not difficult to prove that the Bolzano-Weierstrass theorem fails if we consider sequences of functions instead of reals. Namely, there exists a uniformly bounded sequence $\langle f_n\rangle_{n\in\omega}$ of real-valued functions defined on $\mathbb{R}$ such that no subsequence of $\langle f_n\rangle_{n\in\omega}$ is pointwise convergent. If we restrict our attention to continuous functions and uniform convergence then the functional counterpart of the Bolzano-Weierstrass theorem also does not hold. However, in this case we have the well known theorem of Arzela-Ascoli which characterizes the sequences for which there is a uniformly convergent subsequence.

Theorem 1.1 (Arzela-Ascoli). *If a sequence $\langle f_n\rangle_{n\in\omega}$ of continuous real-valued functions defined on $[0,1]$ is uniformly bounded and equicontinuous then there is a subsequence $\langle f_{n_k}\rangle_{k\in\omega}$ which is uniformly convergent.*

In Section 3 we show (Theorem 3.1) that the ideal version of Arzela-Ascoli theorem holds for every ideal with the BW property.

Mazurkiewicz [20] proved that if we drop the equicontinuity assumption in the Arzela-Ascoli theorem then one can always find a subsequence that is uniformly convergent on a perfect set.

Theorem 1.2 (Mazurkiewicz). *If $\langle f_n\rangle_{n\in\omega}$ is a uniformly bounded sequence of continuous real-valued functions defined on $\mathbb{R}$ then there is a subsequence $\langle f_{n_k}\rangle_{k\in\omega}$ and a perfect set $P \subseteq \mathbb{R}$ such that $\langle f_{n_k} \upharpoonright P\rangle_{k\in\omega}$ is uniformly convergent.*

In Section 4 we show that the ideal version of Mazurkiewicz's theorem holds for every ideal which can be extended to an F_σ ideal (Theorem 4.1)—in particular, for every analytic P-ideal with the BW property (Corollary 4.2). However, we do not know if the answer is positive for every ideal with the BW property (Problem 4.3).

Helly [10] proved that if we replace equicontinuity by monotonicity in the Arzela-Ascoli theorem then one can always find a subsequence that is pointwise convergent.

Theorem 1.3 (Helly). *If $\langle f_n\rangle_{n\in\omega}$ is a uniformly bounded sequence of monotone real-valued functions defined on $\mathbb{R}$ then there is a subsequence $\langle f_{n_k}\rangle_{k\in\omega}$ which is pointwise convergent.*

The ideal version of Helly's theorem was already investigated by Kojman in [16] and [15]. He proved that the ideal version of Helly's theorem holds for two ideals which were defined in a combinatorial manner. Namely, for van der Waerden ideal (of all sets which does not contain an arbitrarily long arithmetic progression) and Hindman ideal (a set $A \subseteq \omega$ is not in the Hindman ideal iff there is an infinite $F \subseteq A$ such that all finite sums of elements of F belongs to A).

In Section 5 we show that the ideal version of Helly's theorem holds for some subclasses of ideals with the BW property (Theorems 5.8 and 5.9). We do not know if the ideal version of Helly's theorem holds for every ideal with the BW property

(Problem 5.10). However, for analytic P-ideals the ideal version of Helly's theorem holds if and only if an ideal has the BW property (Theorem 5.8).

In Section 5 we also provide sufficient conditions on spaces and ideals to have the BW property (Corollary 5.6)—a variation on a theorem of Kojman [16]. We use this as a tool in the proof of the ideal version of Helly's theorem.

2. Preliminaries

The set of natural numbers we denote by the symbol ω. The cardinality of a set X is denoted by $|X|$. For any set A by $A^{<\omega}$ we denote the set $\bigcup_{n\in\omega} A^n$, where A^n is the set of all functions from $\{0, 1, 2, \ldots, n-1\}$ to A. If $x \in A^\omega$ then $x \upharpoonright n \in A^n$ is the restriction of x to the first n coordinates.

An ideal on ω is a family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ (where $\mathcal{P}(\omega)$ denotes the power set of ω) which is closed under taking subsets and finite unions. By Fin we denote the ideal of all finite subsets of ω. If not explicitly said we assume that all considered ideals are proper ($\neq \mathcal{P}(\omega)$) and contain all finite sets.

If $\mathcal{I}$ is an ideal on ω, then $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ and $\mathcal{I}^* = \{\omega \setminus A : A \in \mathcal{I}\}$. $\mathcal{I}^+$ is called the *coideal* and $\mathcal{I}^*$ is called the *dual filter*.

If $\mathcal{I}$ is an ideal on ω and $A \in \mathcal{I}^+$, then the *restriction* of $\mathcal{I}$ to A, denoted by $\mathcal{I} \upharpoonright A$, is the ideal on ω given by $\mathcal{I} \upharpoonright A = \{B \subseteq \omega : B \cap A \in \mathcal{I}\}$.

An ideal $\mathcal{I}$ is a *P-ideal* if for every sequence $\langle A_n \rangle_{n\in\omega}$ of sets from $\mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \setminus A \in \text{Fin}$ for all n, i.e. A_n is almost contained in A for each n.

2.1. **Analytic ideals.** By identifying sets of naturals with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the Cantor-space topology and therefore we can assign the topological complexity to the ideals of sets of integers. In particular, an ideal $\mathcal{I}$ is F_σ (*analytic*) if it is an F_σ subset of the Cantor space (if it is a continuous image of a G_δ subset of the Cantor space, respectively.)

