POINTWISE VERSUS EQUAL (QUASI-NORMAL) CONVERGENCE VIA IDEALS

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Abstract. We prove a characterization showing when the ideal pointwise convergence does not imply the ideal equal (aka quasi-normal) convergence. The characterization is expressed in terms of a cardinal coefficient related to the bounding number $\mathfrak{b}$. We also prove a characterization showing when the ideal equal limit is unique.

1. Introduction

By an *ideal on a set* X we mean a nonempty family of subsets of X closed under taking finite unions and subsets of its elements. By $\mathrm{Fin}(X)$ we denote the ideal of all finite subsets of X (for $X = \mathbb{N}$ we write $\mathrm{Fin} = \mathrm{Fin}(\mathbb{N})$). Moreover, we also assume that ideals are *proper* (i.e. $X \notin \mathcal{I}$) and contain all finite subsets of X (i.e. $\mathrm{Fin}(X) \subseteq \mathcal{I}$).

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. We say that a sequence (x_n) of reals is *$\mathcal{I}$-convergent* to $x \in \mathbb{R}$ if the set $\{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$. We write $(x_n) \xrightarrow{\mathcal{I}} x$ in this case. For $\mathcal{I} = \mathrm{Fin}$ we obtain the ordinary convergence.

Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. Let f_n ($n \in \mathbb{N}$) and f be real-valued functions defined on a set X. We say that the sequence (f_n) is *$(\mathcal{I}, \mathcal{J})$-equally convergent* to f if there exists a sequence of positive reals $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$ such that $\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for every $x \in X$ ([13]). We write $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ in this case. For $\mathcal{I} = \mathcal{J} = \mathrm{Fin}$ we obtain the equal convergence introduced by Á. Császár and M. Laczkovich [9] and write $(f_n) \xrightarrow{e} f$ instead of $(f_n) \xrightarrow{(\mathrm{Fin},\mathrm{Fin})-e} f$. The equal convergence was independently introduced by Z. Bukovská (see [2] and [1]), where she used the name "quasi-normal convergence" instead of "equal convergence". In [4], [3] the authors use the equal convergence in the theory of trigonometrical series. In [11] and [14] the authors considered the ideal equal convergence in some special cases. Namely, in [11] the authors considered $(\mathcal{I}, \mathcal{I})$-equal convergence, whereas in [14] the authors considered $(\mathcal{I}, \mathrm{Fin})$-equal convergence.

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Let f_n ($n \in \mathbb{N}$) and f be real-valued functions defined on a set X. We say that the sequence (f_n) is *$\mathcal{I}$-pointwise convergent* to f if $\{n : |f_n(x) - f(x)| \geq \varepsilon\} \in \mathcal{I}$ for every $x \in X$ and $\varepsilon > 0$. We write $(f_n) \xrightarrow{\mathcal{I}} f$ in this case. For $\mathcal{I} = \mathrm{Fin}$ we obtain the pointwise convergence and write $(f_n) \to f$.

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If $\mathcal{I} = \mathcal{J} = \mathrm{Fin}$, then it is not difficult to show that the equal convergence implies the pointwise convergence. However the same is not true for arbitrary ideals $\mathcal{I}, \mathcal{J}$.

Theorem 1.1 ([13, Proposition 4.4])**.** *Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

1. *For every sequence of real-valued functions defined on X, if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ then $(f_n) \xrightarrow{\mathcal{I}} f$.*
2. *$\mathcal{J} \subseteq \mathcal{I}$.*

If $\mathcal{I} = \mathcal{J} = \mathrm{Fin}$, then it also is not difficult to show that the pointwise convergence does not imply the equal convergence. In [11, Example 3.1] the authors showed that if $\mathcal{I}$ is a countably generated ideal, then the $\mathcal{I}$-pointwise convergence does not imply the $(\mathcal{I},\mathcal{I})$-equal convergence, and in [13, Example 4.7] we showed that if $|X| \geq \mathfrak{c}$ and $\mathcal{J} \subseteq \mathcal{I}$ then the $\mathcal{I}$-pointwise convergence does not imply the $(\mathcal{I},\mathcal{J})$-equal convergence. (The relationship between pointwise convergence and equal convergence in the realm of continuous functions was already done in the literature, for instance in [7] and [8] the authors consider the case of the ordinary convergence (i.e. $\mathcal{I} = \mathrm{Fin}$), whereas in [10] and [6] the authors generalized the previous results to the case of the ideal convergence.)

The aim of this paper is to prove the following theorem that gives a sufficient and necessary condition when the $\mathcal{I}$-pointwise convergence implies the $(\mathcal{I},\mathcal{J})$-equal convergence (see Section 5 for the proof).

Theorem 1.2. *Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. For every set X the following are equivalent.*

1. *For every sequence (f_n) of real-valued functions defined on X, if $(f_n) \xrightarrow{\mathcal{I}} f$ then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$.*
2. *For every family $\{E_n^\alpha : n \in \mathbb{N}, \alpha < |X|\} \subseteq \mathcal{I}$ such that $E_n^\alpha \cap E_k^\alpha = \emptyset$ for $n \neq k, \alpha < |X|$ there exists a partition $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{J}$ of $\mathbb{N}$ such that*
$$\bigcup_{n\in\mathbb{N}} \left(A_n \cap \bigcup_{i\leq n} E_i^\alpha \right) \in \mathcal{I}$$
for every $\alpha < |X|$.

In Sections 2, 3 and 4 we introduced tools we need to prove the above-mentioned characterization. For instance we define a cardinal coefficient $\mathfrak{b}(\mathcal{I},\mathcal{J})$ which is related to the bounding number $\mathfrak{b}$. In Section 6 we prove a characterization showing when the $(\mathcal{I},\mathcal{J})$-equal limit is unique.

2. Some properties of ideals

2.1. **Orthogonal ideals.** We say that ideals $\mathcal{I}, \mathcal{J}$ on X are *orthogonal* if there is a set $A \subseteq X$ such that $A \in \mathcal{I}$ and $X \setminus A \in \mathcal{J}$.

Example 2.1.
1. Two distinct maximal ideals are orthogonal.
2. If $\mathcal{I} \subseteq \mathcal{J}$, then $\mathcal{I}$, $\mathcal{J}$ are not orthogonal.
3. The ideal $\mathrm{Fin}(X)$ is not orthogonal to any ideal on X.

Proof. Straightforward. □

Example 2.2. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on X and Y respectively. The ideals

$$\begin{aligned}\mathcal{I}\oplus\mathcal{P}(Y) &= \{A\subseteq X\times\{0\}\cup Y\times\{1\} : \{x\in X : (x,0)\in A\}\in\mathcal{I}\},\\ \mathcal{P}(X)\oplus\mathcal{J} &= \{B\subseteq X\times\{0\}\cup Y\times\{1\} : \{y\in Y : (y,1)\in B\}\in\mathcal{J}\}\end{aligned}$$

are orthogonal.

