ON IDEAL EQUAL CONVERGENCE

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Abstract. We consider ideal equal convergence of a sequence of functions. This is a generalization of equal convergence introduced by Császár and Laczkovich [1]. Our definition of ideal equal convergence encompasses two different kinds of ideal equal convergence introduced in [3] and [4]. We also solve a few problems posed in [3].

1. Introduction

Let f_n $(n \in \mathbb{N})$ and f be real-valued functions defined on a set X. We say that the sequence (f_n) is *equally convergent* to f if there exists a sequence of positive reals $(\varepsilon_n) \to 0$ such that for every $x \in X$ there is N with $|f_n(x) - f(x)| < \varepsilon_n$ for every $n > N$. The notion of equal convergence was introduced by Császár and Laczkovich in [1]. It is known that equal convergence is weaker than uniform convergence and stronger than pointwise convergence, i.e. if (f_n) is uniformly convergent to f then (f_n) is equally convergent to f; and if (f_n) is equally convergent to f then (f_n) is pointwise convergent to f.

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ (i.e. $\mathcal{I}$ is a family of subsets of $\mathbb{N}$ closed under taking finite unions and subsets of its elements). We say that a sequence of reals (x_n) is *$\mathcal{I}$-convergent* to $x \in \mathbb{R}$ if $\{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$ (see e.g. [6]). We write $(x_n) \xrightarrow{\mathcal{I}} x$ in this case.

The notion of equal convergence was generalized in [3] and [4] with the aid of ideals on $\mathbb{N}$. However the authors of both papers generalized it in different ways. Let $\mathcal{I}$ be an ideal of subsets of $\mathbb{N}$.

- In [3] the authors says that (f_n) is *$\mathcal{I}$-equally convergent* to f if there exists a sequence of positive reals $(\varepsilon_n) \xrightarrow{\mathcal{I}} 0$ such that $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for every $x \in X$
- In [4] the authors says that (f_n) is *$\mathcal{I}$-equally convergent* to f if there exists a sequence of positive reals $(\varepsilon_n) \to 0$ such that $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for every $x \in X$.

The only difference between these two definitions of ideal equal convergence is the requirement that the sequence (ε_n) is either convergent (in the classical meaning) to zero or it is convergent to zero with respect to the ideal $\mathcal{I}$. It is easy to see that if (f_n) is $\mathcal{I}$-equally convergent to f in the sense of [4], then it is also $\mathcal{I}$-equally convergent to f in a sense of [3].

Date: February 24, 2014.

2010 *Mathematics Subject Classification.* Primary: 40A35. Secondary: 40A30, 26A03, 54A20.

Key words and phrases. ideal, P-ideal, filter, ideal convergence, statistical convergence, filter convergence, $\mathcal{I}$-convergence, convergence of a sequence of functions, equal convergence, uniform convergence, pointwise convergence, sigma-uniform convergence.

In this paper we introduce one more kind of ideal equal convergence (Section 3) which encompasses ideal equal convergence in senses of [3] and [4].

In Section 3 we examine relationships between our ideal equal convergence and ideal equal convergence introduced in [3] and [4].

In Section 4 we examine relationships between ideal uniform, ideal equal and ideal pointwise convergence. Among other we show (Example 4.7) that ideal pointwise convergence does not imply ideal equal convergence (this solves a problem posed in [3]).

In Section 5 we examine relationships between ideal equal convergence and ideal σ-uniform convergence. For instance we prove (Corollary 5.4) that ideal equal convergence implies ideal σ-uniform convergence if and only if the ideal is countably generated (this solves a problem posed in [3]).

In Section 6 we consider convergence of big (in a sense of ideals) subsequences (so-called filter convergence). Among other we prove (Corollary 6.5) that filter equal convergence is equivalent to filter σ-uniform convergence if and only if the ideal is a P-ideal (this solves a problem posed in [3]).

In Section 7 we examine relationships between ideal convergence and filter convergence. For instance we prove (Corollary 7.14) that ideal equal convergence does not imply filter equal convergence (this solves one more problem posed in [3]).

2. Preliminaries

2.1. **Ideals.** An *ideal* on $\mathbb{N}$ is a family of subsets of $\mathbb{N}$ closed under taking finite unions and subsets of its elements. In the sequel we assume that ideals contain all finite sets. The ideal of all finite subsets of $\mathbb{N}$ is denoted by Fin. We can define the notion of ideal on any set in a similar way.

For an ideal $\mathcal{I}$, we write $\mathcal{I}^* = \{A : \mathbb{N} \setminus A \in \mathcal{I}\}$ and called it the *filter dual to $\mathcal{I}$*.

Let $\mathcal{G}$ be a family of subsets of $\mathbb{N}$. The smallest ideal containing $\mathcal{G}$ is called the *ideal generated* by $\mathcal{G}$. An ideal is *countably generated* if it is generated by a countable family $\mathcal{G}$. Note that an ideal $\mathcal{I}$ is countably generated if and only if there are $A_1, A_2, \ldots \in \mathcal{I}$ such that for every $A \in \mathcal{I}$ there is $n \in \mathbb{N}$ with $A \subseteq A_n$.

Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. An ideal $\mathcal{I}$ is a *$P(\mathcal{J})$-ideal* if for any sets $A_1, A_2, \ldots \in \mathcal{I}$ there is a set $A \in \mathcal{I}$ such that $A_n \setminus A \in \mathcal{J}$ for every $n \in \mathbb{N}$ ([7, Definition 3.10] where the authors call it $AP(\mathcal{I}, \mathcal{J})$). Note that $P(\text{Fin})$-ideals are also called *P-ideals*. It is easy to see that every ideal $\mathcal{I}$ is always $P(\mathcal{I})$-ideal.

2.2. **Ideal convergence.** Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. We say that a sequence of reals (x_n) is

- *$\mathcal{I}$-convergent* to $x \in \mathbb{R}$ if $\{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$ (see e.g. [6]) We write $(x_n) \xrightarrow{\mathcal{I}} x$ in this case.
- *$(\mathcal{I}^*, \mathcal{J})$-convergent* to x if there is a set $F \in \mathcal{I}^*$ such that the subsequence $(x_n)_{n\in F}$ is $\mathcal{J}$-convergent to x ([7, Definition 3.2], where the authors call it $\mathcal{I}^{\mathcal{J}}$-convergence). We write $(x_n) \xrightarrow{(\mathcal{I}^*, \mathcal{J})} x$ in this case.

Note that $(\mathcal{I}^*, \text{Fin})$-convergence is also called *$\mathcal{I}^*$-convergence* (see e.g. [6]).

Theorem 2.1 ([7, Theorems 3.11 and 3.12])**.** *Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence of reals (x_n), if $(x_n) \xrightarrow{\mathcal{I}} x$, then $(x_n) \xrightarrow{(\mathcal{I}^*, \mathcal{J})} x$.*
(2) *$\mathcal{I}$ is a $P(\mathcal{J})$-ideal.*

3. Ideal equal convergence

Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. Let f_n ($n \in \mathbb{N}$) and f be real-valued functions defined on a set X. We say that the sequence (f_n) is $(\mathcal{I},\mathcal{J})$-*equally convergent* to f if there exists a sequence of positive reals $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$ such that $\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for every $x \in X$. We write $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ in this case.

For $\mathcal{I} = \mathcal{J} = \mathrm{Fin}$ we obtain the equal convergence introduced by Császár and Laczkovich [1] and write $(f_n) \xrightarrow{e} f$ instead of $(f_n) \xrightarrow{(\mathrm{Fin},\mathrm{Fin})-e} f$.

In [3] and [4] the authors also introduced a kind of ideal equal convergence (see Section 1). It is easy to see that ideal equal convergence introduced in [3] is equivalent to $(\mathcal{I},\mathcal{I})$-equal convergence, and ideal equal convergence introduced in [4] is equivalent to $(\mathcal{I},\mathrm{Fin})$-equal convergence.

The following fact can be easily checked.