A map $\phi \colon \mathcal{P}(\omega) \to [0, \infty]$ is a *submeasure on* ω if

$$\phi(\emptyset) = 0,$$

$$\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B),$$

for all $A, B \subseteq \omega$. It is *lower semicontinuous* (we will write *lsc* for short) if for all $A \subseteq \omega$ we have

$$\phi(A) = \lim_{n\to\infty} \phi(A \cap \{0, 1, \ldots, n-1\}).$$

Let

$$\text{Fin}(\phi) = \{A \subseteq \omega \ : \ \phi(A) < \infty\}.$$

All F_σ ideals are characterized by the following theorem of Mazur.

Theorem 2.1 ([19]). *The following conditions are equivalent for an ideal $\mathcal{I}$ on ω.*

(1) *$\mathcal{I}$ is an F_σ ideal;*
(2) *$\mathcal{I} = Fin(\phi)$ for some lsc submeasure ϕ on ω.*

All analytic P-ideals were characterized in similar way by Solecki (see [23]).

Below we present a few examples of analytic ideals. More examples can be found in Farah's book [6].

Example 2.2. The ideal of sets of density 0

$$\mathcal{I}_d = \left\{A \subseteq \omega \ : \ \limsup_{n\to\infty} \frac{|A \cap \{0, 1, \ldots, n-1\}|}{n} = 0\right\},$$

is an analytic P-ideal.

Example 2.3 (Mazur [19])**.** For $f : \omega \to \mathbb{R}^+$ such that $\sum_{n\in\omega} f(n) = +\infty$ we define the *summable ideal* by

$$\mathcal{I}_f = \left\{ A \subseteq \omega : \sum_{n\in A} f(n) < \infty \right\}.$$

Every summable ideal is an F_σ P-ideal.

Example 2.4. The ideal of arithmetic progressions free sets

$$\mathcal{W} = \{W \subseteq \omega \ : \ W \text{ does not contain arithmetic progressions of all lengths}\}$$

is an F_σ ideal which is not a P-ideal. The fact that $\mathcal{W}$ is an ideal follows from the non-trivial theorem of van der Waerden. This ideal was firstly considered by Kojman in [16].

2.2. **Ideal convergence.** Let X be a topological Hausdorff space (in the sequel we assume all our topological spaces to be Hausdorff) and $\mathcal{I}$ be an ideal on ω. Let $A \in \mathcal{I}^+$ and $x_n \in X$ for each $n \in A$. We say that the sequence $\langle x_n \rangle_{n\in A}$ is *$\mathcal{I}$-convergent* to $x \in X$ if

$$\{n \in A : x_n \notin U\} \in \mathcal{I}$$

for every open neighborhood U of x.

The following key definitions were introduced in [8]. We say that the pair $(X, \mathcal{I})$ has

(1) the *BW property* (*Bolzano-Weierstrass property*) if every sequence $\langle x_n \rangle_{n\in\omega} \subseteq X$ has an $\mathcal{I}$-convergent subsequence $\langle x_n \rangle_{n\in A}$ with $A \in \mathcal{I}^+$;
(2) the *hBW property* (*hereditary Bolzano-Weierstrass property*) if $(X, \mathcal{I} \restriction A)$ has the BW property for every $A \in \mathcal{I}^+$;
(3) the *FinBW property* (*finite Bolzano-Weierstrass property*) if every sequence $\langle x_n \rangle_{n\in\omega} \subseteq X$ has a convergent subsequence $\langle x_n \rangle_{n\in A}$ with $A \in \mathcal{I}^+$;
(4) the *hFinBW property* (*hereditary finite Bolzano-Weierstrass property*) if $(X, \mathcal{I} \restriction A)$ has the FinBW property for every $A \in \mathcal{I}^+$.

We will write $(X, \mathcal{I}) \in$ BW, if the pair $(X, \mathcal{I})$ has the BW property. An ideal $\mathcal{I}$ is called a *BW ideal* (equivalently, $\mathcal{I} \in$ BW, or $\mathcal{I}$ has the *BW property*) if the pair $([0, 1], \mathcal{I}) \in$ BW (and similarly for other properties). It is easy to show the following fact.

Fact 2.5. (1) *$\mathcal{I} \in BW$ ($\mathcal{I} \in FinBW$, respectively) if and only if every bounded sequence $\langle x_n \rangle_{n\in\omega}$ of reals has an $\mathcal{I}$-convergent (convergent, respectively) subsequence $\langle x_n \rangle_{n\in A}$ for some $A \in \mathcal{I}^+$.*
(2) *$\mathcal{I} \in hBW$ ($\mathcal{I} \in hFinBW$, respectively) if and only if for every $A \in \mathcal{I}^+$ and for every bounded sequence $\langle x_n \rangle_{n\in A}$ of reals there is $B \subseteq A$, $B \in \mathcal{I}^+$ such that the subsequence $\langle x_n \rangle_{n\in B}$ is $\mathcal{I}$-convergent (convergent, respectively).*

By the well-known Bolzano-Weierstrass theorem, the ideal Fin has the FinBW property. For the discussion and applications of these properties see [8], where we examine all BW-like properties. In particular, it is known that the ideal $\mathcal{I}_d$ of sets of density 0 does not have the BW property, every F_σ ideal has the hFinBW property and every maximal ideal has the hBW property.

It is not difficult to prove (see e.g. [17] or [8, Section 2.1]) that if a P-ideal has the BW property then it also has the FinBW property (and similarly for hBW and hFinBW properties). We will use this fact in the sequel to obtain convergence instead of $\mathcal{I}$-convergence for P-ideals.