Proof. Take $A = Y\times\{1\}\in\mathcal{I}\oplus\mathcal{P}(Y)$. Then $(X\times\{0\}\cup Y\times\{1\})\setminus A = X\times\{0\}\in\mathcal{P}(X)\oplus\mathcal{J}$. ☐

2.2. **$P(\mathcal{J})$-ideals.** Let $\mathcal{I},\mathcal{J}$ be ideals on X. We say that $\mathcal{I}$ is a *$P(\mathcal{J})$-ideal* if for every countable family $\{A_n : n\in\mathbb{N}\}\subseteq\mathcal{I}$ there exists $A\in\mathcal{I}$ such that $A_n\setminus A\in\mathcal{J}$ for any $n\in\mathbb{N}$. If $\mathcal{J}=\mathrm{Fin}(X)$ then $P(\mathcal{J})$-ideals are called *P-ideals.*

Example 2.3.
1. If $\mathcal{I}$ is a P-ideal, then $\mathcal{I}$ is a $P(\mathcal{J})$-ideal for every $\mathcal{J}$.
2. If $\mathcal{I}$ and $\mathcal{J}$ are orthogonal ideals, then $\mathcal{I}$ is a $P(\mathcal{J})$-ideal (and $\mathcal{J}$ is a $P(\mathcal{I})$-ideal).
3. If $\mathcal{I}\subseteq\mathcal{J}$, then $\mathcal{I}$ is a $P(\mathcal{J})$-ideal. (In particular, $\mathcal{I}$ is a $P(\mathcal{I})$-ideal for every $\mathcal{I}$.)

Proof. Straightforward. ☐

2.3. **Ideals with $\mathrm{W}(\mathcal{I},\mathcal{J})$ property.** Let $\mathcal{I},\mathcal{J}$ be ideals on X. By $\mathrm{W}(\mathcal{I},\mathcal{J})$ we denote the following sentence: For every partition $\{A_n : n\in\mathbb{N}\}\subseteq\mathcal{J}$ of X there exists $S\notin\mathcal{I}$ such that $A_n\cap S\in\mathcal{I}$ for every $n\in\mathbb{N}$.

Example 2.4.
1. If $\mathcal{I},\mathcal{J}$ are ideals such that $\mathcal{J}$ is a $P(\mathcal{I})$-ideal and $\mathcal{I},\mathcal{J}$ are not orthogonal, then $\mathrm{W}(\mathcal{I},\mathcal{J})$ holds.
2. If $\mathcal{I},\mathcal{J}$ are distinct maximal ideals, then $\mathcal{I}$ is a $P(\mathcal{J})$-ideal but $\mathrm{W}(\mathcal{I},\mathcal{J})$ does not hold.
3. $\mathrm{W}(\mathcal{I},\mathcal{I})$, $\mathrm{W}(\mathrm{Fin},\mathcal{J})$ and $\mathrm{W}(\mathcal{I},\mathrm{Fin})$ always hold.
4. If $\mathcal{J}\subseteq\mathcal{I}$, then $\mathrm{W}(\mathcal{I},\mathcal{J})$ holds.

Proof. Straightforward. ☐

Example 2.5. If $\mathcal{I}$ and $\mathcal{J}$ are orthogonal ideals on a countable set X, then $\mathrm{W}(\mathcal{I},\mathcal{J})$ does not hold.

Proof. Let $A\in\mathcal{J}$ be such that $X\setminus A\in\mathcal{I}$. Then $\{A\}\cup\{\{b\} : b\in X\setminus A\}$ is the required partition. ☐

The following example shows that Example 2.5 does not reverse.

Example 2.6. For ideals $\mathcal{I},\mathcal{J}$ on $\mathbb{N}$ we define ideals on $\mathbb{N}\times\mathbb{N}$ in the following way:

$$\begin{aligned}\emptyset\times\mathcal{I} &= \{A\subseteq\mathbb{N}\times\mathbb{N} : \{k : (n,k)\in A\}\in\mathcal{I}\text{ for every }n\in\mathbb{N}\},\\ \mathcal{J}\times\emptyset &= \{A\subseteq\mathbb{N}\times\mathbb{N} : \{n : \{k : (n,k)\in A\}\neq\emptyset\}\in\mathcal{J}\}.\end{aligned}$$

The ideals $\emptyset\times\mathcal{I}$ and $\mathcal{J}\times\emptyset$ are not orthogonal and $\mathrm{W}(\emptyset\times\mathcal{I},\mathcal{J}\times\emptyset)$ does not hold.

Proof. If $A\in\mathcal{J}\times\emptyset$, then there is $n\in\mathbb{N}$ such that $A\cap(\{n\}\times\mathbb{N})=\emptyset$, so $\mathbb{N}\times\mathbb{N}\setminus A\notin\emptyset\times\mathcal{I}$. Thus $\emptyset\times\mathcal{I}$ and $\mathcal{J}\times\emptyset$ are not orthogonal.

Now we show that $\mathrm{W}(\emptyset\times\mathcal{I},\mathcal{J}\times\emptyset)$ does not hold. Let $A_n=\{n\}\times\mathbb{N}\in\mathcal{J}\times\emptyset$ for $n\in\mathbb{N}$. Then $\{A_n : n\in\mathbb{N}\}$ is the required partition. Indeed, let $S\subseteq\mathbb{N}\times\mathbb{N}$ be such that $A_n\cap S\in\emptyset\times\mathcal{I}$ for every $n\in\mathbb{N}$. Then $\{k : (n,k)\in S\}\in\mathcal{I}$ for every $n\in\mathbb{N}$, hence $S\in\emptyset\times\mathcal{I}$. ☐

We say that an ideal $\mathcal{I}$ on X *contains an isomorphic copy* of an ideal $\mathcal{J}$ on Y if there is a bijection $\phi : X \to Y$ such that $\phi^{-1}[A] \in \mathcal{I}$ for every $A \in \mathcal{J}$.

Proposition 2.7. *Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. If $\mathrm{W}(\mathcal{I}, \mathcal{J})$ does not hold, then either $\mathcal{I}, \mathcal{J}$ are orthogonal or $\mathcal{I}$ contains an isomorphic copy of the ideal $\emptyset \times Fin$.*

Proof. Let $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{J}$ be a partition of $\mathbb{N}$ such that if $S \notin \mathcal{I}$, then $S \cap A_n \notin \mathcal{I}$ for some $n \in \mathbb{N}$. Let $A = \{n : A_n \text{ is infinite}\}$.

Suppose that A is finite. Let $B = \bigcup\{A_n : n \in A\} \in \mathcal{J}$. We will show that $\mathbb{N} \setminus B \in \mathcal{I}$ (so $\mathcal{I}$ and $\mathcal{J}$ are orthogonal). Suppose that $S = \mathbb{N} \setminus B \notin \mathcal{I}$. Then $S \cap A_n \notin \mathcal{I}$ for some $n \in \mathbb{N} \setminus A$. But if $n \in \mathbb{N} \setminus A$, then A_n is finite, hence $S \cap A_n$ is finite, a contradiction.

Now suppose that A is infinite. Let $A = \{a_n : n \in \mathbb{N}\}$ be an injective enumeration of A. Let $\phi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection such that

$$\phi\left[A_{a_0} \cup \bigcup\{A_n : n \in \mathbb{N} \setminus A\}\right] = \{0\} \times \mathbb{N}$$

and $\phi[A_{a_n}] = \{n\} \times \mathbb{N}$ for $n \geq 1$. We claim that $\phi^{-1}[B] \in \mathcal{I}$ for every $B \in \emptyset \times \mathrm{Fin}$ (and this will show that $\mathcal{I}$ contains an isomorphic copy of the ideal $\emptyset \times \mathrm{Fin}$).

