On the difference property of families of measurable functions

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Abstract

We show that, generally, families of measurable functions do not have the difference property under some assumption. And we also show that there are natural classes of functions which do not have the difference property in ZFC. This extends the result of Erdős concerning the family of Lebesgue measurable functions.

1 Introduction

Erdős showed that the family of Lebesgue measurable functions does not have the difference property if we assume the Continuum Hypothesis (see e.g. [8]). His proof works for both the family of functions with the Baire property and the family of Borel functions as well. In that proof he used two key facts. The first one was as follows. There is Lebesgue non-measurable $\mathcal{N}$-almost invariant set (under the Continuum Hypothesis). This set was first constructed by Sierpiński in [13]. And then he used the fact that the family of Lebesgue measurable sets has the weak Ostrowski property.

On the other hand M. Laczkovich in [8] proved that the family of Lebesgue measurable function has the weak difference property. And in [9] he showed, using the previous result, that the family of Lebesgue measurable functions has the difference property under some set theoretic assumption. In [10] he extended the result about the weak difference property on any family

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of real valued functions defined on any compact metric Abelian group and measurable with respect to the Haar measure.

In this paper we extend the result of Erdős on the families of functions which do not have the weak Ostrowski property.

2 Preliminaries

All functions, considered in this paper, are defined on some group and are real valued. We assume also that groups are Abelian.

Let G be a group. For a function $f\colon G \to \mathbb{R}$ and an $h \in G$ we define the *difference function* $\Delta_h f\colon G \to \mathbb{R}$ by $\Delta_h f(x) = f(x+h) - f(x)$. A function $A\colon G \to \mathbb{R}$ is called *homomorphism* if it satisfies Cauchy's functional equation $A(x+y) = A(x) + A(y)$ for all $x, y \in G$.

The notion of the difference property dates back to the paper [1] of de Bruijn. Recall that a class of functions $\mathcal{F} \subset \mathbb{R}^G$ has the *difference property* if every function $f\colon G \to \mathbb{R}$ such that $\Delta_h f \in \mathcal{F}$ for each $h \in G$ is of the form $f = g + A$ where $g \in \mathcal{F}$ and A is a homomorphism.

A set A has the Baire property if it is the symmetric difference of an open set and a meager set. We say that a function $f\colon X \to Y$ has the Baire property if $f^{-1}(U)$ has the Baire property for every open set U. By $\mathcal{B}(X)$ we will denote the family of sets with the Baire property in X and by $\mathcal{M}(X)$ the family of meager sets. Or simply $\mathcal{B}$ and $\mathcal{M}$ if it does not lead to misunderstandings.

We say that a set A is (s)-measurable (Marczewski measurable) if every perfect set P has a perfect subset Q which is a subset of A or misses A (we assume that empty set is not perfect). And by (s) we will denote the class of (s)-measurable sets. We write $A \in (s_0)$ if every perfect set P has a perfect subset Q which misses A. It is known that (s) is a σ-algebra and (s_0) is a σ-ideal. A function $f : \mathbb{R} \to \mathbb{R}$ is (s)-measurable if the preimage of any open subset is (s)-measurable. We will use the following characterization.

Theorem 2.1 (Marczewski [11]) *A function $f : \mathbb{R} \to \mathbb{R}$ is (s)-measurable if and only if every perfect set $P \subset \mathbb{R}$ has a perfect subset Q such that $f|Q$ is continuous.*

By a measure on X we mean a countably additive, nonnegative, nonzero extended real-valued function defined on a σ-algebra $\mathcal{A}$ of subsets of X. By

m we will denote the Lebesgue measure defined on $\mathbb{R}$. By $\mathcal{L}(\mu)$ we will denote the family of μ-measurable sets and by $\mathcal{N}(\mu)$ the family of μ-measure zero sets. Or simply $\mathcal{L}$ and $\mathcal{N}$ if it does not lead to misunderstandings.

A measure μ on X is called:

- *diffused* (or continuous) if $\mu(\{x\}) = 0$ for every $x \in X$;
- *finite* if $\mu(X) < +\infty$;
- *σ-finite* if X is a countable union of sets of finite measure;

Measures μ and ν on X are called *equivalent* if

1. $(\forall A \subset X)(A$ is μ-measurable if and only if A is ν-measurable$)$,
2. $(\forall A \subset X)(\mu(A) = 0$ if and only if $\nu(A) = 0)$.