2.2.1. *Equivalent conditions for BW-like properties.*

Proposition 2.6. *If Y is a continuous image of X or Y is a closed subset of X then $(X,\mathcal{I}) \in BW \implies (Y,\mathcal{I}) \in BW$.*

Proof. Straightforward. □

Proposition 2.7. *The following conditions are equivalent.*

(1) *$\mathcal{I} \in BW$.*
(2) *$(2^\omega,\mathcal{I}) \in BW$.*
(3) *$(X,\mathcal{I}) \in BW$ for every uncountable compact metric space X.*
(4) *$(X,\mathcal{I}) \in BW$ for some uncountable compact metric space X.*

Proof. (1) ⇒ (2). Note that 2^ω is homeomorphic to a closed subset of $[0,1]$ and apply Proposition 2.6.

(2) ⇒ (3). Note that X is a continuous image of 2^ω (see e.g. [14, Thm. 4.18]) and apply Proposition 2.6.

(3) ⇒ (4). It is obvious.

(4) ⇒ (2). Note that X contains a closed subset $C \subseteq X$ that is homeomorphic to 2^ω (see e.g. [14, Thm. 6.2]) and apply Proposition 2.6.

(2) ⇒ (1). Note that $[0,1]$ is a continuous image of 2^ω (see e.g. [14, Thm. 4.18]) and apply Proposition 2.6. □

Corollary 2.8. *If $\mathcal{I} \in BW$ then $(X,\mathcal{I}) \in BW$ for every compact metric space X.*

Proof. By [14, Thm. 4.18] X is a continuous image of 2^ω. Thus, Propositions 2.7 and 2.6 show that $(X,\mathcal{I}) \in$ BW. □

Remark. It is not difficult to see that in Propositions 2.6, 2.7 and Corollary 2.8 the BW property can be replaced by the FinBW, hBW and hFinBW properties.

Applying Proposition 2.7 to the Hilbert cube $[0,1]^\omega$, we get the following fact.

Proposition 2.9. *Let $\mathcal{I}$ be a BW ideal on ω. If $\langle x_{n,m}\rangle_{n,m\in\omega}$ is a double sequence with values in $[0,1]$ then there exists $A \in \mathcal{I}^+$ such that $\langle x_{n,m}\rangle_{n\in A}$ is $\mathcal{I}$-convergent for every $m \in \omega$.*

In the sequel we will use the following characterization of the BW property which is a slight generalization of Proposition 3.3 from [8].

Proposition 2.10. *An ideal $\mathcal{I}$ has the BW property if and only if for every finite branching tree $T \subseteq \omega^{<\omega}$, and sets $\{A_s : s \in T\}$ fulfilling the following conditions*

(T1) *$A_\emptyset = \omega$,*
(T2) *$A_s = \bigcup\{A_{s^\frown r} \ : \ s^\frown r \in T\}$,*
(T3) *$A_{s^\frown r_1} \cap A_{s^\frown r_2} = \emptyset$, for every distinct r_1, r_2 such that $s^\frown r_1, s^\frown r_2 \in T$,*

there exist $x \in \omega^\omega$ and $B \subseteq \omega$, $B \in \mathcal{I}^+$ such that $B \setminus A_{x\restriction n} \in \mathcal{I}$ for all n.

2.3. **Ideal convergence in functional spaces.** Let X be a topological Hausdorff space and $\mathcal{I}$ be an ideal on ω.

We say that a sequence $\langle f_n : X \to \mathbb{R}\rangle_{n\in\omega}$ is *pointwise $\mathcal{I}$-convergent* if $\langle f_n(x)\rangle_{n\in\omega}$ is $\mathcal{I}$-convergent for every $x \in X$. Equivalently, if $\langle f_n\rangle_{n\in\omega}$ is $\mathcal{I}$-convergent in the topological space $\mathbb{R}^X$ with the product topology.

A sequence $\langle f_n : X \to \mathbb{R}\rangle_{n\in\omega}$ is *uniformly $\mathcal{I}$-convergent* to $f : X \to \mathbb{R}$ if $\{n \in \omega : |f_n(x) - f(x)| > \varepsilon \text{ for some } x\} \in \mathcal{I}$ for every $\varepsilon > 0$. For a compact space X and continuous functions f_n, it is equivalent to say that $\langle f_n\rangle_{n\in\omega}$ is $\mathcal{I}$-convergent in the metric space $\mathcal{C}(X)$ of all continuous real-valued functions with the *sup*-metric. The classical argument shows that the limit of a uniformly $\mathcal{I}$-convergent sequence of continuous functions is continuous.

3. Arzela-Ascoli Theorem

Theorem 3.1 (Ideal version of Arzela-Ascoli Theorem)**.** *Let $\mathcal{I}$ be an ideal on ω. The following conditions are equivalent.*

1. *$\mathcal{I}$ is a BW ideal (FinBW ideal, respectively).*
2. *For every uniformly bounded and equicontinuous sequence $\langle f_n\rangle_{n\in\omega}$ of continuous real-valued functions defined on $[0,1]$ there exists $A \in \mathcal{I}^+$ such that $\langle f_n\rangle_{n\in A}$ is uniformly $\mathcal{I}$-convergent (uniformly convergent, respectively).*

Proof. (1) $\Rightarrow$ (2). Let X be the space of all continuous real-valued functions defined on $[0,1]$ with *sup*-metric. Let $Y = \mathrm{cl}_X(\{f_n : n \in \omega\})$. It is easy to show that Y consists of uniformly bounded and equicontinuous functions. Thus, by ordinary Arzela-Ascoli theorem (Theorem 1.1), Y is a compact metric space. Then by Corollary 2.8 $(Y,\mathcal{I}) \in \mathrm{BW}$ ($(Y,\mathcal{I}) \in \mathrm{FinBW}$, respectively), so there exists $A \in \mathcal{I}^+$ such that $\langle f_n\rangle_{n\in A}$ is $\mathcal{I}$-convergent (convergent, respectively) in Y. Thus the subsequence $\langle f_n\rangle_{n\in A}$ is uniformly $\mathcal{I}$-convergent (uniformly convergent, respectively).