Let $B \in \emptyset \times \mathrm{Fin}$. If $n \in \mathbb{N} \setminus A$, then $A_n \cap \phi^{-1}[B] \subseteq A_n \in \mathrm{Fin} \subseteq \mathcal{I}$. Now suppose that $n \in A$. There is $k \in \mathbb{N}$ such that $n = a_k$. Let $B_i = \{m : (i, m) \in B\} \in \mathrm{Fin}$ for every $i \in \mathbb{N}$. Then

$$A_n \cap \phi^{-1}[B] = A_n \cap \phi^{-1}\left[\bigcup_i \{i\} \times B_i\right] = A_n \cap \bigcup_i \phi^{-1}[\{i\} \times B_i] =$$

$$A_n \cap \phi^{-1}[\{k\} \times B_k] \subseteq \phi^{-1}[\{k\} \times B_k] \in \mathrm{Fin} \subseteq \mathcal{I}.$$

Thus $A_n \cap \phi^{-1}[B] \in \mathcal{I}$ for every $n \in \mathbb{N}$, so $\phi^{-1}[B] \in \mathcal{I}$. □

Of course, Proposition 2.7 does not reverse (for if $\mathcal{I} = \mathcal{J} = \emptyset \times \mathrm{Fin}$, then $\mathcal{I}$ contains an isomorphic copy of $\emptyset \times \mathrm{Fin}$, but $\mathrm{W}(\mathcal{I}, \mathcal{J})$ holds).

3. THE SENTENCE $\mathrm{B}(\mathcal{I}, \mathcal{J}, \kappa)$

Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$ and κ be a cardinal number. By $\mathrm{B}(\mathcal{I}, \mathcal{J}, \kappa)$ we denote the following sentence:

For every family $\{E_n^\alpha : n \in \mathbb{N}, \alpha < \kappa\} \subseteq \mathcal{I}$ such that $E_n^\alpha \cap E_k^\alpha = \emptyset$ for $n \neq k, \alpha < \kappa$ there exists a partition $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{J}$ of $\mathbb{N}$ such that

$$\bigcup_{n \in \mathbb{N}} \left(A_n \cap \bigcup_{i \leq n} E_i^\alpha\right) \in \mathcal{I}$$

for every $\alpha < \kappa$.

Proposition 3.1. (1) $\mathrm{B}(\mathcal{I}, \mathcal{J}, 0)$ *always holds.*
(2) *If $\kappa \leq \lambda$ then $\mathrm{B}(\mathcal{I}, \mathcal{J}, \lambda) \Rightarrow \mathrm{B}(\mathcal{I}, \mathcal{J}, \kappa)$.*
(3) *If $\lambda \geq \mathfrak{c}$ then $\mathrm{B}(\mathcal{I}, \mathcal{J}, \lambda) \iff \mathrm{B}(\mathcal{I}, \mathcal{J}, \mathfrak{c})$.*
(4) *$\mathrm{B}(\mathcal{I}, \mathcal{J}, \mathfrak{c}) \iff \mathrm{B}(\mathcal{I}, \mathcal{J}, \kappa)$ for every κ.*
(5) *If $\mathrm{B}(\mathcal{I}, \mathcal{J}, 1)$ holds then $\mathrm{B}(\mathcal{I}, \mathcal{J}, \kappa)$ holds for every $\kappa \leq \aleph_0$.*

Proof. Since (1), (2), (3) and (4) are easy to show, we will only prove (5).

By (2) it is enough to show that $B(\mathcal{I}, \mathcal{J}, \aleph_0)$ holds. Take a family $\{E_n^\alpha : n \in \mathbb{N}, \alpha < \aleph_0\} \subseteq \mathcal{I}$ such that $E_n^\alpha \cap E_k^\alpha = \emptyset$ for $n \neq k, \alpha < \aleph_0$.

For $n \in \mathbb{N}$ we define

$$F_n = \bigcup_{\alpha \leq n} \bigcup_{i \leq n} E_i^\alpha \setminus \bigcup_{i<n} F_i.$$

Then $F_n \in \mathcal{I}$ for $n \in \mathbb{N}$ and $F_n \cap F_k = \emptyset$ for $n \neq k$. Thus, by $\mathrm{B}(\mathcal{I}, \mathcal{J}, 1)$ there is a partition $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{J}$ of $\mathbb{N}$ such that

$$\bigcup_{n \in \mathbb{N}} \left(A_n \cap \bigcup_{i \leq n} F_i \right) \in \mathcal{I}.$$

Now we show that $\bigcup_{n \in \mathbb{N}} \left(A_n \cap \bigcup_{i \leq n} E_i^\alpha \right) \in \mathcal{I}$ for every $\alpha < \aleph_0$.

Let $\alpha < \aleph_0$. It is easy to see that $\bigcup_{i \leq n} E_i^\alpha \subseteq \bigcup_{i \leq n} F_i$ for every $n \geq \alpha$. Thus

$$\bigcup_{n \in \mathbb{N}} \left(A_n \cap \bigcup_{i \leq n} E_i^\alpha \right) \subseteq \bigcup_{n < \alpha} \left(A_n \cap \bigcup_{i \leq n} E_i^\alpha \right) \cup \bigcup_{n \geq \alpha} \left(A_n \cap \bigcup_{i \leq n} F_i \right) \in \mathcal{I}.$$

□

Proposition 3.2. *If* $\mathcal{J}_1 \subseteq \mathcal{J}_2$, *then* $\mathrm{B}(\mathcal{I}, \mathcal{J}_1, \kappa) \Rightarrow \mathrm{B}(\mathcal{I}, \mathcal{J}_2, \kappa)$ *for every* κ.

Proof. Straightforward. □

Example 3.3 **($\mathcal{I}_1 \subseteq \mathcal{I}_2$ and $\mathrm{B}(\mathcal{I}_1, \mathcal{J}, \kappa) \not\Rightarrow \mathrm{B}(\mathcal{I}_2, \mathcal{J}, \kappa)$).** Let $\mathcal{I}_1 = \mathcal{J} = \mathrm{Fin}$ and $\mathcal{I}_2$ be any non P-ideal. Then, by Proposition 3.7, $\mathrm{B}(\mathcal{I}_1, \mathcal{J}, 1)$ holds, whereas $\mathrm{B}(\mathcal{I}_2, \mathcal{J}, 1)$ does not hold.

Example 3.4 **($\mathcal{I}_1 \subseteq \mathcal{I}_2$ and $\mathrm{B}(\mathcal{I}_2, \mathcal{J}, \kappa) \not\Rightarrow \mathrm{B}(\mathcal{I}_1, \mathcal{J}, \kappa)$).** Let $\mathcal{I}_1 = \mathrm{Fin}$ and $\mathcal{I}_2, \mathcal{J}$ be two distinct maximal ideals. Then, by Example 2.4(2), $\mathrm{W}(\mathcal{I}_2, \mathcal{J})$ does not hold, hence, by Proposition 3.6, $\mathrm{B}(\mathcal{I}_2, \mathcal{J}, \mathfrak{c})$ holds. On the other hand, by Proposition 3.12, $\mathrm{B}(\mathcal{I}_1, \mathcal{J}, \mathfrak{c})$ does not hold.

Example 3.5 **($\mathcal{J}_1 \subseteq \mathcal{J}_2$ and $\mathrm{B}(\mathcal{I}, \mathcal{J}_2, \kappa) \not\Rightarrow \mathrm{B}(\mathcal{I}, \mathcal{J}_1, \kappa)$).** Let $\mathcal{I}$ be any non P-ideal. Let $\mathcal{J}_1 = \mathrm{Fin}$ and $\mathcal{J}_2 = \mathcal{I}$. Then, by Proposition 3.7, $\mathrm{B}(\mathcal{I}, \mathcal{J}_2, 1)$ holds, whereas $\mathrm{B}(\mathcal{I}, \mathcal{J}_1, 1)$ does not hold.