Proposition 2.2 *Every σ-finite measure is equivalent to some finite measure.*

By a *universal measure* on X we mean diffused and finite measure defined on $P(X)$.

Let κ be a cardinal. A measure μ is called *κ-additive* if $\mu(\bigcup \mathcal{F}) = \sum_{F \in \mathcal{F}} \mu(F)$ for every disjoint family $\mathcal{F}$ such that $|\mathcal{F}| < \kappa$.

We say that an uncountable cardinal κ is *real-valued measurable* if there exists a κ-additive, universal measure on κ. And κ is *measurable* if there exists a two-valued, κ-additive, universal measure on κ.

We will use the following well-known theorems.

Theorem 2.3 *If there is a universal measure on a set X then there is a real-valued measurable cardinal $\leq |X|$. And if there is a universal two-valued measure on a set X then there is a measurable cardinal $\leq |X|$.*

Theorem 2.4 *There is no measurable cardinal less than or equal to $\mathfrak{c}$.*

We say that $\mathcal{J} \subset P(X)$ is an *ideal* on X if

1. $\emptyset \in \mathcal{J}$,
2. $(\forall A, B \in \mathcal{J})(A \cup B \in \mathcal{J})$,

3. $(\forall A \in \mathcal{J})(\forall B \subset A)(B \in \mathcal{J})$.

An ideal $\mathcal{J}$ on X is called:

- *proper* if $X \notin \mathcal{J}$;
- *prime* if for every $A \subset X$, either $A \in \mathcal{J}$ or $X \setminus A \in \mathcal{J}$;
- *σ-ideal* (or countable complete) if $\mathcal{J}$ is closed under countable unions of sets from $\mathcal{J}$ (i.e. $\bigcup_{n=0}^{+\infty} A_n \in \mathcal{J}$ whenever $A_n \in \mathcal{J}$ for every n);
- *κ-complete* if $\mathcal{J}$ is closed under unions of less than κ sets from $\mathcal{J}$ (i.e. $\bigcup \mathcal{F} \in \mathcal{J}$ for every family $\mathcal{F} \subset \mathcal{J}$ such that $|\mathcal{F}| < \kappa$);
- *κ-saturated* if every disjoint family $\mathcal{F} \subset P(X) \setminus \mathcal{J}$ has size $< \kappa$.

We say that an uncountable cardinal κ is *quasi-measurable* if there is a proper ω_1-saturated κ-complete ideal on κ containing all of the singletons.

We can easily prove the following propositions.

Proposition 2.5 *If there is a prime σ-ideal on X containing all of the singletons then there is a two-valued universal measure on X.*

Proposition 2.6 *If there is a proper ω_1-saturated σ-ideal on X containing all of the singletons then there is a quasi-measurable cardinal $\leq |X|$.*

If $\mathcal{J}$ is an ideal on X we define the following cardinal coefficient: $\mathrm{non}(J) = \min\{|A| : A \subset X \wedge A \notin \mathcal{J}\}$.

We shall need one more theorem concerning measurable cardinals (see e.g. [4]).

Theorem 2.7 *If there is a real-valued measurable cardinal then* $\mathrm{non}(\mathcal{N}(m)) = \omega_1$. *In particular, as an immediate consequence we get that there is no real-valued measurable cardinals less than or equal to* $\mathrm{non}(\mathcal{N}(m))$.

If $\mathcal{A}$ is a σ-algebra of subsets of X and $\mathcal{J} \subset \mathcal{A}$ is a σ-ideal on X then we say that a pair $(\mathcal{A}, \mathcal{J})$ satisfies the *c.c.c* if every disjoint family $\mathcal{F} \subset \mathcal{A} \setminus \mathcal{J}$ is countable.

Let $\mathcal{A}$ be a family of subsets of a group G. Then we use the following notation $-A = \{-a : a \in A\}, A - g = \{a - g : a \in A\}, A - A = \{a - b : a, b \in A\}$ and $-\mathcal{A} = \{-A : A \in \mathcal{A}\}$ for $g \in G, A \in \mathcal{A}$. Moreover, we say that a family $\mathcal{A}$ is *invariant under translations* if $A - g \in \mathcal{A}$ for every $A \in \mathcal{A}$ and $g \in G$ and *invariant under reflections* if $-A \in \mathcal{A}$ for every $A \in \mathcal{A}$.