Fact 3.1. *Let (f_n) be a sequence of real-valued functions defined on a set X. Let $\mathcal{I}_0, \mathcal{I}_1, \mathcal{J}_0, \mathcal{J}_1$ be ideals on $\mathbb{N}$.*

1. *If $\mathcal{I}_0 \subseteq \mathcal{I}_1$, then $(f_n) \xrightarrow{(\mathcal{I}_0,\mathcal{J})-e} f$ implies $(f_n) \xrightarrow{(\mathcal{I}_1,\mathcal{J})-e} f$.*
2. *If $\mathcal{J}_0 \subseteq \mathcal{J}_1$, then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J}_0)-e} f$ implies $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J}_1)-e} f$.*

Theorem 3.2. *Let X be a nonempty set. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

1. *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$.*
2. *$\mathcal{J} \subseteq \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2). Let $A \in \mathcal{J}$. Let

$$f_n(x) = \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{if } n \notin A \end{cases}$$

for $x \in X$ and $n \in \mathbb{N}$. Let $f(x) = 0$ for $x \in X$. Then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$. Indeed, let

$$\varepsilon_n = \begin{cases} 2 & \text{if } n \in A, \\ \frac{1}{n+1} & \text{if } n \notin A \end{cases}$$

for $n \in \mathbb{N}$. Then $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$ and $\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} = \emptyset \in \mathcal{I}$ for every $x \in X$.

By the assumption we obtain that $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$. Let $(\eta_n) \xrightarrow{\mathcal{I}} 0$ be such that $\{n : |f_n(x) - f(x)| \geq \eta_n\} \in \mathcal{I}$ for every $x \in X$. Then $B = \{n : \eta_n \geq 1\} \in \mathcal{I}$ and $A \setminus B \subseteq \{n : |f_n(x) - f(x)| \geq \eta_n\} \in \mathcal{I}$ so $A \in \mathcal{I}$.

(2) $\Rightarrow$ (1). Follows from Fact 3.1(2).

□

Since we assume that ideals contain all finite subsets of $\mathbb{N}$, so we obtain the following corollary.

Corollary 3.3. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Let (f_n) be a sequence of real-valued functions defined on a set X. If $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$ then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$.*

Theorem 3.4. *Let X be a nonempty set. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence* (f_n) *of real-valued functions defined on a set* X*, if* $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ *then* $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$.

(2) $\mathcal{I}$ *is a* $P(\mathcal{J})$*-ideal.*

Proof. (1) $\Rightarrow$ (2). Let $B_k \in \mathcal{I}$ ($k \in \mathbb{N}$). Let $A_1 = B_1$, $A_k = B_k \setminus (B_1 \cup ... \cup B_{k-1})$, $k \in \mathbb{N} \setminus \{1\}$. Obviously $A_k \in \mathcal{I}$ for every $k \in \mathbb{N}$. We define $f_n : X \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \frac{1}{k+1} & \text{if } n \in A_k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{if } n \notin A_k \text{ for every } k \in \mathbb{N}. \end{cases}$$

Let $f(x) = 0$ for every $x \in X$. It is easy to see that $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$. Indeed, let

$$\varepsilon_n = \begin{cases} \frac{1}{k} & \text{if } n \in A_k \text{ for some } k \in \mathbb{N}, \\ \frac{1}{n} & \text{if } n \notin A_k \text{ for every } k \in \mathbb{N}. \end{cases}$$

Then $(\varepsilon_n) \xrightarrow{\mathcal{I}} 0$ and $\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} = \emptyset \in \mathcal{I}$. Thus by the assumption we have $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$. Let $(\eta_n) \xrightarrow{\mathcal{J}} 0$ be such that $\{n : |f_n(x) - f(x)| \geq \eta_n\} \in \mathcal{I}$ for every $x \in X$. Fix $x \in X$ and set $A = \{n : |f_n(x) - f(x)| \geq \eta_n\} \in \mathcal{I}$. Let $k \in \mathbb{N}$. Then $A_k \setminus A = \{n \in A_k : |f_n(x) - f(x)| < \eta_n\} = \{n \in A_k : \frac{1}{k+1} < \eta_n\} \in \mathcal{J}$. Then $B_1 \setminus A = A_1 \setminus A \in \mathcal{J}$ and $B_k \setminus A \subseteq (A_1 \cup A_2 \cup ... \cup A_k) \setminus A \in \mathcal{J}$. Thus $\mathcal{I}$ is a $P(\mathcal{J})$-ideal.

(2) $\Rightarrow$ (1). Suppose that $\mathcal{I}$ is a $P(\mathcal{J})$-ideal and $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$. Let $(\varepsilon_n) \xrightarrow{\mathcal{I}} 0$ be such that $\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for every $x \in X$. Since $\mathcal{I}$ is a $P(\mathcal{J})$-ideal, so by Theorem 2.1 we obtain $(\varepsilon_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})} 0$. Take $F \in \mathcal{I}^*$ such that $(\varepsilon_n)_{n \in F} \xrightarrow{\mathcal{J}} 0$. Define a sequence (η_n) by

$$\eta_n = \begin{cases} \varepsilon_n & \text{if } n \in F, \\ \frac{1}{n} & \text{if } n \notin F. \end{cases}$$

It is easy to see that $(\eta_n) \xrightarrow{\mathcal{J}} 0$ and the set $\{n : |f_n(x) - f(x)| \geq \eta_n\} \in \mathcal{I}$ for every $x \in X$. Thus $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$.

□

Corollary 3.5. *Let* X *be a nonempty set. Let* $\mathcal{I}$ *be an ideal on* $\mathbb{N}$*. The following are equivalent.*

(1) *For every sequence* (f_n) *of real-valued functions defined on a set* X*, if* $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ *then* $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$.

(2) $\mathcal{I}$ *is a* P*-ideal.*

Thus $\mathcal{I}$-equal convergence in the sense of [3] is equivalent to $\mathcal{I}$-equal convergence in the sense of [4] if and only if $\mathcal{I}$ is a P-ideal.

4. Ideal convergence: uniform, equal and pointwise

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. A sequence $(f_n)_{n \in \mathbb{N}}$ of real-valued functions defined on X is

- $\mathcal{I}$*-uniformly convergent* to f if for every $\varepsilon > 0$ the set $\{n : |f_n(x) - f(x)| \geq \varepsilon \text{ for some } x\} \in \mathcal{I}$. We write $(f_n)_n \xrightarrow{\mathcal{I}-u} f$ for short.
- $\mathcal{I}$*-pointwise convergent* to f if for every $\varepsilon > 0$ and every $x \in X$ the set $\{n : |f_n(x) - f(x)| \geq \varepsilon\} \in \mathcal{I}$. We write $(f_n)_n \xrightarrow{\mathcal{I}} f$ for short.

Note that $(f_n)_n \xrightarrow{\mathcal{I}-u} f$ is equivalent to $(\sup_{x\in X}|f_n(x)-f(x)|)_n \xrightarrow{\mathcal{I}} 0$.

For $\mathcal{I} = \mathrm{Fin}$ we obtain the usual uniform and pointwise convergence and write $(f_n)_n \xrightarrow{u} f$ and $(f_n)_n \to f$ instead of $(f_n)_n \xrightarrow{\mathrm{Fin}-u} f$ and $(f_n)_n \xrightarrow{\mathrm{Fin}} f$, respectively.

It is not difficult to show that if $(f_n) \xrightarrow{u} f$ then $(f_n) \xrightarrow{e} f$; and if $(f_n) \xrightarrow{e} f$ then $(f_n) \to f$. Moreover it is known that these implications do not reverse.

Below we examine relationships between ideal equal convergence and ideal uniform and ideal pointwise convergence. For instance, we provide necessary and sufficient condition for $\mathcal{I}$ and $\mathcal{J}$ so that $(\mathcal{I},\mathcal{J})$-equal convergence is between $\mathcal{I}$-uniform and $\mathcal{I}$-pointwise convergence (Corollary 4.6).

In [3, Theorem 3.1] the authors proved that if $(f_n) \xrightarrow{\mathcal{I}-u} f$, then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$.

