THE REAPING AND SPLITTING NUMBERS OF NICE IDEALS

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Abstract. We examine the splitting number \( s(B) \) and the reaping number \( r(B) \) of quotient Boolean algebras \( B = \mathcal{P}(\omega)/I \) over \( F_\sigma \) ideals and analytic \( P \)-ideals. For instance we prove that under Martin’s axiom \( s(\mathcal{P}(\omega)/I) = \epsilon \) for all \( F_\sigma \) ideals and analytic \( P \)-ideals with BW property (and one cannot drop the assumption about BW property). On the other hand we prove that under Martin’s axiom \( r(\mathcal{P}(\omega)/I) = \epsilon \) for all \( F_\sigma \) ideals and analytic \( P \)-ideals (in this case we do not need the assumption about BW property). We also provide applications of these characteristics to the ideal convergence of sequences of real-valued functions defined on reals.

1. Introduction

Let \( B \) be a Boolean algebra. A set \( S \) is a splitting set for \( B \) if for every nonzero \( b \in B \) there is an \( s \in S \) such that \( b \cdot s \neq 0 \neq b \cdot (-s) \). A set \( D \subseteq B \setminus \{0\} \) is weakly dense if for every \( b \in B \) there is \( d \in D \) such that \( d \leq b \) or \( d \leq -b \). By the splitting number of \( B \) we mean the cardinal \( s(B) = \min\{|S| : S \text{ is a splitting set for } B\} \), and by the reaping number of \( B \) we mean \( r(B) = \min\{|D| : D \text{ is weakly dense in } B\} \). Many results on \( s(B) \) and \( r(B) \) for various Boolean algebras can be found in [23].

In the sequel we assume that if \( I \) is an ideal on \( \omega \) then \( [\omega]^{<\omega} \subseteq I \) and \( \omega \notin I \).

For a set \( A \subseteq \omega \) we put \( A^0 = A \) and \( A^1 = \omega \setminus A \).

Let \( I \) be an ideal on \( \omega \). By \( I^+ = \mathcal{P}(\omega) \setminus I \) we denote the coideal of \( I \). A set \( A \) \( I \)-splits \( B \) if both \( B \cap A^0, B \cap A^1 \in I^+ \). A family \( \mathcal{R} \subseteq I^+ \) is \( I \)-unsplittable if no single set \( I \)-splits all members of \( \mathcal{R} \). An \( I \)-splitting family is a family \( S \subseteq \mathcal{P}(\omega) \) such that each \( A \in I^+ \) is \( I \)-split by at least one \( S \in S \).

In this paper we are interested in the splitting and reaping numbers of quotient Boolean algebras of the form \( B = \mathcal{P}(\omega)/I \) where \( I \) is an \( F_\sigma \) ideal or analytic \( P \)-ideal on \( \omega \) (see Section 2 for definitions of \( F_\sigma \) and analytic \( P \)-ideals). We write then \( s(I) = s(\mathcal{P}(\omega)/I) \) and \( r(I) = r(\mathcal{P}(\omega)/I) \). In this case the definitions of \( s(I) \) and \( r(I) \) can be rephrased in the following manner:

\[
s(I) = \min\{|S| : S \subseteq I^+, S \text{ is an } I \text{-splitting family}\},
\]

and

\[
r(I) = \min\{|\mathcal{R}| : \mathcal{R} \subseteq I^+, \mathcal{R} \text{ is } I \text{-unsplittable}\}.
\]

In the case of the ideal \( I = \text{Fin} \) of all finite subsets of \( \omega \), we obtain the classical cardinal characteristics of the continuum: \( s = s(\text{Fin}) \) and \( r = r(\text{Fin}) \) (see e.g. [2])

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and \([27]\)). It is well known that \(s\) and \(t\) are uncountable and if we assume Martin’s Axiom (MA) then \(s = t = \mathfrak{c}\).

In Section 3 we show that \(s(\mathcal{I}), t(\mathcal{I})\) are uncountable for every \(F_{\sigma}\) ideal (Proposition 3.1) and we prove that if we assume MA then \(s(\mathcal{I}) = t(\mathcal{I}) = \mathfrak{c}\) for every \(F_{\sigma}\) ideal (Theorem 3.2).