Let $\mathcal{A}$ denote σ-algebra of subsets of a group (a topological group if necessary) G and $\mathcal{J} \subset \mathcal{A}$ denote σ-ideal on G.

We say that a set $A \subset G$ is *$\mathcal{J}$-almost invariant* if $(A + g)\Delta A \in \mathcal{J}$ for every $g \in G$. Moreover, we say that a pair $(\mathcal{A}, \mathcal{J})$ has

1. the *Steinhaus property* (SP) if for every set $A \in \mathcal{A} \setminus \mathcal{J}$ the set $A - A$ contains an open neighbourhood of 0.

2. the *Ostrowski property* (OP) if every homomorphism bounded on a set from $\mathcal{A} \setminus \mathcal{J}$ is continuous.

3. the *weak Ostrowski property* (WOP) if every homomorphism bounded on a set from $\mathcal{A} \setminus \mathcal{J}$ is $\mathcal{A}$-measurable.

We will use $S(\mathcal{A}, \mathcal{J})$ to denote the following sentence.

There exists a set $A \subset G$ such that A is $\mathcal{J}$-almost invariant and $A \notin \mathcal{A}$.

And $S^*(\mathcal{A}, \mathcal{J})$ to denote the following sentence.

There exists a set $A \subset G$ such that A is $\mathcal{J}$-almost invariant, $A \notin \mathcal{A}$ and $A = -A$.

3 Pairs with(out) the SP, OP, WOP

Firstly, we have well-known theorems which explain the names Ostrowski Property and Steinhaus Property. This is the case of Lebesgue measure on $\mathbb{R}$.

Theorem 3.1 (Steinhaus [14]) *The pair $(\mathcal{L}, \mathcal{N})$ has the Steinhaus property.*

Theorem 3.2 (Ostrowski [12]) *The pair $(\mathcal{L}, \mathcal{N})$ has the Ostrowski property.*

Proposition 3.3

1. *If a pair $(\mathcal{A}, \mathcal{J})$ has the Ostrowski property then it has the weak Ostrowski property (if all open sets are $\mathcal{A}$-measurable).*

2. *If a pair $(\mathcal{A}, \mathcal{J})$ has the Steinhaus property then it has the Ostrowski property.*

Proof. The first is trivial. For the second suppose that f is a homomorphism bounded on a set $A \in \mathcal{A} \setminus \mathcal{J}$. Then, there is an open neighbourhood $U \subset A - A$ on which f is bounded as well. The additivity of f implies that f is continuous at point 0 (see e.g. [10, p. 4]). Then using the additivity of f once more we get that f is continuous everywhere. ■

We have extensions of the above theorems which are also well-known. For proofs of below theorems see e.g. [2, pp. 173-174].

Theorem 3.4 *Let G be a locally compact topological group and let μ be a left invariant Haar measure on G. The pair $(\mathcal{L}, \mathcal{N})$ has the Steinhaus property.*

Theorem 3.5 *Let G be a topological group. The pair $(\mathcal{B}, \mathcal{M})$ has the Steinhaus property.*

But the SP, OP, WOP are not very common among σ-algebras.

Proposition 3.6 *A pair $((s), (s_0))$ does not have the weak Ostrowski property (thus neither has the Ostrowski and Steinhaus properties).*

Proof. We will prove that the pair $((s), (s_0))$ does not have the weak Ostrowski property. Then from Proposition 3.3 follows that the pair $((s), (s_0))$ does not have the Ostrowski and Steinhaus properties.

Let $P \subset \mathbb{R}$ be a perfect (i.e., nonempty, closed, with no isolated points) set, which is linearly independent over rationals. Let $B \subset P$ be a Bernstein subset of P (i.e., B and $P \setminus B$ do not contain a nonempty perfect (in P) set). Now we define a function $A_1 \colon P \to \mathbb{R}$ by $A_1(x) = 0$ for $x \in B$ and $A_1(x) = 1$ for $x \in P \setminus B$. Since P is linearly independent over rationals, we can extend A_1 to an additive function $A \colon \mathbb{R} \to \mathbb{R}$ such that $A|P = A_1$. We see that the function A is bounded on the set $P \in (s) \setminus (s_0)$. Suppose that A is (s)-measurable. Then, by Theorem 2.1, there is a perfect set $D \subset P$ such that $A|D$ is continuous. Take an $x \in D \cap B$ (it can be done because B is Bernstein set in P). Then we can find a sequence $(x_n)_{n\in\omega}$ such that $\lim_{n\to\infty} x_n = x$ and $x_n \in D \setminus B$ since $P \setminus B$ is Bernstein set too. But $A(x_n) = 1$ for all n and $A(x) = 0$ so $A|D$ is not continuous, a contradiction. Thus A is not (s)-measurable. ■