Proposition 4.1. *Let X be a nonempty set. Let $\mathcal{I},\mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

1. *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\mathcal{I}-u} f$, then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$.*
2. *$\mathcal{I}$ is a $P(\mathcal{J})$-ideal.*

Proof. (1) ⇒ (2). Let $B_k \in \mathcal{I}$ ($k \in \mathbb{N}$). Let $A_1 = B_1$, $A_k = B_k \setminus (B_1 \cup ... \cup B_{k-1})$, $k \in \mathbb{N}\setminus\{1\}$. Obviously $A_k \in \mathcal{I}$ for every $k \in \mathbb{N}$. We define $f_n : X \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \frac{1}{k+1} & \text{if } n \in A_k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{if } n \notin A_k \text{ for every } k \in \mathbb{N}. \end{cases}$$

Let $f(x) = 0$ for every $x \in X$. It is easy to see that $(f_n) \xrightarrow{\mathcal{I}-u} f$. Indeed, let $\varepsilon > 0$. Let $K \in \mathbb{N}$ be such that $\frac{1}{k+1} < \varepsilon$ for every $k > K$. Then $\{n : |f_n(x) - f(x)| \geq \varepsilon\} \subseteq A_0 \cup A_1 \cup \cdots \cup A_K \in \mathcal{I}$ for every $x \in X$. Thus by the assumption we have $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$. Now proceeding as in the proof of Theorem 3.4 we finish the proof.

(2) ⇒ (1). Let $(f_n) \xrightarrow{\mathcal{I}-u} f$. By [3, Theorem 3.1] $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$. If $\mathcal{I}$ is a $P(\mathcal{J})$-ideal then by Theorem 3.4, $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$. □

Corollary 4.2. *Let X be a nonempty set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent.*

1. *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\mathcal{I}-u} f$, then $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$.*
2. *$\mathcal{I}$ is a P-ideal.*

The following example shows that $(\mathcal{I},\mathcal{J})$-equal convergence does not imply $\mathcal{I}$-uniform convergence in general.

Example 4.3. Let $\mathcal{I},\mathcal{J}$ be ideals on $\mathbb{N}$. Let X be an infinite set. Let $x_n \in X$ ($n \in \mathbb{N}$) be distinct elements of X. Let (f_n) be a sequence defined by $f_n(x) = \chi_{\{x_n\}}(x)$ for $x \in X$. Let $f(x) = 0$ for all $x \in X$. Then it is not difficult to see that $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ and $\neg((f_n)_n \xrightarrow{\mathcal{I}-u} f)$.

Proposition 4.4. *Let X be a nonempty set. Let $\mathcal{I},\mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

1. *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ then $(f_n) \xrightarrow{\mathcal{I}} f$.*

(2) $\mathcal{J} \subseteq \mathcal{I}$.

Proof. (1) $\Rightarrow$ (2). Let $A \in \mathcal{J}$. Let

$$f_n(x) = \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{if } n \notin A \end{cases}$$

for $x \in X$ and $n \in \mathbb{N}$. Let $f(x) = 0$ for $x \in X$. Then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ (see the proof of Theorem 3.2). By the assumption we obtain that $(f_n) \xrightarrow{\mathcal{I}} f$. Let $\varepsilon = \frac{1}{2}$ and $x \in X$. Then $A = \{n : |f_n(x) - f(x)| \geq \varepsilon\} \in \mathcal{I}$.

(2) $\Rightarrow$ (1). Let $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$. There exists a sequence $\varepsilon_n \xrightarrow{\mathcal{J}} 0$ such that $A_x = \{n : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for every $x \in X$. Fix $\varepsilon > 0$ and $x \in X$. Then $B_\varepsilon = \{n : \varepsilon_n \geq \varepsilon\} \in \mathcal{J} \subseteq \mathcal{I}$. We have $\{n : |f_n(x) - f(x)| \geq \varepsilon\} \subseteq A_x \cup B_\varepsilon$, so $(f_n) \xrightarrow{\mathcal{I}} f$.

□

Corollary 4.5. *Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. Let (f_n) be a sequence of real-valued functions defined on a set X.*

(1) *If $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ then $(f_n) \xrightarrow{\mathcal{I}} f$.*
(2) *If $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$ then $(f_n) \xrightarrow{\mathcal{I}} f$.*

Corollary 4.6. *Let X be a nonempty set. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\mathcal{I}-u} f$ then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ and if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ then $(f_n) \xrightarrow{\mathcal{I}} f$.*
(2) *$\mathcal{J} \subseteq \mathcal{I}$ and $\mathcal{I}$ is a $P(\mathcal{J})$-ideal.*

In [3, Example 3.1] the authors prove that for any countably generated ideal $\mathcal{I}$ there is a sequence (f_n) of real-valued functions defined on $\mathbb{R}$ such that $(f_n) \xrightarrow{\mathcal{I}} f$ and $\neg((f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f)$. And they write a problem if one can remove the assumption that $\mathcal{I}$ is countably generated in their example. Below we solve the problem in the affirmative (Example 4.7).

Example 4.7. Let $|X| \geq \mathfrak{c}$. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$ such that $\mathcal{J} \subseteq \mathcal{I}$ and $\mathbb{N} \notin \mathcal{I}$. There exists a sequence of real-valued functions defined on X such that $(f_n) \xrightarrow{\mathcal{I}} f$ and $\neg((f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f)$.

Proof. Let $(s_\alpha)_{\alpha<\mathfrak{c}}$ be an enumeration of all sequences of positive reals (ε_n) such that $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$. Let $x_\alpha \in X$ ($\alpha < \mathfrak{c}$) be distinct points of X.

We define $f_n : X \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} s_\alpha(n) & \text{if } x = x_\alpha \text{ for some } \alpha < \mathfrak{c}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : X \to \mathbb{R}$ be given by $f(x) = 0$ for every $x \in X$.

It is easy to see that $(f_n) \xrightarrow{\mathcal{I}} f$. Now we show that $\neg((f_n)_n \xrightarrow{(\mathcal{I},\mathcal{J})-e} f)$.

Suppose that $(f_n)_n \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$. Let (ε_n) be a sequence of positive reals such that $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$ and $\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for every $x \in X$. Let $\alpha < \mathfrak{c}$ be such that $s_\alpha = (\varepsilon_n)$. Then $\{n : |f_n(x_\alpha) - f(x_\alpha)| \geq \varepsilon_n\} = \mathbb{N} \notin \mathcal{I}$, a contradiction. □

5. Ideal σ-uniform convergence

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. A sequence (f_n) of real-valued functions defined on X is $\sigma-\mathcal{I}$-*uniformly convergent* to $f : X \to \mathbb{R}$ if there are sets $X_k \subseteq X$ ($k \in \mathbb{N}$) such that $X = \bigcup_{k\in\mathbb{N}} X_k$ and $(f_n \restriction X_k) \xrightarrow{\mathcal{I}-u} f \restriction X_k$ for every $k \in \mathbb{N}$. We write $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$ in this case.

For $\mathcal{I} = \text{Fin}$ we obtain the σ-uniform convergence introduced in [2] and write $(f_n) \xrightarrow{\sigma-u} f$ instead of $(f_n) \xrightarrow{\sigma-\text{Fin}-u} f$.

It is easy to see that for every ideal $\mathcal{I}$, if $(f_n) \xrightarrow{\mathcal{I}-u} f$, then $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$; and if $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$, then $(f_n) \xrightarrow{\mathcal{I}} f$

In [2] the authors proved that the equal convergence and σ-uniform convergence are the same, i.e. $(f_n) \xrightarrow{e} f \iff (f_n) \xrightarrow{\sigma-u} f$. Below we examine relationships between ideal equal convergence and ideal σ-uniform convergence.

In [3, Theorem 3.2] the authors proved that for every ideal $\mathcal{I}$, if $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$ then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$.

Proposition 5.1. *Let X be a nonempty set. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$, then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$.*
(2) *$\mathcal{I}$ is a $P(\mathcal{J})$-ideal.*

Proof. (1) $\Rightarrow$ (2). Let $B_k \in \mathcal{I}$ ($k \in \mathbb{N}$). Let $A_1 = B_1$, $A_k = B_k \setminus (B_1 \cup ... \cup B_{k-1})$, $k \in \mathbb{N} \setminus \{1\}$. Obviously $A_k \in \mathcal{I}$ for every $k \in \mathbb{N}$. We define $f_n : X \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \frac{1}{k+1} & \text{if } n \in A_k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{if } n \notin A_k \text{ for every } k \in \mathbb{N}. \end{cases}$$

Let $f(x) = 0$ for every $x \in X$. In the proof of Proposition 4.1 we showed that $(f_n) \xrightarrow{\mathcal{I}-u} f$. Hence $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$. By the assumption we have $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$. Now proceeding as in the proof of Theorem 3.4 we finish the proof.