(2) $\Rightarrow$ (1). Let $\langle x_n\rangle_{n\in\omega}$ be a bounded sequence of reals. For each $n \in \omega$ let $f_n : [0,1] \to \mathbb{R}$ be a constant function given by $f_n(x) = x_n$. By (2) there exists $A \in \mathcal{I}^+$ such that $\langle f_n\rangle_{n\in A}$ is uniformly $\mathcal{I}$-convergent (uniformly convergent, respectively). Then the subsequence $\langle x_n\rangle_{n\in A}$ is $\mathcal{I}$-convergent (convergent, respectively). □

Remark. It is not difficult to show that the above theorem is also true if we consider functions f_n defined on a compact metric space K with values in a metric space X. More precisely in a such generalization the condition of uniform boundedness should be replaced by the condition that $\bigcup_{n\in\omega} f_n(K)$ is a relatively compact subspace in X.

4. Mazurkiewicz's Theorem

Theorem 4.1 (Ideal version of Mazurkiewicz's Theorem)**.** *Let $\mathcal{I}$ be an ideal on ω that can be extended to an F_σ ideal. If $\langle f_n\rangle_{n\in\omega}$ is a sequence of uniformly bounded continuous real-valued functions defined on $\mathbb{R}$ then there exists $A \in \mathcal{I}^+$ and a perfect set $P \subseteq \mathbb{R}$ such that the subsequence $\langle f_n \upharpoonright P\rangle_{n\in A}$ is uniformly convergent.*

Proof. It is enough to prove the theorem for any F_σ ideal $\mathcal{I}$. So, by Theorem 2.1, we can assume that $\mathcal{I} = \mathrm{Fin}(\phi)$ for an lsc submeasure ϕ on ω.

Let $r \in \mathbb{R}$ be such that $|f_n(x)| \le r$ for every $x \in \mathbb{R}$ and $n \in \omega$. Let $\{I_s : s \in 2^{<\omega}\}$ be a family of open subsets of $[-r,r]$ such that

1. $I_\emptyset = [-r,r]$,
2. $\{I_s : s \in 2^n\}$ is an open cover of $[-r,r]$ for every $n \in \omega$,

(3) For every $\varepsilon > 0$ there is $n \in \omega$ such that $\operatorname{diam}(I_s) < \varepsilon$ for every $s \in 2^n$,
(4) $I_s = I_{s^\frown 0} \cup I_{s^\frown 1}$ for every $s \in 2^{<\omega}$.

For each $m, n \in \omega$ and $x \in \mathbb{R}$ let $p(m,n,x)$ be the least $s \in 2^n$ such that $f_m(x) \in I_s$.

Let $\mathbb{Q} = \{q_n : n \in \omega\}$. For $t_0, \dots, t_{n-1} \in 2^n$ we define

$$B(t_0, \dots, t_{n-1}) = \{m \in \omega \ : \ p(m,n,q_i) = t_i \text{ for every } i \in n\},$$

i.e. for each $m \in B(t_0, \dots, t_{n-1})$ and $i \in n$, $f_m(q_i) \in I_{t_i}$. Let

$$\mathcal{B}^n = \{B(t_0, \dots, t_{n-1}) \ : \ t_0, \dots, t_{n-1} \in 2^n\}.$$

Clearly, $\mathcal{B}^n$ is a partition of ω. Moreover, it is not difficult to check that $\mathcal{B}^{n+1}$ is a refinement of $\mathcal{B}^n$. Thus, $\bigcup_{n\in\omega} \mathcal{B}^n$ forms a finite-branching tree with respect to the inclusion.

By Proposition 2.10 there exists a sequence of sets $\langle Z_n \rangle_{n\in\omega}$, and $Z \in \mathcal{I}^+$ such that $Z_n \in \mathcal{B}^n$, $Z_n \supseteq Z_{n+1}$ and $Z \setminus Z_n \in \mathcal{I}$ for every $n \in \omega$.

For each $n \in \omega$ let $z(n,0), \dots, z(n,n-1) \in 2^n$ be such that

$$Z_n = B(z(n,0), \dots, z(n,n-1)).$$

It follows that

$$I_{z(n+1,i)} \subseteq I_{z(n,i)}$$

for every $i \in n$ and $n \in \omega$.

For each $n \in \omega$ let $A_n \subseteq (Z \cap Z_n) \setminus n$ be a finite set such that $\phi(A_n) > n$ (sets A_n exist by semicontinuity of ϕ and the fact that $\phi(Z \cap Z_n) = \infty$).