Theorem 3.7 *There exists an invariant under translations and reflections extension of the Lebesgue measure which does not have the Steinhaus property.*

Proof. See [6, p. 148, Proposition 2]. ■

Theorem 3.8 *There exists an invariant under reflections extension of the Lebesgue measure which does not have the weak Ostrowski property.*

Proof. This extension was constructed in [7, Example 2]. That extension is not complete and the completion of this measure may have the weak Ostrowski property. But one can change a little the definition of that measure to get a complete measure which does not have the weak Ostrowski property. ■

But we do not know if there exists an invariant under translations extension of the Lebesgue measure which does not have the (weak) Ostrowski property.

4 Generalization of Erdős' result

Let $\mathcal{A}$ denote σ-algebra of subsets of a group G and $\mathcal{J} \subset \mathcal{A}$ denote σ-ideal on G.

4.1 Trivial generalization

In order to repeat the Erdős' proof (showing that the family of Lebesgue measurable functions does not have the difference property) for a family of $\mathcal{A}$-measurable functions we only need that the pair $(\mathcal{A}, \mathcal{J})$ has the weak Ostrowski property and the sentence $S(\mathcal{A}, \mathcal{J})$ is true. Thus we get a theorem, essentially due to Erdős.

Theorem 4.1 *Suppose that a pair $(\mathcal{A}, \mathcal{J})$ has the weak Ostrowski property. If $S(\mathcal{A}, \mathcal{J})$ holds then the family of $\mathcal{A}$-measurable functions does not have the difference property.*

Proof. The proof is the same as Erdős' one but we sketch it here for the sake of completeness. By $S(\mathcal{A}, \mathcal{J})$ we have a set $S \subset G$ such that S is $\mathcal{J}$-almost invariant and $S \notin \mathcal{A}$. Put $f = \chi_S$. Now we show that the function f is a witness for a lack of the difference property for the family of $\mathcal{A}$-measurable functions.

Since $\{x \in G : \Delta_h f \neq 0\} = \{x \in G : \chi_{S-h}(x) \neq \chi_S(x)\} = (S-h)\Delta S \in \mathcal{J}$ we get that the function $\Delta_h f$ is $\mathcal{A}$-measurable for every $h \in G$. Now suppose that $f = g + A$ where g is $\mathcal{A}$-measurable and A is a homomorphism. Then there is n such that a set $B = g^{-1}([-n, n])$ is in $\mathcal{A} \setminus \mathcal{J}$ thus homomorphism A is bounded on B since f and g are bounded on B. By the weak Ostrowski property, function A is $\mathcal{A}$-measurable. Thus the function $f = \chi_S$ is $\mathcal{A}$-measurable as a sum of two $\mathcal{A}$-measurable functions. But the set S is not $\mathcal{A}$-measurable, a contradiction. ■

Corollary 4.2 *Let G be a locally compact topological group and let μ be a left invariant Haar measure on G. If $S(\mathcal{L}, \mathcal{N})$ holds then the family of $\mathcal{L}$-measurable functions does not have the difference property.*

Proof. It follows from Theorems 3.4 and 4.1. ■

Corollary 4.3 *Let G be a topological group. If $S(\mathcal{B}, \mathcal{M})$ holds then the family of $\mathcal{B}$-measurable functions does not have the difference property.*

Proof. It follows from Theorems 3.5 and 4.1. ■

4.2 Less trivial generalization

Now we shall show that we do not need the weak Ostrowski property for a pair $(\mathcal{A}, \mathcal{J})$ in order to prove that a family of $\mathcal{A}$-measurable functions does not have the difference property (in some cases).