(2) $\Rightarrow$ (1). Let $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$. By [3, Theorem 3.2] $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$. If $\mathcal{I}$ is a $P(\mathcal{J})$-ideal then by Theorem 3.4, $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$. □

Corollary 5.2. *Let X be a nonempty set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$, then $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$.*
(2) *$\mathcal{I}$ is a P-ideal.*

In [3, Theorem 3.2] the authors proved that if $\mathcal{I}$ is a countably generated ideal then $(f_n)_n \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ implies $(f_n)_n \xrightarrow{\sigma-\mathcal{I}-u} f$. They also state a problem if the later implication holds for every ideal. Below we answer the problem in the negative (Corollary 5.4).

Theorem 5.3. *Let $|X| \geq \mathfrak{c}$. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ then $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$.*

(2) *$\mathcal{I}$ is countably generated and $\mathcal{J} \subseteq \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{I} = \{A_\alpha : \alpha < \mathfrak{c}\}$. Let $x_\alpha \in X$ ($\alpha < \mathfrak{c}$) be distinct elements of X. We define functions $f_n : X \to \mathbb{R}$ ($n \in \mathbb{N}$) by

$$f_n(x) = \begin{cases} 1 & \text{if } x = x_\alpha \wedge n \in A_\alpha \text{ for some } \alpha < \mathfrak{c}, \\ 0 & \text{if } x = x_\alpha \wedge n \notin A_\alpha \text{ for some } \alpha < \mathfrak{c}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : X \to \mathbb{R}$ be given by $f(x) = 0$ for every $x \in X$.

It is not difficult to check that $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$. So, by the assumption we have $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$. Hence, there are sets $X_k \subseteq X$ ($k \in \mathbb{N}$) such that $X = \bigcup_{k\in\mathbb{N}} X_k$ and $(f_n \restriction X_k)_n \xrightarrow{\mathcal{I}-u} f \restriction X_k$ for every $k \in \mathbb{N}$.

For every $k \in \mathbb{N}$ we define

$$C_k = \left\{ n : |f_n(x) - f(x)| \geq \frac{1}{2} \text{ for some } x \in X_k \right\}.$$

It is easy to see that $C_k \in \mathcal{I}$ for all $k \in \mathbb{N}$. We will show that $\mathcal{I}$ is generated by the family $\{C_k : k \in \mathbb{N}\}$ (hence $\mathcal{I}$ is countably generated).

Let $A \in \mathcal{I}$. Let $\alpha < \mathfrak{c}$ be such that $A = A_\alpha$. Let $k \in \mathbb{N}$ be such that $x_\alpha \in X_k$. Then $A = \{n : f_n(x_\alpha) = 1\} \subseteq C_k$.

Finally we have to show that $\mathcal{J} \subseteq \mathcal{I}$. If for every sequence (f_n), $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ implies $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$, then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ also implies $(f_n) \xrightarrow{\mathcal{I}} f$. Thus by Proposition 4.4 we obtain $\mathcal{J} \subseteq \mathcal{I}$.

(2) $\Rightarrow$ (1). Let $\{C_k : k \in \mathbb{N}\} \subseteq \mathcal{I}$ be such that for every $A \in \mathcal{I}$ there exists $k \in \mathbb{N}$ with $A \subseteq C_k$. Let $(f_n)_n \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$. Then there is a sequence of positive reals $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$ such that for every $x \in X$ we have $A_x = \{n : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$. For every $k \in \mathbb{N}$ we define

$$X_k = \{x \in X : |f_n(x) - f(x)| < \varepsilon_n \text{ for all } n \in \mathbb{N} \setminus C_k\}.$$

It is easy to see that $X = \bigcup_{k\in\mathbb{N}} X_k$. Indeed, let $x \in X$. Then there is $k \in \mathbb{N}$ with $A_x \subseteq C_k$, so $x \in X_k$.

Now we will show that $(f_n \restriction X_k)_n \xrightarrow{\mathcal{I}-u} f \restriction X_k$ for every $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Let $\varepsilon > 0$. Let $B_\varepsilon = \{n : \varepsilon_n \geq \varepsilon\} \in \mathcal{J} \subseteq \mathcal{I}$. Then $\{n : |f_n(x) - f(x)| \geq \varepsilon\} \subseteq C_k \cup B_\varepsilon \in \mathcal{I}$ for every $x \in X_k$. $\square$

Corollary 5.4. *Let $|X| \geq \mathfrak{c}$. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ then $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$.*

(2) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$ then $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$.*

(3) *$\mathcal{I}$ is countably generated.*

It is easy to see that if X is countable then $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f \iff (f_n) \xrightarrow{\mathcal{I}} f$. Thus the following proposition follows from Proposition 4.4.

Proposition 5.5. *Let X be a countable set. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence* (f_n) *of real-valued functions defined on a set* X*, if* $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ *then* $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$*.*

(2) $\mathcal{J} \subseteq \mathcal{I}$*.*

Remark. In connection with Theorem 5.3 and Proposition 5.5, the question arises what happens when X is uncountable and $\neg$CH is assumed.

6. Filter convergence: uniform, equal, σ-uniform and pointwise

Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. A sequence (f_n) of real-valued functions defined on X is

- $\mathcal{I}^*$*-uniformly convergent* to $f : X \to \mathbb{R}$ if there exists a set $F \in \mathcal{I}^*$ with $(f_n)_{n\in F} \xrightarrow{u} f$. We write $(f_n) \xrightarrow{\mathcal{I}^*-u} f$ in this case.
- $\mathcal{I}^*$*-pointwise convergent* to $f : X \to \mathbb{R}$ if there exists a set $F \in \mathcal{I}^*$ with $(f_n)_{n\in F} \to f$. We write $(f_n) \xrightarrow{\mathcal{I}^*} f$ in this case.
- $(\mathcal{I}^*, \mathcal{J})$*-equally convergent* to $f : X \to \mathbb{R}$ if there exists a set $F \in \mathcal{I}^*$ with $(f_n)_{n\in F} \xrightarrow{(\mathrm{Fin},\mathcal{J})-e} f$. We write $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$ in this case.
- $\sigma - \mathcal{I}^*$*-uniformly convergent* to $f : X \to \mathbb{R}$ if there exist X_k $(k \in \mathbb{N})$ with $X = \bigcup_{k\in\mathbb{N}} X_k$ and $(f_n \restriction X_k) \xrightarrow{\mathcal{I}^*-u} f \restriction X_k$ for every $k \in \mathbb{N}$. We write $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$ in this case.

The notions of $\mathcal{I}^*$-uniform, $\mathcal{I}^*$-pointwise and $\sigma - \mathcal{I}^*$-uniform convergence were introduced in [3]. In [3] the authors introduced also $\mathcal{I}^*$-equal convergence, which is equivalent to $(\mathcal{I}^*, \mathrm{Fin})$-equal convergence in our notation.

Proposition 6.1. *Let* X *be a nonempty set. Let* $\mathcal{I}, \mathcal{J}$ *be ideals on* $\mathbb{N}$*. Then for every sequence* (f_n) *of real-valued functions defined on a set* X*, if* $(f_n) \xrightarrow{\mathcal{I}^*-u} f$ *then* $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$*.*

Proof. Let $(f_n) \xrightarrow{\mathcal{I}^*-u} f$. There exists $F \in \mathcal{I}^*$ such that $(f_n)_{n\in F} \xrightarrow{u} f$. By Proposition 4.1 we have $(f_n)_{n\in F} \xrightarrow{(\mathrm{Fin},\mathcal{J})-e} f$ and hence $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$. □

Example 6.2. Let X be a nonempty set, $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$ such that $\mathcal{I} \subseteq \mathcal{J}$ and $\mathcal{I} \neq \mathcal{J}$. Then there exists a sequence (f_n) of real-valued functions defined on a set X such that $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$ and $\neg((f_n) \xrightarrow{\mathcal{I}^*-u} f)$.