In Section 4 we prove that \(t(\mathcal{I})\) is uncountable for every analytic \(P\)-ideal (Proposition 4.1) and we also prove that if we assume MA then \(t(\mathcal{I}) = \mathfrak{c}\) for every analytic \(P\)-ideal (Theorem 4.2).

In [9] the authors proved that \(s(\mathcal{I}) = \omega\) \iff the ideal \(\mathcal{I}\) does not have BW property (see Section 2 for the definition of BW property). We prove that if we assume MA then \(s(\mathcal{I}) = \mathfrak{c}\) for analytic \(P\)-ideals with BW property (Theorem 4.3).

The splitting, reaping and other cardinal characteristics (e.g. \(a, p\) and \(t\)) of the quotient Boolean algebras \(\mathcal{P}(\omega)/\mathcal{I}\) were already considered in some papers, see e.g. \([1], [3], [8], [13], [15], [16]\) and \([26]\).

In Section 5 we apply the results on \(s(\mathcal{I})\) and \(t(\mathcal{I})\) to the ideal convergence of sequences of real-valued functions defined on reals.

2. Preliminaries

2.1. Nice ideals. By identifying sets of natural numbers with their characteristic functions, we equip \(\mathcal{P}(\omega)\) with the Cantor-space topology and therefore we can assign topological complexity to ideals of sets of integers. In particular, an ideal \(\mathcal{I}\) is an \(F_{\sigma}\) (resp. analytic) subset of the Cantor space.

An ideal \(\mathcal{I}\) is a \(P\)-ideal if for every countable family \(\{A_n : n \in \omega\} \subseteq \mathcal{I}\) there is \(A \in \mathcal{I}\) such that \(A_n \setminus A\) is finite for every \(n \in \omega\).

A map \(\phi: \mathcal{P}(\omega) \to [0, \infty]\) is a submeasure on \(\omega\) if \(\phi(\emptyset) = 0\) and \(\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B)\) for all \(A, B \subseteq \omega\). In the sequel we assume that \(\phi(\{n\}) < \infty\) for every submeasure \(\phi\) and \(n \in \omega\). A submeasure \(\phi\) is lower semicontinuous (we will write lsc for short) if for all \(A \subseteq \omega\) we have \(\phi(A) = \lim_{n \to \infty} \phi(A \cap \{0, 1, \ldots, n-1\})\). For a submeasure \(\phi\) we write

\[\text{Fin}(\phi) = \{A \subseteq \omega : \phi(A) < \infty\}\]

and

\[\text{Exh}(\phi) = \left\{A \subseteq \omega : \|A\|_\phi = 0\right\},\]

where \(\|A\|_\phi = \lim_{n \to \infty} \phi(A \setminus \{0, 1, \ldots, n-1\})\).

Theorem 2.1 ([21],[25]). Let \(\mathcal{I}\) be an ideal on \(\omega\) (not necessarily proper).

1. \(\mathcal{I}\) is an \(F_{\sigma}\) ideal \iff \(\mathcal{I} = \text{Fin}(\phi)\) for some lsc submeasure \(\phi\) on \(\omega\).

2. \(\mathcal{I}\) is an analytic \(P\)-ideal \iff \(\mathcal{I} = \text{Exh}(\phi)\) for some lsc submeasure \(\phi\) on \(\omega\).

2.1.1. Examples. For many examples of nice ideals see e.g. [16] or [7]. Below we list some of them.

1. The ideal \(\text{Fin}\) is an \(F_{\sigma}\) \(P\)-ideal.

2. Let \(f: \omega \to [0, \infty)\) be such that \(\sum_{n \in \omega} f(n) = \infty\). The summandable ideal generated by \(f\)

\[\mathcal{I}_f = \left\{A \subseteq \omega : \sum_{n \in A} f(n) < \infty\right\}\]
is an $F_\sigma$ ideal ([21]).