Theorem 4.4 *Let $\mathcal{A}$ be an invariant under reflections σ-algebra on a group G and $\mathcal{J} \subset \mathcal{A}$ be σ-ideal on G. If $S^*(\mathcal{A}, \mathcal{J})$ holds then the family of $\mathcal{A}$-measurable functions does not have the difference property.*

Proof. Let $A \subset G$ be a set such that $A \notin \mathcal{A}$, $A = -A$ and $(A+g)\Delta A \in \mathcal{J}$ for all $g \in G$. Such a set A exists by an assumption that the sentence $S^*(\mathcal{A}, \mathcal{J})$ is true. Let $f = \chi_A$. We will show that the function f is a witness for a lack of the difference property of the family of $\mathcal{A}$-measurable functions.
At first, it is easy to see that a function $\Delta_g f$ is $\mathcal{A}$-measurable for every $g \in G$. Now suppose that $f = k + h$ where k is $\mathcal{A}$-measurable and h is a homomorphism. We define a function F by a formula $F(x) = f(x) + f(-x)$. Then we have $F(x) = (k(x)+h(x))+(k(-x)+h(-x)) = k(x)+k(-x)$ so the function F is $\mathcal{A}$-measurable (since $\mathcal{A}$=-$\mathcal{A}$).
But on the other hand we have $F(x) = f(x) + f(-x) = \chi_A(x) + \chi_A(-x) = \chi_A(x) + \chi_{-A}(x) = 2\chi_A(x)$. And, since $A \notin \mathcal{A}$, we get that the function F is not $\mathcal{A}$-measurable.

We have reached a contradiction which shows that the function f is not a sum of suitable functions. ■

5 On the sentence $S^*(\mathcal{A},\mathcal{J})$ and $S(\mathcal{A},\mathcal{J})$

We have seen that in both generalizations, section 4.1 and 4.2, a weak point is an assumption that the sentence $S(\mathcal{A},\mathcal{J})$ or $S^*(\mathcal{A},\mathcal{J})$ is true. Hence we shall examine these sentences in order to check when they are true.

Let $\mathcal{A}$ be a σ-algebra on a group G and $\mathcal{J}$ be a σ-ideal on G. Let $|G| = \kappa$ and $G = \{g_\alpha : \alpha < \kappa\}$ be an enumeration of elements of the group G. By G_α we denote a group generated by a set $\{g_\beta : \beta < \alpha\}$. For any $\alpha < \kappa$ let $Q_\alpha = G_{\alpha+1} \setminus G_\alpha$. Now for every set $T \subset \kappa$ let $A_T = \bigcup_{\alpha\in T} Q_\alpha$.

Lemma 5.1 *If* $\mathrm{non}(\mathcal{J}) = |G|$ *then a set* A_T *is* $\mathcal{J}$*-almost invariant and* $A_T = -A_T$ *for every set* $T \subset \kappa$.

Proof. It is easy to see that for every set $T \subset \kappa$ and every $g \in G$ the set $(A_T + g) \setminus A_T$ is in $\mathcal{J}$. Indeed, take any set $T \subset \kappa$ and any element g of the group G. Then

$$(A_T + g) \setminus A_T = \bigcup_{\alpha\in T}(Q_\alpha + g) \setminus \bigcup_{\alpha\in T} Q_\alpha = \bigcup_{\alpha\in T}[(G_{\alpha+1} + g) \setminus (G_\alpha + g)] \setminus \bigcup_{\alpha\in T} Q_\alpha.$$

But there is $\beta < \kappa$ such that for every $\alpha > \beta$ we have $g \in G_\alpha$ so

$$(A_T+g)\setminus A_T = \Big[\bigcup_{T\ni\alpha\le\beta}(Q_\alpha+g)\cup\bigcup_{T\ni\alpha>\beta}(G_{\alpha+1}\setminus G_\alpha)\Big]\setminus\bigcup_{\alpha\in T} Q_\alpha \subset \bigcup_{\alpha\le\beta}(Q_\alpha+g).$$

And the last set is in $\mathcal{J}$, by an assumption that $\mathrm{non}(\mathcal{J}) = |G|$, so the set $(A_T+g)\setminus A_T$ is in $\mathcal{J}$. Similarly we get that a set $A_T \setminus (A_T+g)$ is in $\mathcal{J}$ hence $(A_T + g)\Delta A_T \in \mathcal{J}$.

Now one can easily check that $A_T = -A_T$ for every subset $T \subset \kappa$. It follows from a fact that $Q_\alpha = -Q_\alpha$ for every $\alpha < \kappa$. ■

The construction of almost invariant sets appears in papers devoted to extensions of invariant measures. And the above construction appeared in [5] and [15] among others.