Proof. Let $A \in \mathcal{J} \setminus \mathcal{I}$. We define $f_n : X \to \mathbb{R}\,(n \in \mathbb{N})$ by

$$f_n(x) = \begin{cases} \frac{1}{n+2} & \text{if } n \notin A, \\ 1 & \text{if } n \in A. \end{cases}$$

Let $f : X \to \mathbb{R}$ be given by $f(x) = 0$ for every $x \in X$. Then $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$. Indeed, let $F = \mathbb{N}$ and

$$\varepsilon_n = \begin{cases} \frac{1}{n+1} & \text{if } n \notin A, \\ 2 & \text{if } n \in A. \end{cases}$$

We have $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$ and $\{n \in F : |f_n(x) - f(x)| \geq \varepsilon_n\} = \emptyset \in \mathrm{Fin}$ for every $x \in X$. We will show that $\neg((f_n) \xrightarrow{\mathcal{I}^*-u} f)$. Let $G \in \mathcal{I}^*$ and $\varepsilon = 1/2$. Then the set $G \cap A$ is infinite, so for every $N \in \mathbb{N}$ there exists $n \geq N$, $n \in G \cap A$ such that $|f_n(x) - f(x)| \geq \varepsilon$ for every $x \in X$. □

In [3, Theorem 3.3] the authors proved that if $\mathcal{I}$ is a P-ideal then for every sequence (f_n), $(f_n) \xrightarrow{(\mathcal{I}^*,\mathrm{Fin})-e} f \iff (f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$. They also posed a problem if one can remove the assumption that $\mathcal{I}$ is a P-ideal in their theorem. Below we answer the problem in the negative (Corollary 6.5).

Proposition 6.3. *Let X be a nonempty set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and $\mathcal{J}$ be a P-ideal. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$ then $(f_n) \xrightarrow{\mathcal{I}^*} f$.*
(2) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$ then $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$.*
(3) *$\mathcal{J} \subseteq \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3). Let $A \in \mathcal{J}$. Let

$$f_n(x) = \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{if } n \notin A \end{cases}$$

for $x \in X$ and $n \in \mathbb{N}$. Let $f(x) = 0$ for $x \in X$. It is easy to see that $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$.

In the first case, from (1) we obtain that $(f_n) \xrightarrow{\mathcal{I}^*} f$. There exists $F \in \mathcal{I}^*$ such that $A \cap F = \{n \in F : |f_n(x) - f(x)| \geq \frac{1}{2}\} \in \mathrm{Fin}$ for every $x \in X$, so $A \in \mathcal{I}$.

In the secend case, from (2) we obtain that $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$. There exist X_k $(k \in \mathbb{N})$ with $X = \bigcup_{k\in\mathbb{N}} X_k$ and $(f_n \restriction X_k) \xrightarrow{\mathcal{I}^*-u} f \restriction X_k$ for every $k \in \mathbb{N}$. Let $\varepsilon = \frac{1}{2}$ and $k \in \mathbb{N}$. Then there exists $F \in \mathcal{I}^*$ such that $(f_n)_{n\in F} \restriction X_k \xrightarrow{u} f \restriction X_k$. Let $x \in X_k$. We obtain $A \cap F = \{n \in F : |f_n(x) - f(x)| \geq \frac{1}{2}\} \in \mathrm{Fin}$, so $A \in \mathcal{I}$.

(3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2). Let $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$. There exists $F \in \mathcal{I}^*$ and $\varepsilon_n \xrightarrow{\mathcal{J}} 0$ such that $\{n \in F : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathrm{Fin}$ for every $x \in X$. Since $\mathcal{J}$ is a P-ideal and $\mathcal{J} \subseteq \mathcal{I}$, we have $\varepsilon_n \xrightarrow{\mathcal{J}^*} 0$, so there exists $G \in \mathcal{I}^*$ such that $(\varepsilon_n)_{n\in G} \to 0$.

Clearly $(f_n)_{n\in F\cap G} \to f$, so $(f_n) \xrightarrow{\mathcal{I}^*} f$. Thus (1) holds.

Let $X_k = \{x \in X : |f_n(x) - f(x)| < \varepsilon_n \text{ for all } n \geq k, n \in F \cap G\}$, $k \in \mathbb{N}$. Clearly $X = \bigcup_{k\in\mathbb{N}} X_k$. We will show that $(f_n)_{n\in F\cap G} \restriction X_k \xrightarrow{u} f \restriction X_k$. Let $\varepsilon > 0$, $k \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that $\varepsilon_n < \varepsilon$ for $n \geq N$, $n \in G$. Then $|f_n(x) - f(x)| < \varepsilon$ for every $n \geq \max\{k, N\}$, $n \in F \cap G$ and $x \in X_k$. We obtain $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$, so (2) holds.

□

Theorem 6.4. *Let X be a nonempty set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$, $\mathcal{J}$ be a P-ideal and $\mathcal{J} \subseteq \mathcal{I}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$ then $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$.*
(2) *$\mathcal{I}$ is a P-ideal.*

Proof. (1) $\Rightarrow$ (2). Let $A_k \in \mathcal{I}$ $(k \in \mathbb{N})$. Let $\emptyset \neq X_k \subseteq X$ $(k \in \mathbb{N})$ be any partition of X. We define $f_n : X \to \mathbb{R}$ $(n \in \mathbb{N})$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in X_k \wedge n \in A_k, k \in \mathbb{N}, \\ 0 & \text{if } x \in X_k \wedge n \notin A_k, k \in \mathbb{N}. \end{cases}$$

Let $f : X \to \mathbb{R}$ be given by $f(x) = 0$ for every $x \in X$. It is not difficult to check that $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$. So, by the assumption we have $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$. There exists $F \in \mathcal{I}^*$ and $\varepsilon_n \xrightarrow{\mathcal{J}} 0$ such that $\{n \in F : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathrm{Fin}$. Since $\mathcal{J}$ is a P-ideal and $\mathcal{J} \subseteq \mathcal{I}$, we have $\varepsilon_n \xrightarrow{\mathcal{J}^*} 0$, so there exists $G \in \mathcal{I}^*$ such that $(\varepsilon_n)_{n\in G} \to 0$. Let $A = \mathbb{N} \setminus (F \cap G)$. Let $k \in \mathbb{N}$ and $x \in X_k$. We obtain $A_k \cap F \cap G = \{n \in F \cap G : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathrm{Fin}$, so $A_k \setminus A \in \mathrm{Fin}$ and $\mathcal{I}$ is a P-ideal.

(2) $\Rightarrow$ (1). Let $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$. There exist X_k ($k \in \mathbb{N}$) with $X = \bigcup_{k\in\mathbb{N}} X_k$ and $F_k \in \mathcal{I}^*$ such that $(f_n)_{n\in F_k} \upharpoonright X_k \xrightarrow{u} f \upharpoonright X_k$ for every $k \in \mathbb{N}$. Since $\mathcal{I}$ is a P-ideal, there exists $F \in \mathcal{I}^*$ such that $F \setminus F_k \in \mathrm{Fin}$ for every $k \in \mathbb{N}$. Clearly $(f_n)_{n\in F} \xrightarrow{\sigma-u} f$, so $(f_n)_{n\in F} \xrightarrow{e} f$ which yields $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$ (cf. Fact 3.1). □

Corollary 6.5. *Let X be a nonempty set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X we have $(f_n) \xrightarrow{(\mathcal{I}^*,Fin)-e} f \iff (f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$.*

(2) *$\mathcal{I}$ is a P-ideal.*

Proposition 6.6. *Let X be a nonempty set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$ then $(f_n) \xrightarrow{\mathcal{I}^*} f$.*

(2) *$\mathcal{I}$ is a P-ideal.*

Proof. (1) $\Rightarrow$ (2). Let A_k, X_k ($k \in \mathbb{N}$), f and f_n ($n \in \mathbb{N}$) be as in the proof of Theorem 6.4. Then $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$. So, by the assumption we have $(f_n) \xrightarrow{\mathcal{I}^*} f$. There exists $F \in \mathcal{I}^*$ such that $(f_n)_{n\in F} \to f$. Let $k \in \mathbb{N}$ and $x \in X_k$. Then $\{n \in F : |f_n(x) - f(x)| \geq \frac{1}{2}\} \in \mathrm{Fin}$. Let $A = \mathbb{N} \setminus F$. Then $A_k \setminus A \in \mathrm{Fin}$, so $\mathcal{I}$ is a P-ideal.