Proposition 3.6. $\mathrm{B}(\mathcal{I}, \mathcal{J}, \kappa)$ *holds for every* κ $\iff$ $\mathrm{W}(\mathcal{I}, \mathcal{J})$ *does not hold.*

Proof. ($\Rightarrow$) Let $\{\{E_n^\alpha : n \in \mathbb{N}\} : \alpha < \mathfrak{c}\}$ be an enumeration of all families $\{E_n : n \in \mathbb{N}\} \subseteq \mathcal{I}$ of disjoint sets. By $\mathrm{B}(\mathcal{I}, \mathcal{J}, \mathfrak{c})$ there is a partition $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{J}$ of $\mathbb{N}$ such that $\bigcup_{n \in \mathbb{N}} \left(A_n \cap \bigcup_{i \leq n} E_i^\alpha \right) \in \mathcal{I}$ for every $\alpha < \mathfrak{c}$.

Now we show that $\mathrm{W}(\mathcal{I}, \mathcal{J})$ does not hold. Let $S \subseteq \mathbb{N}$ be such that $A_n \cap S \in \mathcal{I}$ for every $n \in \mathbb{N}$. Let $\alpha < \mathfrak{c}$ be such that $E_n^\alpha = A_n \cap S$ for every $n \in \mathbb{N}$. Then

$$S = \bigcup_{n \in \mathbb{N}} (A_n \cap E_n^\alpha) \subseteq \bigcup_{n \in \mathbb{N}} \left(A_n \cap \bigcup_{i \leq n} E_i^\alpha \right) \in \mathcal{I}.$$

($\Leftarrow$) By $\neg \mathrm{W}(\mathcal{I}, \mathcal{J})$ there is a partition $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{J}$ of $\mathbb{N}$ such that for every $S \notin \mathcal{I}$ there is $n \in \mathbb{N}$ with $A_n \cap S \notin \mathcal{I}$.

For a cardinal number κ take a family $\{E_n^\alpha : n \in \mathbb{N}, \alpha < \kappa\} \subseteq \mathcal{I}$ such that $E_n^\alpha \cap E_k^\alpha = \emptyset$ for $n \neq k, \alpha < \kappa$. To finish the proof it is enough to show that

$$\bigcup_{n \in \mathbb{N}} \left(A_n \cap \bigcup_{i \leq n} E_i^\alpha \right) \in \mathcal{I}$$

for every $\alpha < \kappa$.

Suppose that $S = \bigcup_{n\in\mathbb{N}} \left(A_n \cap \bigcup_{i\le n} E_i^\alpha\right) \notin \mathcal{I}$ for some $\alpha < \kappa$. By $\neg\,\mathrm{W}(\mathcal{I},\mathcal{J})$ there exists $n_0 \in \mathbb{N}$ such that $A_{n_0} \cap S \notin \mathcal{I}$. On the other hand $A_{n_0} \cap S = A_{n_0} \cap \bigcup_{i\le n_o} E_i^\alpha \in \mathcal{I}$, a contradiction. ☐

Proposition 3.7. $\mathrm{B}(\mathcal{I},\mathcal{J},1) \iff \mathcal{I}$ *is a* $P(\mathcal{J})$*-ideal.*

Proof. ($\Rightarrow$) Let $E_n \in \mathcal{I}$ for $n \in \mathbb{N}$. Without loss of generality we can assume that $E_n \cap E_k = \emptyset$ for $n \neq k$. Since $\mathrm{B}(\mathcal{I},\mathcal{J},1)$ holds, so there is a partition $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{J}$ of $\mathbb{N}$ such that

$$E = \bigcup_{n\in\mathbb{N}} \left(A_n \cap \bigcup_{i\le n} E_i\right) \in \mathcal{I}.$$

We will show that $E_n \setminus E \in \mathcal{J}$ for every $n \in \mathbb{N}$ (hence $\mathcal{I}$ is a $P(\mathcal{J})$-ideal).

Let $l \in E_n \setminus E$. Let $m \in \mathbb{N}$ be such that $l \in A_m$. Since $l \notin E$, so $l \notin \bigcup_{i\le m} E_i$. Thus $m < n$. Hence $E_n \setminus E \subseteq \bigcup_{i<n} A_i \in \mathcal{J}$.

($\Leftarrow$) Let $\{E_n : n \in \mathbb{N}\} \subseteq \mathcal{I}$ be a disjoint family. Since $\mathcal{I}$ is a $P(\mathcal{J})$-ideal, so there is $E \in \mathcal{I}$ with $E_n \setminus E \in \mathcal{J}$ for every $n \in \mathbb{N}$. Let $\mathbb{N} \setminus (\bigcup_{n\ge1} E_n \setminus E) = \{l_n : n \in \mathbb{N}\}$. Let $A_n = (E_{n+1} \setminus E) \cup \{l_n\} \in \mathcal{J}$ for $n \in \mathbb{N}$. Then $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{J}$ is a partition of $\mathbb{N}$ and

$$\bigcup_{n\in\mathbb{N}} \left(A_n \cap \bigcup_{i\le n} E_i\right) = \bigcup_{n\in\mathbb{N}} \left(((E_{n+1} \setminus E) \cup \{l_n\}) \cap \bigcup_{i\le n} E_i\right)$$

$$= \bigcup_{n\in\mathbb{N}} (\{l_n\} \cap \bigcup_{i\le n} E_i) \subseteq E_0 \cup E \in \mathcal{I},$$

hence $\mathrm{B}(\mathcal{I},\mathcal{J},1)$ holds. ☐

Corollary 3.8. $\mathrm{B}(\mathcal{I},\mathit{Fin},1) \iff \mathcal{I}$ *is a P-ideal.*

Corollary 3.9. $\mathrm{B}(\mathcal{I},\mathcal{J},\kappa)$ *holds for every* $\kappa \le \aleph_0 \iff \mathcal{I}$ *is a* $P(\mathcal{J})$*-ideal.*

Proof. Apply Proposition 3.7 and Proposition 3.1(5). ☐

Proposition 3.10. *If* $\mathcal{I}$ *is a* $P(\mathcal{J})$*-ideal, then* $\mathrm{B}(\mathcal{I},\mathcal{I},\kappa) \Rightarrow \mathrm{B}(\mathcal{I},\mathcal{J},\kappa)$ *for every* κ.

Proof. Let $\{E_n^\alpha : n \in \mathbb{N}, \alpha < \kappa\} \subseteq \mathcal{I}$ be such that $E_n^\alpha \cap E_k^\alpha = \emptyset$ for $n \neq k, \alpha < \kappa$. By $\mathrm{B}(\mathcal{I},\mathcal{I},\kappa)$ there exists a partition $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{I}$ of $\mathbb{N}$ such that

$$\bigcup_{n\in\mathbb{N}} \left(A_n \cap \bigcup_{i\le n} E_i^\alpha\right) \in \mathcal{I}$$

for every $\alpha < \kappa$. Since $\mathcal{I}$ is a $P(\mathcal{J})$-ideal, so there exists $A = \{a_n : n \in \mathbb{N}\} \in \mathcal{I}$ such that $A_n \setminus A \in \mathcal{J}$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ define $B_n = (A_n \setminus A) \cup \{a_n\}$. Clearly $\{B_n : n \in \mathbb{N}\} \subseteq \mathcal{J}$ is a partition of $\mathbb{N}$.