Now we define a family of natural numbers $\{n_s \in \omega : s \in 2^{<\omega}\}$, open intervals $\{U_s : s \in 2^{<\omega}\}$ and a sequence $\langle k_n \in \omega : n \in \omega\rangle$ such that

(1) $U_{s^\frown 0}, U_{s^\frown 1} \subseteq U_s$ and $\operatorname{cl}(U_{s^\frown 0}) \cap \operatorname{cl}(U_{s^\frown 1}) = \emptyset$ for every $s \in 2^{<\omega}$,
(2) $\lim_{n\to\infty} \operatorname{diam}(U_{\alpha\restriction n}) = 0$ for every $\alpha \in 2^\omega$,
(3) $k_n = \max\{n_s : s \in 2^n\}$,
(4) $f_i[\operatorname{cl}(U_s)] \subseteq I_{z(k_n, n_s)}$ for every $s \in 2^n$ and $i \in A_{k_n}$.

Let $n_\emptyset = 0$, $U_\emptyset = \mathbb{R}$ and $k_0 = 0$. Suppose that we have already constructed n_s, U_s and k_i for each $s \in 2^{\leq n}$ and $i \in n$.

For each $s \in 2^n$ let $n_{s^\frown 0}, n_{s^\frown 1} \in \omega$ be such that $q_{n_{s^\frown 0}} \in U_s$ and $q_{n_{s^\frown 1}} \in U_s \setminus \{q_{n_{s^\frown 0}}\}$. Then we put $k_{n+1} = \max\{n_s : s \in 2^{n+1}\}$. By continuity of functions $\langle f_i : i \in \omega\rangle$ the set

$$W_{s^\frown j} = \bigcap_{i\in A_{k_{n+1}}} f_i^{-1}\left[I_{z(k_{n+1}, n_{s^\frown j})}\right] \cap U_s$$

is an open neighbourhood of $q_{n_{s^\frown j}}$ for every $s \in 2^n$ and $j \in \{0,1\}$. Now for every $s \in 2^n$ and $j \in \{0,1\}$ it is enough to take $U_{s^\frown j}$ such that $\operatorname{cl}(U_{s^\frown j}) \subseteq W_{s^\frown j}$, and $\operatorname{cl}(U_{s^\frown 0}) \cap \operatorname{cl}(U_{s^\frown 1}) = \emptyset$. This finishes the construction of n_s, U_s and k_n.

Let

$$P = \bigcap_{n\in\omega} \bigcup_{s\in 2^n} \operatorname{cl}(U_s) \text{ and } \{x_\alpha\} = \bigcap_{n\in\omega} \operatorname{cl}(U_{\alpha\restriction n}).$$

For $n \in \omega$ we define $\phi_{k_n} : P \to \mathbb{R}$ by

$$\phi_{k_n}(x) = \sum_{s\in 2^n} a_s \cdot \chi_{\operatorname{cl}(U_s)}(x),$$

where a_s is an arbitrarily taken point of $I_{z(k_n,n_s)}$. Since functions ϕ_{k_n} are continuous hence by ordinary Mazurkiewicz's theorem there exists a perfect set $Q \subseteq P$ and a subsequence $\langle k_{l_n}\rangle_{n\in\omega}$ such that the subsequence $\langle \phi_{k_{l_n}} \restriction Q : n \in \omega\rangle$ is uniformly convergent.

Let $X = \bigcup_{n\in\omega} A_{k_{l_n}} \in \mathcal{I}^+$ and $f : Q \to \mathbb{R}$ be given by

$$f(x) = \lim_{n\to\infty} \phi_{k_{l_n}}(x).$$

We claim that $\langle f_i \restriction Q\rangle_{i\in X}$ is uniformly convergent to f.

Fix $\varepsilon > 0$. There is $N \in \omega$ such that

1. $|f(x) - \phi_{k_{l_n}}(x)| < \varepsilon/2$ for every $n > N$ and $x \in Q$, and
2. $\operatorname{diam}(I_s) < \varepsilon/2$ for every $s \in 2^{k_{l_n}}$ and $n > N$.

We will show that

$$|f_i(x) - f(x)| < \varepsilon$$

for every $n > N$, $i \in A_{k_{l_n}}$ and $x \in Q$.

Let $n > N$, $i \in A_{k_{l_n}}$ and $x \in Q$. There is $\alpha \in 2^\omega$ with $x = x_\alpha$. Since $x_\alpha \in \mathrm{cl}(U_{\alpha\restriction l_n})$ and $f_i[\mathrm{cl}(U_{\alpha\restriction l_n})] \subseteq I_{z(k_{l_n}, n_{\alpha\restriction l_n})}$ so $f_i(x_\alpha) \in I_{z(k_{l_n}, n_{\alpha\restriction l_n})}$. On the other hand, $\phi_{k_{l_n}}(x_\alpha) \in I_{z(k_{l_n}, n_{\alpha\restriction l_n})}$, hence $|f_i(x_\alpha) - f(x_\alpha)| < \varepsilon$.

Finally,

$$\{i \in X \ : \ |f_i(x) - f(x)| \geq \varepsilon \text{ for some } x \in Q\} \subseteq A_{k_{l_0}} \cup \cdots \cup A_{k_{l_N}}$$

and the latter set is finite. □

Remark. It is not difficult to show that the above theorem also holds for real-valued functions defined on a Polish space.

The next corollary follows from the fact that every analytic P-ideal with the BW property can be extended to an F_σ ideal (see [8, Theorem 4.2]).

Corollary 4.2. *Let $\mathcal{I}$ be an analytic P-ideal on ω. The following conditions are equivalent.*

1. *$\mathcal{I}$ is a FinBW ideal.*
2. *For every uniformly bounded sequence $\langle f_n\rangle_{n\in\omega}$ of continuous real-valued functions defined on $\mathbb{R}$ there is $A \in \mathcal{I}^+$ and a perfect set $P \subseteq \mathbb{R}$ such that the subsequence $\langle f_n \restriction P\rangle_{n\in A}$ is uniformly convergent.*

Problem 4.3. Let $\mathcal{I}$ be an ideal on ω. Are the following conditions equivalent?