(2) $\Rightarrow$ (1). Let $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$. There exist X_k ($k \in \mathbb{N}$) with $X = \bigcup_{k\in\mathbb{N}} X_k$ and $F_k \in \mathcal{I}^*$ such that $(f_n)_{n\in F_k} \upharpoonright X_k \xrightarrow{u} f \upharpoonright X_k$ for every $k \in \mathbb{N}$. $\mathcal{I}$ is a P-ideal, so there exists $F \in \mathcal{I}^*$ such that $F \setminus F_k \in \mathrm{Fin}$ for every $k \in \mathbb{N}$. Clearly $(f_n)_{n\in F} \to f$, so $(f_n) \xrightarrow{\mathcal{I}^*} f$. □

7. Ideal convergence versus filter convergence

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and (f_n) be a sequence of real-valued functions defined on a set X. It is easy to see that if $(f_n) \xrightarrow{\mathcal{I}^*-u} f$ then $(f_n) \xrightarrow{\mathcal{I}-u} f$ and if $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$ then $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$.

Proposition 7.1. *Let X be a nonempty set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\mathcal{I}-u} f$ then $(f_n) \xrightarrow{\mathcal{I}^*-u} f$.*

(2) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$ then $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$.*

(3) *$\mathcal{I}$ is a P-ideal.*

Proof. (1) $\Rightarrow$ (3) ((2) $\Rightarrow$ (3)). Let $y, y_n \in \mathbb{R}$ ($n \in \mathbb{N}$) with $(y_n) \xrightarrow{\mathcal{I}} y$. We define $f, f_n : X \to \mathbb{R}$ ($n \in \mathbb{N}$) by $f_n(x) = y_n$, $f(x) = y$ for every $x \in X$. It is not difficult to see that $(f_n) \xrightarrow{\mathcal{I}-u} f$ ($(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$, resp.). So, by the assumption $(f_n) \xrightarrow{\mathcal{I}^*-u} f$ ($(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$, resp.), hence $(y_n) \xrightarrow{\mathcal{I}^*} y$. By Theorem 2.1 we obtain that $\mathcal{I}$ is a P-ideal.

(3) $\Rightarrow$ (1). Let $(f_n) \xrightarrow{\mathcal{I}-u} f$ and $k \in \mathbb{N}$. There exists $A_k \in \mathcal{I}$ such that $|f_n(x) - f(x)| < \frac{1}{k+1}$ for every $n \in \mathbb{N} \setminus A_k$ and $x \in X$. Since $\mathcal{I}$ is a P-ideal, there exists $A \in \mathcal{I}$ such that $A_k \setminus A \in \mathrm{Fin}$. Let $F = \mathbb{N} \setminus A$ and fix $\varepsilon > 0$. There exists k such that $\frac{1}{k+1} < \varepsilon$. We obtain $\{n \in F : |f_n(x) - f(x)| \geq \varepsilon \text{ for every } x \in X\} \subseteq A_k \cap F \in \mathrm{Fin}$, so $(f_n) \xrightarrow{\mathcal{I}^*-u} f$.

(3) $\Rightarrow$ (2). Let $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$. There are sets $X_k \subseteq X$ ($k \in \mathbb{N}$) such that $X = \bigcup_{k\in\mathbb{N}} X_k$ and $(f_n \upharpoonright X_k) \xrightarrow{\mathcal{I}-u} f \upharpoonright X_k$ for every $k \in \mathbb{N}$. By the implication "(3)$\Rightarrow$ (1)", we obtain $(f_n \upharpoonright X_k) \xrightarrow{\mathcal{I}^*-u} f \upharpoonright X_k$ for every $k \in \mathbb{N}$, so $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$. □

It is easy to see that for every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\mathcal{I}^*} f$ then $(f_n) \xrightarrow{\mathcal{I}} f$.

Proposition 7.2. *Let X be a countable set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\mathcal{I}} f$ then $(f_n) \xrightarrow{\mathcal{I}^*} f$.*
(2) *$\mathcal{I}$ is a P-ideal.*

Proof. (1) $\Rightarrow$ (2). Let $y, y_n \in \mathbb{R}$ ($n \in \mathbb{N}$) with $(y_n) \xrightarrow{\mathcal{I}} y$. We define $f, f_n : X \to \mathbb{R}$ ($n \in \mathbb{N}$) by $f_n(x) = y_n$, $f(x) = y$ for every $x \in X$. It is not difficult to see that $(f_n) \xrightarrow{\mathcal{I}} f$. So, by the assumption $(f_n) \xrightarrow{\mathcal{I}^*} f$ hence $(y_n) \xrightarrow{\mathcal{I}^*} y$. By Theorem 2.1 we obtain that $\mathcal{I}$ is a P-ideal.

(2) $\Rightarrow$ (1). Let $X = \{x_k : k \in \mathbb{N}\}$ and $(f_n) \xrightarrow{\mathcal{I}} f$. For every $k, l \in \mathbb{N}$ we have $A_{k,l} = \{n : |f_n(x_k) - f(x_k)| \geq \frac{1}{l+1}\} \in \mathcal{I}$. Since $\mathcal{I}$ is a P-ideal, there exists $A \in \mathcal{I}$ such that $A_{k,l} \setminus A \in \mathrm{Fin}$. Let $F = \mathbb{N} \setminus A$. Fix $\varepsilon > 0$ and $k \in \mathbb{N}$. There exists $l \in \mathbb{N}$ such that $\frac{1}{l+1} < \varepsilon$. We obtain $\{n \in F : |f_n(x_k) - f(x_k)| \geq \varepsilon\} = A_{k,l} \cap F \in \mathrm{Fin}$, so $(f_n) \xrightarrow{\mathcal{I}^*} f$. □

Remark. The implication (2) $\Rightarrow$ (1) in Proposition 7.2 can be generalized: If $\mathcal{I}$ is a P-ideal on $\mathbb{N}$ and $|X| < \mathrm{add}^*(\mathcal{I})$ then (1) is true. Here

$$\mathrm{add}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall X \in \mathcal{I})(\exists A \in \mathcal{A})\ A \setminus X \text{ is infinite}\}$$

(see e.g. [5]). The proof remains the same. For P-ideals we clearly have $\mathrm{add}^*(\mathcal{I}) > \omega$.

Proposition 7.3. *Let X be a set such that $|X| \geq \mathfrak{c}$. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\mathcal{I}} f$ then $(f_n) \xrightarrow{\mathcal{I}^*} f$.*
(2) *There exists $F \in \mathcal{I}^*$ such that if $A \in \mathcal{I}$ then $F \cap A$ is finite.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{I} = \{A_\alpha : \alpha < \mathfrak{c}\}$. Let $x_\alpha \in X$ ($\alpha < \mathfrak{c}$) be distinct points of X. We define functions $f_n : X \to \mathbb{R}$ ($n \in \mathbb{N}$) by

$$f_n(x) = \begin{cases} 1 & \text{if } x = x_\alpha \wedge n \in A_\alpha, \\ 0 & \text{if } x = x_\alpha \wedge n \notin A_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : X \to \mathbb{R}$ be given by $f(x) = 0$ for every $x \in X$.

It is easy to see that $(f_n) \xrightarrow{\mathcal{I}} f$. By (1), $(f_n) \xrightarrow{\mathcal{I}^*} f$. Let $F \in \mathcal{I}^*$ be such that $(f_n)_{n\in F} \to f$. Fix $A \in \mathcal{I}$. Then $A = A_\alpha \in \mathcal{I}$ for some $\alpha < \mathfrak{c}$. Since $(f_n(x_\alpha))_{n\in F} \to 0$, so $F \cap A_\alpha = \{n \in F : f_n(x_\alpha) = 1\} \in \mathrm{Fin}$.