Let $\alpha < \kappa$. Then

$$\bigcup_{n\in\mathbb{N}} \left(B_n \cap \bigcup_{i\le n} E_i^\alpha\right) = \bigcup_{n\in\mathbb{N}} \left(((A_n \setminus A) \cup \{a_n\}) \cap \bigcup_{i\le n} E_i^\alpha\right)$$

$$\subseteq A \cup \bigcup_{n\in\mathbb{N}} \left(A_n \cap \bigcup_{i\leq n} E_i^\alpha \right) \in \mathcal{I}.$$

Hence $\mathrm{B}(\mathcal{I},\mathcal{J},\kappa)$ holds. ☐

Corollary 3.11. *If $\mathcal{I}$ is a P-ideal, then $\mathrm{B}(\mathcal{I},\mathcal{I},\kappa) \iff \mathrm{B}(\mathcal{I},Fin,\kappa)$ for every κ.*

Proof. It follows from Proposition 3.10 and Proposition 3.2. ☐

For $f,g:\mathbb{N}\to\mathbb{N}$ we write $f\leq^* g$ if $\{n : f(n) > g(n)\}$ is finite. The *bounding number* $\mathfrak{b}$ is defined in the following way

$$\mathfrak{b} = \min\left\{|\mathcal{F}| : \mathcal{F}\subseteq \mathbb{N}^{\mathbb{N}} \wedge \neg\left(\exists g\in\mathbb{N}^{\mathbb{N}} \forall f\in\mathcal{F}\,(f\leq^* g)\right)\right\},$$

and it is known that $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{c}$ (see e.g. [5, p. 196]).

Proposition 3.12. *If $\mathcal{J}$ is an arbitrary ideal, then $\mathrm{B}(Fin,\mathcal{J},\kappa) \iff \kappa < \mathfrak{b}$.*

Proof. ($\Rightarrow$) Suppose, for the sake of contradiction, that $\kappa\geq\mathfrak{b}$. Let $\{f_\alpha\in\mathbb{N}^{\mathbb{N}} : \alpha < \mathfrak{b}\}$ be such that there is no $g:\mathbb{N}\to\mathbb{N}$ such that $f_\alpha \leq^* g$ for every $\alpha<\mathfrak{b}$. Without loss of generality we can assume that each f_α is strictly increasing. For every $\alpha<\mathfrak{b}$ we define $E_0^\alpha = \{i\in\mathbb{N} : i\leq f_\alpha(1)\}$, and $E_n^\alpha = \{i\in\mathbb{N} : f_\alpha(n) < i \leq f_\alpha(n+1)\}$ for $n\geq 1$. By Proposition 3.1(2), $\mathrm{B}(\mathrm{Fin},\mathcal{J},\mathfrak{b})$ holds, so there is a partition $\{A_n : n\in\mathbb{N}\}\subseteq\mathcal{J}$ of $\mathbb{N}$ such that

$$\bigcup_{n\in\mathbb{N}} \left(A_n \cap \bigcup_{i\leq n} E_i^\alpha \right) \in \mathrm{Fin}$$

for every $\alpha<\mathfrak{b}$. Since the sets A_n are pairwise disjoint, so for every $\alpha<\mathfrak{b}$ there is $N_\alpha\in\mathbb{N}$ such that $A_n\cap\bigcup_{i\leq n}E_i^\alpha=\emptyset$ for every $n>N_\alpha$. Let $(k_n)_n$ be an increasing sequence of all indices such that A_{k_n} are nonempty. For every $n\in\mathbb{N}$ choose a point $a_{k_n}\in A_{k_n}$ and define the function $g:\mathbb{N}\to\mathbb{N}$ by $g(n)=a_{k_n}$. We claim that $f_\alpha\leq^* g$ for every $\alpha<\mathfrak{b}$ (and this is a contradiction with the choice of f_α).

Let $\alpha<\mathfrak{b}$. Since $\bigcup_{i\leq n}E_i^\alpha = \{i\in\mathbb{N} : i\leq f_\alpha(n+1)\}$ and $A_n\cap\bigcup_{i\leq n}E_i^\alpha=\emptyset$ for every $n>N_\alpha$, so $A_{k_n}\cap\bigcup_{i\leq n}E_i^\alpha=\emptyset$ for every $n>N_\alpha$, hence $g(n) > f_\alpha(n+1) > f_\alpha(n)$ for every $n>N_\alpha$.

($\Leftarrow$) Let $\kappa<\mathfrak{b}$. Let $\{E_n^\alpha : n\in\mathbb{N}, \alpha<\kappa\}\subseteq\mathrm{Fin}$ be such that $E_n^\alpha\cap E_k^\alpha=\emptyset$ for $n\neq k, \alpha<\kappa$. For every $\alpha<\kappa$ we define $f_\alpha:\mathbb{N}\to\mathbb{N}$ by $f_\alpha(n)=\max\left(\bigcup_{i\leq n}E_i^\alpha\cup\{0\}\right)$. Since $\kappa<\mathfrak{b}$, so there exists $g:\mathbb{N}\to\mathbb{N}$ such that $f_\alpha\leq^* g$ for every $\alpha<\kappa$. Without loss of generality we can assume that g is strictly increasing. Let $A_0=\{i\in\mathbb{N} : i\leq g(1)\}$ and $A_n=\{i\in\mathbb{N} : g(n)<i\leq g(n+1)\}$ for $n\geq 1$. Then $\{A_n : n\in\mathbb{N}\}$ is a partition of $\mathbb{N}$ and $A_n\in\mathrm{Fin}\subseteq\mathcal{J}$. We claim that

$$\bigcup_{n\in\mathbb{N}} \left(A_n \cap \bigcup_{i\leq n} E_i^\alpha \right) \in \mathrm{Fin}$$

for every $\alpha<\kappa$ (and this will show that $\mathrm{B}(\mathrm{Fin},\mathcal{J},\kappa)$ holds).

Let $\alpha<\kappa$. Since $f_\alpha\leq^* g$, so there exists $N\in\mathbb{N}$ such that $f_\alpha(n)\leq g(n)$ for every $n\geq N$. Since $f_\alpha(n)=\max\left(\bigcup_{i\leq n}E_i^\alpha\cup\{0\}\right)$ and $g(n)<\min A_n$, so

$$A_n\cap\bigcup_{i\leq n}E_i^\alpha=\emptyset$$

for $n \geq N$. Thus

$$\bigcup_{n\in\mathbb{N}} \left(A_n \cap \bigcup_{i\leq n} E_i^\alpha \right) = \bigcup_{n<N} \left(A_n \cap \bigcup_{i\leq n} E_i^\alpha \right) \in \mathrm{Fin}.$$

□

Proposition 3.13. *If $\mathcal{I}$ is a P-ideal and $\mathcal{J}$ is an arbitrary ideal, then* $\mathrm{B}(\mathcal{I},\mathcal{J},\kappa)$ *holds for every $\kappa < \mathfrak{b}$.*

Proof. Let $\kappa < \mathfrak{b}$. Let $\{E_n^\alpha : n\in\mathbb{N}, \alpha<\kappa\} \subseteq \mathcal{I}$ be such that $E_n^\alpha \cap E_k^\alpha = \emptyset$ for $n\neq k, \alpha<\kappa$. Since $\mathcal{I}$ is a P-ideal, so for every $\alpha<\kappa$ there is $E_\alpha \in \mathcal{I}$ such that $E_n^\alpha \setminus E_\alpha$ is finite for every $n\in\mathbb{N}$. For every $\alpha<\kappa$ we define $f_\alpha : \mathbb{N}\to\mathbb{N}$ by $f_\alpha(n) = \max\left(\bigcup_{i\leq n} E_i^\alpha \setminus E_\alpha \cup \{0\}\right)$. Since $\kappa<\mathfrak{b}$, so there exists $g:\mathbb{N}\to\mathbb{N}$ such that $f_\alpha \leq^* g$ for every $\alpha<\kappa$. Without loss of generality we can assume that g is strictly increasing. Let $A_0 = \{i\in\mathbb{N} : i\leq g(1)\}$ and $A_n = \{i\in\mathbb{N} : g(n) < i \leq g(n+1)\}$ for $n\geq 1$. Then $\{A_n : n\in\mathbb{N}\}$ is a partition of $\mathbb{N}$ and $A_n \in \mathrm{Fin} \subseteq \mathcal{J}$. We claim that

$$\bigcup_{n\in\mathbb{N}} \left(A_n \cap \bigcup_{i\leq n} E_i^\alpha \right) \in \mathcal{I}$$

for every $\alpha<\kappa$ (and this will show that $\mathrm{B}(\mathcal{I},\mathrm{Fin},\kappa)$ holds).