We see that to prove the sentence $S^*(\mathcal{A},\mathcal{J})$ we have to show that there is a set T such that A_T is not in $\mathcal{A}$. But it will be done for every case separately. We see that we had to assume that $\mathrm{non}(\mathcal{J}) = |G|$. Moreover, for some cases we will need to assume something more. It is no wonder that we need some assumptions since the Erdős theorem is proved under the Continuum Hypothesis (and under the Continuum Hypothesis our assumptions will be fulfilled). Moreover, we prove that there are some families for which those assumptions are fulfilled in ZFC.

5.1 Measure case

Theorem 5.2 *Let μ be a σ-finite measure on a group G. If $|G|$ is less than the first real-valued measurable cardinal and* $\mathrm{non}(\mathcal{N}) = |G|$ *then $S^*(\mathcal{L}, \mathcal{N})$.*

Proof. By Lemma 5.1 we see that to prove the theorem we must find at least one set $T \subset \kappa$ such that the set A_T is not $\mathcal{L}$-measurable. Suppose that for every set $T \subset \kappa$ the set A_T is $\mathcal{L}$-measurable. By Proposition 2.2 we have a finite measure ν which is equivalent to μ. Then we can define measure $\tau \colon P(\kappa) \to [0, +\infty]$ by $\tau(T) = \nu(A_T)$. This is diffused, finite measure which measures all subsets of κ so by Theorem 2.3 there is a real-valued measurable cardinal $\leq \kappa = |G|$, a contradiction. Hence there is a set $T_0 \subset \kappa$ such that the set A_{T_0} is not ν-measurable. Thus, the set A_{T_0} is not $\mathcal{L}$-measurable, as well. ■

Corollary 5.3 *If a measure μ is invariant under reflections then under the assumptions of Theorem 5.2 we get that the family of $\mathcal{L}$-measurable function does not have the difference property.*

Proof. Apply Theorems 4.4 and 5.2. ■

5.2 Category case

Theorem 5.4 *Let G be a topological group. If $|G|$ is less than the first measurable cardinal and* $\mathrm{non}(\mathcal{M}) = |G|$ *then $S^*(\mathcal{B}, \mathcal{M})$.*

Proof. We see that to prove the theorem we must find at least one set $T \subset \kappa$ such that the set A_T is not $\mathcal{B}$-measurable again. In this case we have to modify the proof. At first we show that there is $T \subset \kappa$ such that $A_T \notin \mathcal{M}$ and $G \setminus A_T \notin \mathcal{M}$. Suppose that for every $T \subset \kappa$ we have $A_T \in \mathcal{M}$ or $G \setminus A_T \in \mathcal{M}$. Then the family $\{T \subset \kappa : A_T \in \mathcal{M}\}$ is a prime σ-ideal on κ containing all of the singletons. Thus by Proposition 2.5 and Theorem 2.3 there is a measurable cardinal $\leq \kappa = |G|$, a contradiction. So there is $T_0 \subset \kappa$ such that $A_{T_0} \notin \mathcal{M}$ and $G \setminus A_{T_0} \notin \mathcal{M}$ what implies that the set A_{T_0} does not have the Baire property. Indeed, suppose that $A_{T_0} \in \mathcal{B}$ than there is $g \in G$ such that $(A_{T_0} + g) \cap (G \setminus A_{T_0}) \notin \mathcal{M}$ hence $(A_{T_0} + g) \setminus A_{T_0} \notin \mathcal{M}$, a contradiction since the set A_{T_0} is $\mathcal{M}$-almost invariant by Lemma 5.1. ■

Corollary 5.5 *Under the assumptions of Theorem 5.4 we get that the family of $\mathcal{B}$-measurable functions does not have the difference property.*

Proof. Apply Theorems 4.4 and 5.4. ■

5.3 c.c.c. case

Now we generalize the measure and category case for a case where there is given a σ-algebra $\mathcal{A}$ and a σ-ideal $\mathcal{J}$ on a group G such that the pair $(\mathcal{A}, \mathcal{J})$ satisfies the c.c.c.