1. $\mathcal{I}$ is a BW ideal (FinBW ideal, respectively).
2. For every uniformly bounded sequence $\langle f_n\rangle_{n\in\omega}$ of continuous real-valued functions defined on $\mathbb{R}$ there is $A \in \mathcal{I}^+$ and a perfect set $P \subseteq \mathbb{R}$ such that the subsequence $\langle f_n \restriction P\rangle_{n\in A}$ is uniformly $\mathcal{I}$-convergent (uniformly convergent, respectively).

By the same argument as in the proof of the implication "(2) ⇒ (1)" of Theorem 3.1, the implication "(2) ⇒ (1)" in above problem is evident. So in fact we are asking only about the converse implication. The authors do not even know an answer for maximal ideals.

Problem 4.4. The proof of Theorem 4.1 uses only the following property of F_σ ideals:

(A) for every decreasing sequence of $Z_n \in \mathcal{I}^+$ $(n \in \omega)$ there are finite sets $A_n \subseteq Z_n$ $(n \in \omega)$ such that $\bigcup_{n\in G} A_n \in \mathcal{I}^+$ for each infinite $G \subseteq \omega$.

The authors do not know any ideal which satisfy condition (A) and does not extend to an F_σ ideal. Since it can be shown that any ideal with the property (A) is hFinBW, for Borel ideals this problem is connected with Problem 6.1.

5. Helly's Theorem

We start this section with reformulation of some known results in the terminology introduced in Section 2. We show a sufficient condition on a space X to have $(X,\mathcal{I}) \in \mathrm{BW}$ for some classes of ideals $\mathcal{I}$. Then these results are used to prove the ideal version of Helly's theorem. However, they also seem to be interesting in their own.

5.1. $\mathrm{P}(\mathcal{I})$**-coideals.** Note that $\mathcal{B}$ is a coideal if and only if it fulfills the following conditions:

(1) $\omega \in \mathcal{B}$;
(2) if $A \in \mathcal{B}$ and $B \supseteq A$ then $B \in \mathcal{B}$;
(3) if $A \in \mathcal{B}$ and $A = A_1 \cup A_2$ then $A_1 \in \mathcal{B}$ or $A_2 \in \mathcal{B}$.

We say that an coideal $\mathcal{B} = \mathcal{J}^+$ is $\mathrm{P}(\mathcal{I})$-*coideal* if for any nonincreasing sequence of sets

$$A_0 \supseteq A_1 \supseteq \cdots$$

from $\mathcal{B}$ there is an $A \in \mathcal{B}$ such that $A \setminus A_i \in \mathcal{I}$ for each $i \in \omega$. If $\mathcal{I} = \mathrm{Fin}$ we say shortly that $\mathcal{B}$ is a P-coideal (in such case the notation "$\mathcal{I}$ satisfies the condition (P)" is used in [24, Section 9].) Note also that there is no direct dependency between the notion of P-ideal and P-coideal.

An example of P-coideal is the property of "being infinite". For every maximal ideal $\mathcal{I}$, the dual ultrafilter $\mathcal{I}^*$ is a $\mathrm{P}(\mathcal{I})$-coideal. If $\mathcal{I}$ is a maximal P-ideal (the existence of such ideals is independent of ZFC axioms) then $\mathcal{I}^*$ is also a P-coideal.

Proposition 5.1 (Folklore)**.** *If $\mathcal{I}$ is an F_σ ideal then $\mathcal{I}^+$ is a* P-*coideal.*

Proof. Let $\mathcal{I} = \mathrm{Fin}(\phi)$ be F_σ, where ϕ is lsc. Let $A_0 \supseteq A_1 \cdots$ be a sequence of sets from $\mathcal{I}^+$, i.e. $\phi(A_i) = \infty$ for each i. For every i fix a finite $B_i \subseteq A_i$ with $\phi(B_i) > i$. Then $A = \bigcup_{i\in\omega} B_i$ does not belong to $\mathcal{I}$ (since $\phi(A) = \infty$), and $A_i \setminus A$ is finite for each i. □

In the sequel we will be interested in the existence of $\mathrm{P}(\mathcal{I})$-coideals disjoint with $\mathcal{I}$ (P-coideals disjoint with $\mathcal{I}$).

Suppose that $\mathcal{I}$ can be extended to an F_σ ideal $\mathcal{J}$. Since $\mathcal{J}^+$ is a P-coideal disjoint with $\mathcal{J}$, $\mathcal{J}^+$ is also P-coideal disjoint with $\mathcal{I}$. Thus we have the following.

Corollary 5.2. *If $\mathcal{I}$ is an ideal which can be extended to an F_σ ideal then there exists* P-*coideal $\mathcal{B}$ disjoint with $\mathcal{I}$.*

The next corollary follows from the fact that every analytic P-ideal $\mathcal{I}$ with the BW property can be extended to an F_σ ideal (see [8, Th. 4.2]).

Corollary 5.3. *For every analytic P-ideal $\mathcal{I}$ with the BW property there exists* P-*coideal $\mathcal{B}$ disjoint with $\mathcal{I}$.*

Remark. There are known more examples of $\mathrm{P}(\mathcal{I})$-coideals disjoint with $\mathcal{I}$. For example, every ideal $\mathcal{I}$ with countably saturated quotients is a $\mathrm{P}(\mathcal{I})$-coideal (for definitions and theorems see e.g. [7]).