(2) $\Rightarrow$ (1). Let $F \in \mathcal{I}^*$ be such that if $A \in \mathcal{I}$ then $F \cap A$ is finite. Let $(f_n) \xrightarrow{\mathcal{I}} f$. Clearly $(f_n)_{n\in F} \to f$, so $(f_n) \xrightarrow{\mathcal{I}^*} f$. $\square$

In [3, Theorem 3.4] the authors proved that for every sequence (f_n) of real-valued functions defined on a set X if $(f_n) \xrightarrow{(\mathcal{I}^*,\mathrm{Fin})-e} f$ then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$.

Proposition 7.4. *Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. Let (f_n) be a sequence of real-valued functions defined on a set X. If $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$, then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$.*

Proof. Let $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$. Let $F \in \mathcal{I}^*$ and $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$ be such that $\{n \in F : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathrm{Fin}$ for every $x \in X$. Then, for every $x \in X$, $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \subseteq \{n \in F : |f_n(x) - f(x)| \geq \varepsilon_n\} \cup (\mathbb{N} \setminus F) \in \mathcal{I}$. Thus $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$. $\square$

Corollary 7.5 ([3, Theorem 3.4])**.** *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and let (f_n) be a sequence of real-valued functions defined on a set X. Then $(f_n) \xrightarrow{(\mathcal{I}^*,Fin)-e} f$ implies $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$ (so $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ as well).*

In [3, Theorem 3.7] the authors proved that if for every sequence (f_n) of real-valued functions defined on a set X if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ implies $(f_n) \xrightarrow{(\mathcal{I}^*,\mathrm{Fin})-e} f$, then $\mathcal{I}$ is a P-ideal.

Proposition 7.6. *Let X be an infinite set. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. If for every sequence (f_n) of real-valued functions defined on a set X, $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ implies $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$, then $\mathcal{I}$ is a $P(\mathcal{J})$-ideal.*

Proof. Let $A_k \in \mathcal{I}$ ($k \in \mathbb{N}$). We will show that there is $A \in \mathcal{I}$ such that $A_k \setminus A \in \mathcal{J}$ for every $\in \mathbb{N}$.

Let $x_k \in X$ ($k \in \mathbb{N}$) be distinct elements of X. We define $f_n : X \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } n \notin A_k, x = x_k, k \in \mathbb{N}, \\ 1 & \text{if } n \in A_k, x = x_k, k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f(x) = 0$ for every $x \in X$. Then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$. Indeed, let $\varepsilon_n = \frac{1}{n+2}$, $n \in \mathbb{N}$. Then $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$. If $x \in X \setminus \{x_k : k \in \mathbb{N}\}$ then $\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} = \emptyset \in \mathcal{I}$. If $x = x_k$ for some $k \in \mathbb{N}$, then $\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} = A_k \in \mathcal{I}$.

By the assumption we obtain that $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$. Let $F \in \mathcal{I}^*$ such that $(f_n)_{n\in F} \xrightarrow{(\mathrm{Fin},\mathcal{J})-e} f$. So, there is a sequence of positive reals $(\eta_n) \xrightarrow{\mathcal{J}} 0$ such that $B_x = \{n \in F : |f_n(x) - f(x)| \geq \eta_n\}$ is finite for every $x \in X$. Let $A = \mathbb{N} \setminus F \in \mathcal{I}$. Let $C = \{n : \eta_n \geq \frac{1}{2}\} \in \mathcal{J}$. Let $k \in \mathbb{N}$. Then $A_k \setminus (A \cup C) \subseteq \{n \in F : |f_n(x_k) - f(x_k)| \geq \eta_n\} = B_{x_k}$, so $A_k \setminus A \subseteq C \cup B_{x_k} \in \mathcal{J}$. □

Corollary 7.7. *Let X be an infinite set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. If for every sequence (f_n) of real-valued functions defined on a set X, $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$ implies $(f_n) \xrightarrow{(\mathcal{I}^*,Fin)-e} f$, then $\mathcal{I}$ is a P-ideal.*

Corollary 7.8 ([3, Theorem 3.7])**.** *Let X be an infinite set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. If for every sequence (f_n) of real-valued functions defined on a set X, $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ implies $(f_n) \xrightarrow{(\mathcal{I}^*,Fin)-e} f$, then $\mathcal{I}$ is a P-ideal.*

In [3, Theorem 3.6] the authors proved that if X is a countable set and $\mathcal{I}$ is a P-ideal then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ implies $(f_n) \xrightarrow{(\mathcal{I}^*,\mathrm{Fin})-e} f$. And they posed the problem if their result remains true when X is uncountable. Below we solve the problem in the negative (see the remark below Corollary 7.14).

Proposition 7.9. *Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. Let X be a set such that $|X| < add^*(\mathcal{I})$. Let (f_n) be a sequence of real-valued functions defined on a set X. If $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ then $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$.*

Proof. Let $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$. Let $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$ such that $A_x = \{n : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for every $x \in X$. Since $|X| < \mathrm{add}^*(\mathcal{I})$, so there is $A \in \mathcal{I}$ such that $A_x \setminus A \in \mathrm{Fin}$ for every $x \in X$.

Let $F = \mathbb{N} \setminus A \in \mathcal{I}^*$. Let $x \in X$. Then the set $\{n \in F : |f_n(x) - f(x)| \geq \varepsilon_n\} \subseteq A_x \setminus A \in \mathrm{Fin}$, so $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$. □

Corollary 7.10. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Let X be a set such that $|X| < add^*(\mathcal{I})$. Let (f_n) be a sequence of real-valued functions defined on a set X. If $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$ then $(f_n) \xrightarrow{(\mathcal{I}^*,Fin)-e} f$.*

The next result is a consequence of Corollaries 3.5 and 7.10.

Corollary 7.11. *Let $\mathcal{I}$ be a P-ideal on $\mathbb{N}$. Let X be a set such that $|X| < add^*(\mathcal{I})$. Let (f_n) be a sequence of real-valued functions defined on a set X. If $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ then $(f_n) \xrightarrow{(\mathcal{I}^*,Fin)-e} f$.*

Remark. In [**?**] (see also [5]) the author proved that it is consistent with ZFC that $\omega_1 < \mathrm{add}^*(\mathcal{I}) = \mathfrak{c}$ for analytic P-ideals. Hence we obtain that it is consistent with ZFC that if $\mathcal{I}$ is an analytic P-ideal, $|X| = \omega_1 < \mathfrak{c}$ and (f_n) is a sequence of real-valued functions defined on a set X such that $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ then $(f_n) \xrightarrow{(\mathcal{I}^*,\mathrm{Fin})-e} f$.

Theorem 7.12. *Let X be a set such that $|X| \geq \mathfrak{c}$. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$ such that $\mathcal{J} \subseteq \mathcal{I}$ and $\mathcal{J}$ is a P-ideal. The following are equivalent.*

(1) *For every sequence* (f_n) *of real-valued functions defined on a set* X*, if* $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ *then* $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$*.*

(2) *There exists* $F \in \mathcal{I}^*$ *such that if* $A \in \mathcal{I}$ *then* $F \cap A$ *is finite.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{I} = \{A_\alpha : \alpha < \mathfrak{c}\}$. Let $x_\alpha \in X$ ($\alpha < \mathfrak{c}$) be distinct points of X. We define functions $f_n : X \to \mathbb{R}$ ($n \in \mathbb{N}$) by

$$f_n(x) = \begin{cases} 1 & \text{if } x = x_\alpha \wedge n \in A_\alpha \text{ for some } \alpha < \mathfrak{c}, \\ 0 & \text{if } x = x_\alpha \wedge n \notin A_\alpha \text{ for some } \alpha < \mathfrak{c}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : X \to \mathbb{R}$ be given by $f(x) = 0$ for every $x \in X$.

It is easy to see that $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$. Indeed, let $\varepsilon_n = \frac{1}{n+2}$. Then $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$ and if $x \neq x_\alpha$ for every $\alpha < \mathfrak{c}$ then $\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} = \emptyset \in \mathcal{I}$; and if $x = x_\alpha$ for some $\alpha < \mathfrak{c}$ then $\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} = A_\alpha \in \mathcal{I}$.