Theorem 5.6 *Let $\mathcal{A}$ be a σ-algebra on a group G and $\mathcal{J}$ be a proper σ-ideal on G such that the pair $(\mathcal{A}, \mathcal{J})$ satisfies the c.c.c. If $|G|$ is less than the first quasi-measurable cardinal and $\mathrm{non}(\mathcal{J}) = |G|$ then $S^*(\mathcal{A}, \mathcal{J})$.*

Proof. Once more we see that to prove the theorem we must find at least one set $T \subset \kappa$ such that the set A_T is not $\mathcal{A}$-measurable. Suppose that $A_T \in \mathcal{A}$ for every $T \subset \kappa$. Let $\mathcal{I} = \{T \subset \kappa : A_T \in \mathcal{J}\}$. Then it is not difficult to check that $\mathcal{I}$ is a proper σ-ideal containing all of the singletons. Since the pair $(\mathcal{A}, \mathcal{J})$ satisfies the c.c.c. we get that the pair $(P(\kappa), \mathcal{I})$ satisfies the c.c.c. as well. But it means that the ideal $\mathcal{I}$ is ω_1-saturated. So by Proposition 2.6 there is a quasi-measurable cardinal $\leq |G|$, a contradiction. So there is a set $T \subset \kappa$ such that $A_T \notin \mathcal{A}$. ■

Corollary 5.7 *If $\mathcal{A}$ is invariant under reflection then under assumptions of Theorem 5.6 the family of $\mathcal{A}$-measurable functions does not have the difference property.*

Proof. Apply Theorems 4.4 and 5.6. ■

Remark. Despite the family of Lebesgue measurable sets and the family of sets with the Baire property satisfy c.c.c. Theorem 5.6 does not imply in general Theorems 5.2 and 5.4. Indeed, if there exists a model with measurable cardinal then there exists a model for Martin's Axiom and with a quasi-measurable cardinal $< \mathfrak{c}$ (see e.g. [4]). In that model there is no real-valued measurable cardinal $\leq \mathfrak{c}$.

5.4 Results in ZFC

Now we show that there are σ-algebras and σ-ideals for which the sentence $S^*(\cdot, \cdot)$ is true in ZFC and, by Theorem 4.4, suitable families of functions does not have the difference property (in ZFC).

5.4.1 (s)-measurable sets

Theorem 5.8 *$S^*((s),(s_0))$ holds in ZFC.*

Proof. We have to construct an (s)-non-measurable set A such that $A = -A$ and $(A+x)\setminus A \in (s_0)$ for every $x \in \mathbb{R}$. It will be a slight modification of Sierpiński construction [13]. Let $\mathbb{R} = \{r_\alpha : \alpha < \mathfrak{c}\}$ and $\{P_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of reals and perfect sets respectively. And let L_α be a linear space over rationals which is spanned by $\{r_\beta : \beta < \alpha\}$. We will construct two sequences $\{x_\alpha : \alpha < \mathfrak{c}\}$ and $\{y_\alpha : \alpha < \mathfrak{c}\}$ as follows. Take

$$x_\alpha \in P_\alpha \setminus (L_\alpha + (\{\pm x_\beta : \beta < \alpha\} \cup \{\pm y_\beta : \beta < \alpha\}))$$

and

$$y_\alpha \in P_\alpha \setminus (L_\alpha + (\{\pm x_\beta : \beta \leq \alpha\} \cup \{\pm y_\beta : \beta < \alpha\})).$$

Now we put $S = \bigcup_{\alpha<\mathfrak{c}}(L_\alpha \pm x_\alpha)$.

We see that $S = -S$. To show that S is not (s)-measurable we check that S is a Bernstein set. One can see that $S \cap P \neq \emptyset$ for every perfect set P since $x_\alpha \in S$ for every $\alpha < \mathfrak{c}$. On the other hand suppose that there is $\beta < \mathfrak{c}$ such that $y_\beta \notin \mathbb{R}\setminus S$. Thus there is α such that $y_\beta \in L_\alpha \pm x_\alpha$. If $\beta \geq \alpha$ then we get a contradiction with the definition of points y_α. So $\alpha > \beta$. But in that case $x_\alpha \in L_\alpha \pm y_\beta$ a contradiction.

Now we will show that S is (s_0)-almost invariant. We only show that $|(S+r)\setminus S| < \mathfrak{c}$ for every $r \in \mathbb{R}$ since all sets of cardinality less than continuum are in (s_0). Take $r \in \mathbb{R}$. Then there is $\beta < \mathfrak{c}$ such that $r = r_\beta$. Since $r_\beta \in L_\alpha$ for every $\alpha > \beta$ we can write $(S+r_\beta)\setminus S = (\bigcup_{\alpha<\mathfrak{c}}(L_\alpha \pm x_\alpha + r_\beta))\setminus S = (\bigcup_{\alpha\leq\beta}(L_\alpha \pm x_\alpha + r_\beta) \cup \bigcup_{\alpha>\beta}(L_\alpha \pm x_\alpha))\setminus S = (\bigcup_{\alpha\leq\beta}(L_\alpha \pm x_\alpha + r_\beta)\setminus S) \cup (\bigcup_{\alpha>\beta}(L_\alpha \pm x_\alpha)\setminus S) = (\bigcup_{\alpha\leq\beta}(L_\alpha \pm x_\alpha + r_\beta)\setminus S) \subset \bigcup_{\alpha\leq\beta}(L_\alpha \pm x_\alpha + r_\beta)$ and the last set is of cardinality less than continuum. ■