5.2. **A sufficient condition.** In [16] the author considers van der Waerden spaces. Using our terminology, we say that a topological space X is a *van der Waerden space* if $(X, \mathcal{W}) \in \mathrm{FinBW}$, where $\mathcal{W}$ is the ideal from Example 2.4. In that paper the author proved [16, Theorem 10] that if a Hausdorff topological space X satisfies the following condition:

$$\text{the closure of every countable set in } X \text{ is compact and first countable} \qquad (\star)$$

then $(X, \mathcal{W}) \in \mathrm{FinBW}$.

Below we show that the condition $(\star)$ is also sufficient for more ideals to have $(X, \mathcal{I}) \in \mathrm{FinBW}$.

It is well known that a space X is compact if and only if every ultrafilter on X is convergent (i.e. there is $x \in X$ such that every neighborhood of x belongs to the ultrafilter). Since every coideal contains an ultrafilter we get the following reformulation of this fact.

Proposition 5.4. *A space X is compact if and only if for every coideal on X there is $x \in X$ such that every neighborhood of x belongs to the coideal.*

Now we are ready to prove a generalization of Kojman's theorem. Presented proof is essentially the same as that in [16] (but see also [2, Theorem 3.4].) We give it here for completeness.

Proposition 5.5. *Let $\mathcal{I}$ be an ideal on ω and $\mathcal{B}$ be the* $\mathrm{P}(\mathcal{I})$*-coideal (*P*-coideal, respectively) disjoint with $\mathcal{I}$. If a Hausdorff topological space X satisfies the condition $(\star)$, then for every $\langle x_n \rangle_{n\in\omega} \subseteq X$ there exists $A \in \mathcal{B}$ such that $\langle x_n \rangle_{n\in A}$ is $\mathcal{I}$-convergent (convergent, respectively).*

Proof. Let $D = \mathrm{cl}_X \{x_n \ : \ n \in \omega\}$. Define a coideal $\mathcal{B}'$ over D by

$$B \in \mathcal{B}' \iff \{n : x_n \in B\} \in \mathcal{B}.$$

By compactness of D and Proposition 5.4, there is a point $x \in D$ with the property that every neighborhood (in D) of x belongs to $\mathcal{B}'$. Fix, by first-countability, a decreasing neighborhood base $\langle u_k \rangle_{k\in\omega}$ at x. Since for each n

$$A_k = \{n \ : \ x_n \in u_k\} \in \mathcal{B},$$

and $\mathcal{B}$ is an $\mathrm{P}(\mathcal{I})$-coideal (P-coideal, respectively), there exists an $A \in \mathcal{B}$ such that $A \setminus A_n \in \mathcal{I}$ ($A \setminus A_n$ is finite, respectively) for each n. Clearly $\langle x_n \rangle_{n\in A}$ is $\mathcal{I}$-convergent (convergent, respectively) to x. □

Corollary 5.6. *Let $\mathcal{I}$ be an ideal on ω such that there exists a* $\mathrm{P}(\mathcal{I})$*-coideal (*P*-coideal, respectively) disjoint with $\mathcal{I}$. If a Hausdorff topological space X satisfies the condition $(\star)$, then $(X, \mathcal{I}) \in BW$ ($(X, \mathcal{I}) \in FinBW$, respectively).*

Corollary 5.7. *Suppose that a Hausdorff topological space X satisfies the condition $(\star)$. If*

(1) *$\mathcal{I}$ can be extended to an F_σ ideal (e.g. $\mathcal{I}$ is an analytic P-ideal with the BW property), or*
(2) *$\mathcal{I}$ can be extended to a maximal P-ideal, or*
(3) *$\mathcal{I}$ is a maximal ideal,*

then $(X, \mathcal{I}) \in BW$ (In the case (1) and (2), $(X, \mathcal{I}) \in FinBW$).

Remark. The case 3 of the above corollary is a stronger version of theorem of Bernstein [2, Theorem 3.4]. The case 1 for a particular F_σ ideal $\mathcal{W}$ is a theorem of Kojman [16, Theorem 10]. Recently, Flašková [9] proved independently the case 1 (for all F_σ ideals).

The class of spaces which satisfy ($\star$) includes all compact metric spaces, all compact linearly ordered topological spaces and every limit ordinal of uncountable cofinality. For more examples of such spaces see [16]. In the next section we will be interested in one particular example, the Helly's space of all monotone functions from $[0,1]$ to $[0,1]$ with the topology induced from the product topology on $[0,1]^{[0,1]}$, which is separable, compact and first-countable.

5.3. **Helly's Theorem.**

Theorem 5.8 (Ideal version of Helly's Theorem)**.** *Let $\mathcal{I}$ be an ideal on ω. Suppose that*

(a) *$\mathcal{I}$ can be extended to an F_σ ideal (e.g. $\mathcal{I}$ is an analytic P-ideal with the BW property), or*
(b) *$\mathcal{I}$ can be extended to a maximal P-ideal.*

Then for every sequence $\langle f_n\rangle_{n\in\omega}$ of uniformly bounded monotone real-valued functions defined on $\mathbb{R}$ there is $A \in \mathcal{I}^+$ such that the subsequence $\langle f_n\rangle_{n\in A}$ is pointwise convergent.