Let $\alpha<\kappa$. Since $f_\alpha \leq^* g$, so there exists $N\in\mathbb{N}$ such that $f_\alpha(n)\leq g(n)$ for every $n\geq N$. Since $f_\alpha(n) = \max\left(\bigcup_{i\leq n} E_i^\alpha \setminus E_\alpha \cup \{0\}\right)$ and $g(n) < \min A_n$, so

$$A_n \cap \left(\bigcup_{i\leq n} E_i^\alpha \setminus E_\alpha \right) = \emptyset$$

for $n\geq N$. Thus

$$\bigcup_{n\in\mathbb{N}} \left(A_n \cap \bigcup_{i\leq n} E_i^\alpha \right) \subseteq \bigcup_{n\in\mathbb{N}} \left(A_n \cap \left(\bigcup_{i\leq n} E_i^\alpha \setminus E_\alpha \right)\right) \cup E_\alpha$$

$$= \bigcup_{n<N} \left(A_n \cap \left(\bigcup_{i\leq n} E_i^\alpha \setminus E_\alpha \right)\right) \cup E_\alpha \in \mathcal{I}.$$

□

4. THE CARDINAL $\mathfrak{b}(\mathcal{I},\mathcal{J})$

Let $\mathcal{I},\mathcal{J}$ be ideals on $\mathbb{N}$. We define the following cardinal coefficient

$$\mathfrak{b}(\mathcal{I},\mathcal{J}) = \min\left(\{\mathfrak{c}^+\} \cup \{\kappa : \mathrm{B}(\mathcal{I},\mathcal{J},\kappa) \text{ does not hold}\}\right).$$

Proposition 4.1. (1) $\mathfrak{b}(\mathcal{I},\mathcal{J}) = 1$ *or* $\mathfrak{b}(\mathcal{I},\mathcal{J}) \geq \aleph_1$.

(2) $\mathfrak{b}(\mathcal{I},\mathcal{J}) = 1 \iff \mathcal{I}$ *is not a $P(\mathcal{J})$-ideal.*

(3) $\mathfrak{b}(\mathcal{I},\mathcal{J}) \geq \aleph_1 \iff \mathcal{I}$ *is a $P(\mathcal{J})$-ideal.*

(4) $\mathfrak{b}(\mathcal{I},\mathcal{J}) = \mathfrak{c}^+ \iff \mathrm{W}(\mathcal{I},\mathcal{J})$ *does not hold.*

(5) $\mathfrak{b}(\mathcal{I},\mathcal{J}) \leq \mathfrak{c} \iff \mathrm{W}(\mathcal{I},\mathcal{J})$ *holds.*

(6) $\aleph_1 \leq \mathfrak{b}(\mathcal{I},\mathcal{J}) \leq \mathfrak{c} \iff \mathcal{I}$ *is a $P(\mathcal{J})$-ideal and* $\mathrm{W}(\mathcal{I},\mathcal{J})$ *holds.*

Proof. (1) follows from Proposition 3.1(1) and Proposition 3.1(5). (2) follows from Proposition 3.7 and Proposition 3.1(1). (3) follows from Corollary 3.9. (4) follows from Proposition 3.6 and Proposition 3.1(3). (5) follows from (4). (6) follows from (3) and (4). ☐

Proposition 4.2. *If* $\mathcal{J}_1 \subseteq \mathcal{J}_2$*, then* $\mathfrak{b}(\mathcal{I},\mathcal{J}_1) \leq \mathfrak{b}(\mathcal{I},\mathcal{J}_2)$*.*

Proof. It follows from Proposition 3.2. ☐

Proposition 4.3. *If* $\mathcal{I}$ *is a* $P(\mathcal{J})$*-ideal, then* $\mathfrak{b}(\mathcal{I},\mathcal{I}) \leq \mathfrak{b}(\mathcal{I},\mathcal{J})$*.*

Proof. It follows from Proposition 3.10. ☐

Corollary 4.4. *If* $\mathcal{I}$ *is a* P*-ideal, then (1)* $\mathfrak{b} \leq \mathfrak{b}(\mathcal{I},\mathcal{J})$*, and (2)* $\mathfrak{b} \leq \mathfrak{b}(\mathcal{I},\mathit{Fin}) = \mathfrak{b}(\mathcal{I},\mathcal{I}) \leq \mathfrak{c}$*.*

Proof. (1) follows from Proposition 3.13. The first inequality in (2) follows from Proposition 3.13, the equality follows from Corollary 3.11, and the second inequality follows from Proposition 4.1(5) and the fact that $\mathrm{W}(\mathcal{I},\mathcal{I})$ holds. ☐

Proposition 4.5. $\mathfrak{b}(\mathit{Fin},\mathcal{J}) = \mathfrak{b}$ *for every ideal* $\mathcal{J}$*. (In particular* $\mathfrak{b}(\mathit{Fin},\mathit{Fin}) = \mathfrak{b}$*.)*

Proof. It follows from Proposition 3.12. ☐

Corollary 4.6. *Assume the Martin's axiom.*

1. $\mathfrak{b}(\mathit{Fin},\mathcal{J}) = \mathfrak{c}$ *for every ideal* $\mathcal{J}$*.*
2. *If* $\mathcal{I}$ *is a* P*-ideal, then* $\mathfrak{b}(\mathcal{I},\mathit{Fin}) = \mathfrak{b}(\mathcal{I},\mathcal{I}) = \mathfrak{c}$*.*
3. *If* $\mathcal{I}$ *is a* P*-ideal and* $\mathrm{W}(\mathcal{I},\mathcal{J})$ *holds, then* $\mathfrak{b}(\mathcal{I},\mathcal{J}) = \mathfrak{c}$*.*

Proof. First recall that $\mathfrak{b} = \mathfrak{c}$ under the Martin's axiom (see e.g. [5, p. 376]). Now, (1) follows from Proposition 4.5, (2) follows from Corollary 4.4, and (3) follows from (2) and Propositions 4.3 and 4.1 ☐

Remark. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. For $f, g : \mathbb{N} \to \mathbb{N}$ we write $f \leq^{\mathcal{I}} g$ if $\{n : f(n) > g(n)\} \in \mathcal{I}$. In [12] the authors introduced an ideal version of the bounding number $\mathfrak{b}(\mathcal{I})$ in the following manner

$$\mathfrak{b}(\mathcal{I}) = \min\left\{|\mathcal{F}| : \mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}} \wedge \neg\left(\exists g \in \mathbb{N}^{\mathbb{N}} \forall f \in \mathcal{F}\, (f \leq^{\mathcal{I}} g)\right)\right\}.$$

It is easy to see that $\mathfrak{b} \leq \mathfrak{b}(\mathcal{I})$. In [12], the authors proved that if $\mathcal{I} \leq_{RB} \mathcal{J}$ (where $\leq_{RB}$ is the Rudin-Blass preorder) then $\mathfrak{b}(\mathcal{I}) = \mathfrak{b}(\mathcal{J})$, and, as a corollary, they obtained that if $\mathcal{I}$ is an analytic P-ideal, then $\mathfrak{b}(\mathcal{I}) = \mathfrak{b}$. By Proposition 4.5, $\mathfrak{b}(\mathrm{Fin},\mathcal{J}) = \mathfrak{b} \leq \mathfrak{b}(\mathcal{J})$, so $\mathfrak{b}(\mathrm{Fin},\mathcal{J}) = \mathfrak{b}(\mathcal{J})$ for analytic P-ideals.