Corollary 5.9 *The family of (s)-measurable functions does not have the difference property.*

Proof. Apply Theorems 4.4 and 5.8. ■

Remark. Corollary 5.9 was already proved in [3]. There, it was proved directly whereas we use general theorem (Theorem 4.4) which is useful in many others cases as well.

Remark. Since the sentence $S^*((s),(s_0))$ is true we get that the sentence $S((s),(s_0))$ is true as well. But we cannot use Theorem 4.1 instead of Theorem 4.4 to prove Corollary 5.9 since by Proposition 3.6 the pair $((s),(s_0))$ does not have the weak Ostrowski property.

5.4.2 Subgroup of the real line with measure

Let X be a subset of the real line which is of positive outer Lebesgue measure. Define $\mathcal{L}_X = \{A \cap X : A \text{ is Lebesgue measurable}\}$ and $m_X : \mathcal{L}_X \to [0, +\infty]$ by $m_X(B) = m^o(B)$ where m^o denotes the Lebesgue outer measure and $B \in \mathcal{L}_X$. Then it is not difficult to check that $\mathcal{L}_X$ is a σ-algebra and m_X is a measure on $\mathcal{L}_X$. We also have $\mathrm{non}(\mathcal{N}(m_X)) = \mathrm{non}(\mathcal{N}(m))$.

Theorem 5.10 *There is a subgroup G of the real line such that the sentence $S^*(\mathcal{L}_G, \mathcal{N}(m_G))$ is true (in ZFC).*

Proof. Let $X \subset \mathbb{R}$ be a Lebesgue non-measurable set of cardinality $\kappa = \mathrm{non}(\mathcal{N}(m))$. Let G be a group generated by the set X. Then we have $|G| = \kappa$.

Since $|G| = \mathrm{non}(\mathcal{N}(m))$ hence by Theorem 2.7 $|G|$ is less than the first real-valued measurable cardinal. Moreover, $\mathrm{non}(\mathcal{N}(m_G)) = \mathrm{non}(\mathcal{N}(m)) = |G|$. Thus by Theorem 5.2 we get that the sentence $S^*(\mathcal{L}_G, \mathcal{N}(m_G))$ is true. ■

Corollary 5.11 *The family of m_G-measurable functions does not have the difference property (in ZFC).*

Proof. Apply Theorems 4.4 and 5.10. ■

Remark. One can see that Theorem 5.10 is true for every Lebesgue non-measurable group of cardinality $\mathrm{non}(\mathcal{N}(m))$.

5.4.3 Subgroup of the real line with topology

Theorem 5.12 *There is a subgroup G of the real line such that the sentence $S^*(\mathcal{B}, \mathcal{M})$ is true (in ZFC).*

Proof. Let $X \subset \mathbb{R}$ be a set without the Baire property of cardinality $\kappa = \mathrm{non}(\mathcal{M}(\mathbb{R}))$. Let G be a group generated by the set $X \cup \mathbb{Q}$. Then we have $|G| = \kappa$.

Since $|G| \leq \mathfrak{c}$ hence by Theorem 2.4 $|G|$ is less than the first measurable cardinal. Moreover, $\mathrm{non}(\mathcal{M}(G)) = \mathrm{non}(\mathcal{M}(\mathbb{R})) = |G|$ since the group G is dense in $\mathbb{R}$. Thus by Theorem 5.4 we get that the sentence $S^*(\mathcal{B}(G), \mathcal{M}(G))$ is true. ■

Corollary 5.13 *The family of functions with the Baire property on G does not have the difference property (in ZFC).*

Proof. Apply Theorems 4.4 and 5.12. ■

Remark. One can see that Theorem 5.12 is true for every dense group without the Baire property of cardinality non($\mathcal{M}(\mathbb{R})$).

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