Proof. Let $\langle f_n\rangle_{n\in\omega}$ be a uniformly bounded sequence of monotone real-valued functions defined $\mathbb{R}$. Then there is $r \in \mathbb{R}$ with $|f_n(x)| \le r$ for every $x \in \mathbb{R}, n \in \omega$. The space of all monotone functions defined on $\mathbb{R}$ with values in $[-r, r]$ with the product topology is compact and first countable, so it satisfies the condition ($\star$) from Section 5.2. From Corollary 5.7 it follows that there is $A \in \mathcal{I}^+$ such that the subsequence $\langle f_n\rangle_{n\in A}$ is pointwise convergent. $\square$

Remark. In case of van der Waerden ideal $\mathcal{W}$ we get the result of Kojman [16, Corollary 13].

For hBW ideals we get a stronger variant of Theorem 5.8.

Theorem 5.9. *Let $\mathcal{I}$ be an ideal on ω. The following conditions are equivalent.*

(1) *$\mathcal{I}$ is an hBW ideal (hFinBW ideal, respectively).*
(2) *For every sequence $\langle f_n\rangle_{n\in\omega}$ of uniformly bounded monotone real-valued functions defined on $\mathbb{R}$ and $B \in \mathcal{I}^+$ there is $A \in \mathcal{I}^+$, $A \subseteq B$ such that the subsequence $\langle f_n\rangle_{n\in A}$ is pointwise $\mathcal{I}$-convergent (pointwise convergent, respectively).*

Proof. Below we show the proof for the hBW property (the case of the hFinBW property can be done in the same manner.)

(1) $\Rightarrow$ (2). By the hBW property of $\mathcal{I}$ we can assume that either all f_n's are non-decreasing, or all f_n's are non-increasing. In the sequel we will consider only the first case.

Fix any countable dense set $D \subseteq \mathbb{R}$. By Proposition 2.9 there exists $A \in \mathcal{I}^+$ such that for each $d \in D$ the sequence $\langle f_n(d)\rangle_{n\in A}$ is $\mathcal{I}$-convergent (say, $f(d)$ is the $\mathcal{I}$-limit of $\langle f_n(d)\rangle_{n\in A}$). So f is a function defined on D. It is easy to see that f is non-decreasing. Hence it is possible to extend the function f to some non-decreasing function $\hat{f}$ defined on $\mathbb{R}$.

We show that if $x \in \mathbb{R}$ is a point of continuity of $\hat{f}$ then $\langle f_n(x)\rangle_{n\in A}$ is $\mathcal{I}$-convergent to $\hat{f}(x)$. Fix $\varepsilon > 0$ and $d_0, d_1 \in D$ such that $d_1 < x < d_2$ and $f(d_0), f(d_1)$ are in the ε-neighborhood of $f(x)$. Since sequences $\langle f_n(d_0)\rangle_{n\in A}, \langle f_n(d_1)\rangle_{n\in A}$ are $\mathcal{I}$-convergent

$$\left\{n \in A \;:\; f_n\left(d_0\right) < \hat{f}\left(x\right) - \varepsilon \text{ or } f_n\left(d_1\right) > \hat{f}\left(x\right) + \varepsilon\right\} \in \mathcal{I}.$$

Since f_n is non-decreasing

$$\left\{n \in A \;:\; \left|f_n\left(x\right) - \hat{f}\left(x\right)\right| \geq \varepsilon\right\} \in \mathcal{I}.$$

Let E be the sets of all discontinuity points of $\hat{f}$. Since $\hat{f}$ is a monotone function, the set E is at most countable. By the hBW property and Proposition 2.9 we can find a set $B \subseteq A$, $B \in \mathcal{I}^+$ such that for each $e \in E$ the sequence $\langle f_n(e)\rangle_{n\in B}$ is $\mathcal{I}$-convergent. Now we change values of the function $\hat{f}$ at points from E into the suitable values of $\mathcal{I}$-limits. Since $B \subseteq A$, the sequence $\langle f_n\rangle_{n\in B}$ is $\mathcal{I}$-convergent at each point of $\mathbb{R}$.

(2) $\Rightarrow$ (1). The proof of this implication can be done in a similar manner as the proof of Theorem 3.1. □

Problem 5.10. Let $\mathcal{I}$ be an ideal on ω. Are the following conditions equivalent?

(1) $\mathcal{I}$ is an BW ideal (FinBW ideal, respectively).
(2) For every sequence $\langle f_n\rangle_{n\in\omega}$ of uniformly bounded monotone real-valued functions defined on $\mathbb{R}$ there is $A \in \mathcal{I}^+$ such that the subsequence $\langle f_n\rangle_{n\in A}$ is pointwise $\mathcal{I}$-convergent (pointwise convergent, respectively).

By the same argument as in the proof of the implication "(2) $\Rightarrow$ (1)" of Theorem 3.1, the implication "(2) $\Rightarrow$ (1)" in above problem is evident. So in fact we are asking only about the converse implication.

6. Final remark

We have formulated the assumptions of Theorems 4.1 and 5.8 in terms of "extendability to an F_σ ideal". There is a connection of these results with the following question of Hrušák (in fact Hrušák formulated his question in terms of Katětov preorder, but by [21, Section 2.7] it can be reformulated in terms of the FinBW property).

Problem 6.1 ([11, Q. 5.16])**.** Let $\mathcal{I}$ be a Borel ideal. Are the following conditions equivalent?

(1) $\mathcal{I}$ is a FinBW ideal.
(2) $\mathcal{I}$ can be extended to an F_σ ideal.

Note that for P-ideals the answer to the above problem is positive ([8, Theorem 4.2]).

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