5. Pointwise convergence versus equal convergence

Theorem 5.1. *Let* $\mathcal{I},\mathcal{J}$ *be ideals on* $\mathbb{N}$*. For every set* X *the following are equivalent.*

1. $\mathrm{B}(\mathcal{I},\mathcal{J},|X|)$ *holds.*
2. *For every sequence* (f_n) *of real-valued functions defined on* X*, if* $(f_n) \xrightarrow{\mathcal{I}} f$ *then* $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$*.*

Proof. (1) $\Rightarrow$ (2) Let $(f_n) \xrightarrow{\mathcal{I}} f$. Let $X = \{x_\alpha : \alpha < \kappa\}$, where $\kappa = |X|$. For every $\alpha < \kappa$ and $n \in \mathbb{N}$ define $\varepsilon_n^\alpha = |f_n(x_\alpha) - f(x_\alpha)| + 1/n$. Let $E_0^\alpha = \{n : \varepsilon_n^\alpha \geq 1\}$ and $E_k^\alpha = \{n : 1/(k+1) \leq \varepsilon_n^\alpha < 1/k\}$ for $k \geq 1$. Note that $E_k^\alpha \in \mathcal{I}$ and $E_k^\alpha \cap E_l^\alpha = \emptyset$ for $k \neq l$. Since $\mathrm{B}(\mathcal{I}, \mathcal{J}, \kappa)$ holds, so there exists a partition $\{A_k : k \in \mathbb{N}\} \subseteq \mathcal{J}$ of $\mathbb{N}$ such that $B_\alpha = \bigcup_{k\in\mathbb{N}} \left(A_k \cap \bigcup_{i\leq k} E_i^\alpha\right) \in \mathcal{I}$ for every $\alpha < \kappa$. Let $\eta_n = 1/(k+1)$ for $n \in A_k$, $k \in \mathbb{N}$. Then $(\eta_n) \xrightarrow{\mathcal{J}} 0$. We will show that $\{n \in \mathbb{N} : |f_n(x_\alpha) - f(x_\alpha)| \geq \eta_n\} \subseteq B_\alpha \in \mathcal{I}$ for every $\alpha < \kappa$ (and this shows that $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$).

Let $n \in \mathbb{N}$ be such that $|f_n(x_\alpha) - f(x_\alpha)| \geq \eta_n$. If $n \in B_\alpha$ then we are done. Suppose that $n \in \mathbb{N}\setminus B_\alpha$. Let $k_0 \in \mathbb{N}$ be such that $n \in A_{k_0}$ (hence $\eta_n = 1/(k_0+1)$). Then $n \notin \bigcup_{i\leq k_0} E_i^\alpha$, so $\varepsilon_n^\alpha < 1/(k_0+1) = \eta_n$. Thus $|f_n(x_\alpha) - f(x_\alpha)| \geq \varepsilon_n^\alpha$, a contradiction.

(2) $\Rightarrow$ (1) Let $X = \{x_\alpha : \alpha < \kappa\}$, where $\kappa = |X|$. Let $\{E_n^\alpha : n \in \mathbb{N}, \alpha < \kappa\} \subseteq \mathcal{I}$ be such that $E_n^\alpha \cap E_k^\alpha = \emptyset$ for $n \neq k, \alpha < \kappa$.

We define functions $f_n : X \to \mathbb{R}$ $(n \in \mathbb{N})$ by

$$f_n(x) = \begin{cases} \frac{1}{k+1} & \text{if } n \in E_k^\alpha \text{ and } x = x_\alpha \text{ for some } \alpha < \kappa, k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : X \to \mathbb{R}$ be given by $f(x) = 0$ for all $x \in X$. It is easy to see that $(f_n) \xrightarrow{\mathcal{I}} f$. Thus $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$, so there exists a sequence of positive reals $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$ such that

$$\{n \in \mathbb{N} : |f_n(x_\alpha) - f(x_\alpha)| \geq \varepsilon_n\} \in \mathcal{I}$$

for every $\alpha < \kappa$.

Let $A_0 = \{n : \varepsilon_n \geq 1/2\}$ and $A_k = \{n : 1/(k+2) \leq \varepsilon_n < 1/(k+1)\}$ for $k \geq 1$. Note that $A_k \in \mathcal{J}$, $\bigcup_{k\in\mathbb{N}} A_k = \mathbb{N}$ and $A_n \cap A_m = \emptyset$ if $n \neq m$.

We claim that

$$\bigcup_{k\in\mathbb{N}} \left(A_k \cap \bigcup_{i\leq k} E_i^\alpha\right) \in \mathcal{I}$$

for every $\alpha < \kappa$ (and that will show that $\mathrm{B}(\mathcal{I}, \mathcal{J}, |X|)$ holds).

Since $\bigcup_{k\in\mathbb{N}} \left(A_k \cap \bigcup_{i\leq k} E_i^\alpha\right) = (A_0 \cap E_0^\alpha) \cup \bigcup_{k\geq 1} \left(A_k \cap \bigcup_{i\leq k} E_i^\alpha\right)$ and $A_0 \cap E_0^\alpha \subseteq E_0^\alpha \in \mathcal{I}$, so it is enough to show that $\bigcup_{k\geq 1} \left(A_k \cap \bigcup_{i\leq k} E_i^\alpha\right) \in \mathcal{I}$.

Let $n \in \bigcup_{k\geq 1} \left(A_k \cap \bigcup_{i\leq k} E_i^\alpha\right)$. Let $k \geq 1$ and $i \leq k$ be such that $n \in A_k \cap E_i^\alpha$. Then $n \in E_i^\alpha$ implies that $f_n(x_\alpha) = 1/(i+1)$. On the other hand, $n \in A_k$ implies that $\varepsilon_n < 1/(k+1)$. Since $i \leq k$, so $|f_n(x_\alpha) - f(x_\alpha)| = 1/(i+1) \geq 1/(k+1) > \varepsilon_n$. Thus

$$\bigcup_{k\geq 1} \left(A_k \cap \bigcup_{i\leq k} E_i^\alpha\right) \subseteq \{n \in \mathbb{N} : |f_n(x_\alpha) - f(x_\alpha)| \geq \varepsilon_n\} \in \mathcal{I}.$$

□

Theorem 5.2. *Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$ such that $\mathrm{W}(\mathcal{I}, \mathcal{J})$ does not hold. For every set X and every sequence (f_n) of real-valued functions defined on X, if $(f_n) \xrightarrow{\mathcal{I}} f$ then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$.*

Proof. It follows from Proposition 3.6 and Theorem 5.1. □

Theorem 5.3. *Let $\mathcal{I},\mathcal{J}$ be ideals on $\mathbb{N}$ such that $\mathrm{W}(\mathcal{I},\mathcal{J})$ holds. The following are equivalent.*

1. $|X| < \mathfrak{b}(\mathcal{I},\mathcal{J})$.
2. *For every sequence (f_n) of real-valued functions defined on X, if $(f_n) \xrightarrow{\mathcal{I}} f$ then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$.*

Proof. (1) ⇒ (2) If $|X| < \mathfrak{b}(\mathcal{I},\mathcal{J})$, then $\mathrm{B}(\mathcal{I},\mathcal{J},|X|)$ holds. Now it is enough to apply Theorem 5.1.