By (1), $(f_n)_n \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$. Let $F \in \mathcal{I}^*$ and $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$ be such that $\{n \in F : |f_n(x) - f(x)| \geq \varepsilon_n\}$ is finite for every $x \in X$.

By Theorem 2.1 there is $H \in \mathcal{J}^*$ such that $(\varepsilon_n)_{n \in H} \to 0$. Then $G = \{n \in H : \varepsilon_n < \frac{1}{2}\} \in \mathcal{J}^*$.

Since $\mathcal{J} \subseteq \mathcal{I}$, so $G \in \mathcal{I}^*$. Thus $F \cap G \in \mathcal{I}^*$. Fix $A \in \mathcal{I}$. We will show that $(F \cap G) \cap A$ is finite. Since $(F \cap G) \cap A \in \mathcal{I}$, so there is $\alpha < \mathfrak{c}$ with $A_\alpha = (F \cap G) \cap A$. Then $\{n \in F : |f_n(x_\alpha) - f(x_\alpha)| \geq \varepsilon_n\} \supseteq \{n \in F \cap G : |f_n(x_\alpha) - f(x_\alpha)| \geq \varepsilon_n\} \supseteq \{n \in A_\alpha : |f_n(x_\alpha) - f(x_\alpha)| \geq \varepsilon_n\} = A_\alpha$. Since $\{n \in F : |f_n(x_\alpha) - f(x_\alpha)| \geq \varepsilon_n\}$ is finite, so A_α is finite too.

(2) $\Rightarrow$ (1). Let $F \in \mathcal{I}^*$ be such that if $A \in \mathcal{I}$ then $F \cap A$ is finite. Let $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$. Let $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$ such that $A_x = \{n : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for every $x \in X$. Then $\{n \in F : |f_n(x) - f(x)| \geq \varepsilon_n\} = F \cap A_x$ is finite for every $x \in X$. Thus $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$. □

Corollary 7.13. *Let* X *be a set such that* $|X| \geq \mathfrak{c}$*. Let* $\mathcal{I}$ *be an ideal on* $\mathbb{N}$*. The following are equivalent.*

(1) *For every sequence* (f_n) *of real-valued functions defined on a set* X*, if* $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$ *then* $(f_n) \xrightarrow{(\mathcal{I}^*,Fin)-e} f$*.*

(2) *There exists* $F \in \mathcal{I}^*$ *such that if* $A \in \mathcal{I}$ *then* $F \cap A$ *is finite.*

Corollary 7.14. *Let* X *be a set such that* $|X| \geq \mathfrak{c}$*. Let* $\mathcal{I}$ *be a* P*-ideal on* $\mathbb{N}$*. The following are equivalent.*

(1) *For every sequence* (f_n) *of real-valued functions defined on a set* X*, if* $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ *then* $(f_n) \xrightarrow{(\mathcal{I}^*,Fin)-e} f$*.*

(2) *There exists* $F \in \mathcal{I}^*$ *such that if* $A \in \mathcal{I}$ *then* $F \cap A$ *is finite.*

Remark. An ideal $\mathcal{I}$ on $\mathbb{N}$ is *dense* if for every infinite $A \subseteq \mathbb{N}$ there is an infinite $B \in \mathcal{I}$ with $B \subseteq A$. It is easy to see that if $\mathcal{I}$ is a dense ideal then for every $F \in \mathcal{I}^*$ there is an infinite $A \in \mathcal{I}$ with $A \subseteq F$. Hence for every dense P-ideal there exists a sequence (f_n) of functions defined on a set X with $|X| \geq \mathfrak{c}$ such that $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ but $\neg((f_n) \xrightarrow{(\mathcal{I}^*,\mathrm{Fin})-e} f)$.

We will characterize ideals that satisfy condition (2) in Theorem 7.12.

We say that ideals $\mathcal{I}$ and $\mathcal{J}$ are *isomorphic* ($\mathcal{I} \simeq \mathcal{J}$) if there exists a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $A \in \mathcal{I} \iff f[A] \in \mathcal{J}$ for every $A \subseteq \mathbb{N}$.

By $\mathrm{Fin} \oplus \mathcal{P}(\mathbb{N})$ we mean the ideal on $\mathbb{N} \times \{0,1\}$ given by $A \in \mathrm{Fin} \oplus \mathcal{P}(\mathbb{N}) \iff \{n \in \mathbb{N} : (n,0) \in A\} \in \mathrm{Fin}$.

Proposition 7.15. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent.*

(1) *There exists $F \in \mathcal{I}^*$ such that if $A \in \mathcal{I}$ then $F \cap A$ is finite.*

(2) *$\mathcal{I}$ is isomorphic to Fin or $\mathrm{Fin} \oplus \mathcal{P}(\mathbb{N})$.*

Proof. (1) $\Rightarrow$ (2). Let $F \in \mathcal{I}^*$ be such that every set $A \subseteq F$ if $A \in \mathcal{I}$ then A is finite. If $\mathbb{N} \setminus F$ is finite then clearly $\mathcal{I} \simeq \mathrm{Fin}$. Suppose that $\mathbb{N} \setminus F$ is infinite. We will show that $\mathcal{I}$ is isomorphic to $\mathrm{Fin} \oplus \mathcal{P}(\mathbb{N})$. Let $g : \mathbb{N} \to F$ and $h : \mathbb{N} \to \mathbb{N} \setminus F$ be bijections. We define function $f : \mathbb{N} \times \{0,1\} \to \mathbb{N}$ by

$$f(n,i) = \begin{cases} g(n) & \text{if } i = 0 \\ h(n) & \text{if } i = 1. \end{cases}$$

Now it is easy to see that f is an isomorphism of ideals $\mathrm{Fin} \oplus \mathcal{P}(\mathbb{N})$ and $\mathcal{I}$.

(2) $\Rightarrow$ (1). Clear for $\mathcal{I} \simeq \mathrm{Fin}$. Suppose that $\mathcal{I} \simeq \mathrm{Fin} \oplus \mathcal{P}(\mathbb{N})$ and $f : \mathbb{N} \to \mathbb{N} \times \{0,1\}$ is an isomorphism between these ideals. Let $G = \{(n,0) : n \in \mathbb{N}\} \subseteq \mathbb{N} \times \{0,1\}$. Then the set $F = f[G]$ is the required one. □

Acknowledgment. The authors would like to thank Tomasz Natkaniec and Piotr Szuca for fruitful discussions and careful reading early version of this article.

The authors are also greatly indebted to the Referee for his helpful suggestions concerning a presentation of results and especially for drawing our attention to the notion of $\mathrm{add}^*(\mathcal{I})$ (for instance the remark below Proposition 7.2 was provided by the referee).

References

1. Á. Császár and M. Laczkovich, *Discrete and equal convergence*, Studia Sci. Math. Hungar. **10** (1975), no. 3-4, 463–472 (1978).
2. ———, *Some remarks on discrete Baire classes*, Acta Math. Acad. Sci. Hungar. **33** (1979), no. 1-2, 51–70, Special issue dedicated to George Alexits on the occasion of his 80th birthday.
3. Pratulananda Das, Sudipta Dutta, and Sudip Kumar Pal, *On $\mathcal{I}$ and $\mathcal{I}^*$-equal convergence and an Egoroff-type theorem*, Mat. Vesnik **66** (2014), no. 2, 165–177.
4. Rafał Filipów and Piotr Szuca, *Three kinds of convergence and the associated $\mathcal{I}$-Baire classes*, J. Math. Anal. Appl. **391** (2012), no. 1, 1–9.
5. Fernando Hernández-Hernández and Michael Hrušák, *Cardinal invariants of analytic P-ideals*, Canad. J. Math. **59** (2007), no. 3, 575–595.
6. Pavel Kostyrko, Tibor Šalát, and Władysław Wilczyński, *$\mathcal{I}$-convergence*, Real Anal. Exchange **26** (2000/01), no. 2, 669–685.
7. Martin Maćaj and Martin Sleziak, *$\mathcal{I}^{\mathcal{K}}$-convergence*, Real Anal. Exchange **36** (2010/11), no. 1, 177–194.

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