(2) ⇒ (1) By Theorem 5.1, $\mathrm{B}(\mathcal{I},\mathcal{J},|X|)$ holds. We have two cases. If $|X| \leq \mathfrak{c}$, then $\mathfrak{b}(\mathcal{I},\mathcal{J}) \geq |X|^+ > |X|$. Now we show that the second case, $|X| > \mathfrak{c}$, cannot happen. If $|X| > \mathfrak{c}$, then $\mathfrak{b}(\mathcal{I},\mathcal{J}) = \mathfrak{c}^+$, hence $|X| < \mathfrak{c}^+$, a contradiction. ☐

Corollary 5.4. *Let $\mathcal{I},\mathcal{J}$ be ideals on $\mathbb{N}$ such that $\mathrm{W}(\mathcal{I},\mathcal{J})$ holds. For every set X with $|X| \geq \mathfrak{c}$ there exists a sequence of real-valued functions defined on X such that $(f_n) \xrightarrow{\mathcal{I}} f$ and $\neg((f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f)$.*

Proof. If $\mathrm{W}(\mathcal{I},\mathcal{J})$ holds, then, by Proposition 4.1, $\mathfrak{b}(\mathcal{I},\mathcal{J}) \leq \mathfrak{c}$. Now apply Theorem 5.3. ☐

Corollary 5.5. *Let X be a nonempty set. Let $\mathcal{I},\mathcal{J}$ be ideals on $\mathbb{N}$ such that $\mathcal{I}$ is not a $P(\mathcal{J})$-ideal. There exists a sequence of real-valued functions defined on X such that $(f_n) \xrightarrow{\mathcal{I}} f$ and $\neg((f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f)$.*

Proof. By Proposition 3.7, $\mathrm{B}(\mathcal{I},\mathcal{J},1)$ does not hold, so by Proposition 3.1(2), $\mathrm{B}(\mathcal{I},\mathcal{J},\kappa)$ does not hold for every $\kappa \geq 1$. Now apply Theorem 5.1. ☐

Corollary 5.6 ([13, Example 4.7])**.** *Let $|X| \geq \mathfrak{c}$. Let $\mathcal{I},\mathcal{J}$ be ideals on $\mathbb{N}$ such that $\mathcal{J} \subseteq \mathcal{I}$. There exists a sequence of real-valued functions defined on X such that $(f_n) \xrightarrow{\mathcal{I}} f$ and $\neg((f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f)$.*

Proof. By Example 2.4(4), if $\mathcal{J} \subseteq \mathcal{I}$, then $\mathrm{W}(\mathcal{I},\mathcal{J})$ holds. Now apply Corollary 5.4. ☐

Corollary 5.7. *Let $|X| < \mathfrak{b}$. Let $\mathcal{I}$ be an P-ideal on $\mathbb{N}$. For every sequence (f_n) of real-valued functions defined on X, if $(f_n) \xrightarrow{\mathcal{I}} f$ then $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$.*

Proof. It follows from Theorem 5.3 and Corollary 4.4. ☐

Corollary 5.8. *The following are equivalent.*

1. $|X| < \mathfrak{b}$.
2. *For every sequence (f_n) of real-valued functions defined on X, if $(f_n) \to f$ then $(f_n) \xrightarrow{e} f$.*

Proof. Take $\mathcal{I} = \mathcal{J} = \mathrm{Fin}$ and apply Theorem 5.3. ☐

Remark. The implication "(1) ⇒ (2)" in Corollary 5.8 also easily follows from [2, Theorem 1.8].

Corollary 5.9. *Let $\mathcal{I},\mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

1. *For every sequence of real-valued functions defined on X, if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ then $(f_n) \xrightarrow{\mathcal{I}} f$, and there exists a sequence of real-valued functions defined on X such that $(f_n) \xrightarrow{\mathcal{I}} f$ and $\neg((f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f)$.*

(2) $\mathcal{J} \subseteq \mathcal{I}$ *and* $|X| \geq \mathfrak{b}(\mathcal{I},\mathcal{J})$.

Proof. Apply Theorems 1.1, 5.3 and note that, by Example 2.4(4), $\mathcal{J} \subseteq \mathcal{I}$ implies that $\mathrm{W}(\mathcal{I},\mathcal{J})$ holds. □

6. The uniqueness of ideal equal limits

Theorem 6.1. *Let $\mathcal{I},\mathcal{J}$ be ideals on $\mathbb{N}$. Let X be a nonempty set. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on X, if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ and $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} g$, then $f = g$.*

(2) *The ideals $\mathcal{I},\mathcal{J}$ are not orthogonal.*

Proof. (1) ⇒ (2) Suppose that the ideals $\mathcal{I},\mathcal{J}$ are orthogonal. Let $A \in \mathcal{I}$ with $\mathbb{N} \setminus A \in \mathcal{J}$. Let $f_n(x) = 0$ for every $n \in \mathbb{N}, x \in X$. Let $f(x) = 0$ for every $x \in X$. Obviously $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$.

Let $g(x) = 1$ for every $x \in X$. Let

$$\varepsilon_n = \begin{cases} \frac{1}{n+1} & \text{for } n \in A, \\ 2 & \text{for } n \in \mathbb{N} \setminus A. \end{cases}$$

Then $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$ and $\{n \in \mathbb{N} : |f_n(x) - g(x)| \geq \varepsilon_n\} = A \in \mathcal{I}$ for every $x \in X$. Thus $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} g$ and $g \neq f$, a contradiction.

¬(1) ⇒ ¬(2) Let (f_n) be a sequence such that there are two distinct functions f, g with $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ and $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} g$.

Let $x_0 \in X$ be such that $f(x_0) \neq g(x_0)$. Let $\varepsilon = |f(x_0) - g(x_0)|/3 > 0$. Let $(\eta_n) \xrightarrow{\mathcal{J}} 0$ and $(\zeta_n) \xrightarrow{\mathcal{J}} 0$ be such that $A_\eta = \{n \in \mathbb{N} : |f_n(x_0) - f(x_0)| \geq \eta_n\} \in \mathcal{I}$ and $A_\zeta = \{n \in \mathbb{N} : |f_n(x_0) - g(x_0)| \geq \zeta_n\} \in \mathcal{I}$.

Let $B_\eta = \{n \in \mathbb{N} : \eta_n \geq \varepsilon\} \in \mathcal{J}$ and $B_\zeta = \{n \in \mathbb{N} : \zeta_n \geq \varepsilon\} \in \mathcal{J}$. Let $B = B_\eta \cup B_\zeta \in \mathcal{J}$.

Now we show that $\mathbb{N} \setminus B \subseteq A_\eta \cup A_\zeta \in \mathcal{I}$ (and this will show that the ideals $\mathcal{I},\mathcal{J}$ are orthogonal). Let $n \in \mathbb{N} \setminus B$ and suppose that $n \notin A_\eta$ and $n \notin A_\zeta$. Then $|f_n(x_0) - f(x_0)| < \eta_n < \varepsilon$ and $|f_n(x_0) - g(x_0)| < \zeta_n < \varepsilon$. Hence $|f(x_0) - g(x_0)| \leq |f(x_0) - f_n(x_0)| + |f_n(x_0) - g(x_0)| < 2\varepsilon$, a contradiction. □

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