(3) The ideal of sets of asymptotic density 0

$$\mathcal{I}_d = \left\{ A \subseteq \omega : \limsup_{n \to \infty} \frac{|A \cap \{0, 1, \ldots, n-1\}|}{n} = 0 \right\}$$

is an analytic $P$-ideal (and it is not an $F_\sigma$ ideal).

(4) Let $f : \omega \to [0, +\infty)$ be such that

$$\sum_{i=0}^{\infty} f(i) = +\infty \text{ and } \lim_{n \to \infty} \frac{\sum_{i \in n} f(i)}{\sum_{i \in n} f(i)} = 0.$$ 

The Erdős-Ulam ideal generated by $f$

$$\mathcal{EU}_f = \left\{ A \subseteq \omega : \lim_{n \to \infty} \frac{\sum_{i \in A \cap n} f(i)}{\sum_{i \in A \cap n} f(i)} = 0 \right\}$$

is an analytic $P$-ideal ([17]). Note that the ideal $\mathcal{I}_d$ is an Erdős-Ulam ideal.

(5) Assume that $I_n$ are pairwise disjoint intervals on $\omega$, and $\mu_n$ is a measure that concentrates on $I_n$. Then $\phi = \sup_n \mu_n$ is a lower semicontinuous submeasure and $\text{Exh}(\phi)$ is called the density ideal generated by $(\mu_n)_n$. It is known that Erdős-Ulam ideals are density ideals.

(6) The van der Waerden ideal

$$W = \{ A \subseteq \omega : A \text{ does not contain arithmetic progressions of arbitrary length} \}$$

is an $F_\sigma$ ideal (and it is not a $P$-ideal).

(7) The eventually different ideal

$$\mathcal{ED} = \{ A \subseteq \omega \times \omega : \exists m, n \in \omega \forall k \geq n (|\{ i \in \omega : (k, i) \in A \}| \leq m) \}$$

is an $F_\sigma$ ideal (and it is not a $P$-ideal).

2.2. Ideal convergence. Let $\mathcal{I}$ be an ideal on $\omega$ and $A \subseteq \omega$. We say that a sequence $(x_n)_{n \in \mathbb{N}}$ of reals is $\mathcal{I}$-convergent to $x \in \mathbb{R}$ if for every $\varepsilon > 0$, $\{ n \in A : |x_n - x| \geq \varepsilon \} \in \mathcal{I}$ for every $\varepsilon > 0$. We say that an ideal $\mathcal{I}$ on $\omega$ has BW property ($\mathcal{I} \in \text{BW}$, for short) if for every bounded sequence $(x_n)_{n \in \mathbb{N}}$ of reals there exists $A \in \mathcal{I}^+$ such that $(x_n)_{n \in A}$ is $\mathcal{I}$-convergent ([9]).

**Proposition 2.2** ([9]).

(1) Every $F_\sigma$ ideal has BW property (hence Fin, summable ideals, $W$ and $\mathcal{ED}$ have BW property as well).

(2) Erdős-Ulam ideals (and $\mathcal{I}_d$) do not have BW property.

(3) A density ideal does not have BW-property if and only if it is an Erdős-Ulam ideal.

**Theorem 2.3** ([9]). Let $\mathcal{I}$ be an ideal on $\omega$. Then $s(\mathcal{I}) = \omega \iff \mathcal{I}$ does not have BW property.

2.3. Big intersections. Below we presents some auxilary results which we will need later (however they seem to be interesting on their own).

**Lemma 2.4.** Let $\mathcal{I}$ be an ideal on $\omega$. There is a function $x : \mathcal{P}(\omega) \to \{0, 1\}$ such that

$$\bigcap \{ A_{x(A)} : A \in \mathcal{A} \} \notin \mathcal{I}$$

for every finite and nonempty family $\mathcal{A} \subseteq \mathcal{P}(\omega)$. 

Proof. Let \( J \) be a maximal ideal such that \( I \subseteq J \). For \( A \in \mathcal{P}(\omega) \) we define
\[
x(A) = \begin{cases} 0 & \text{if } A \notin J \\ 1 & \text{if } A \in J \end{cases}.
\]
Since \( A^{x(A)} \notin J \) for every \( A \) and \( J \) is a maximal ideal, so \( \bigcap \{ A^{x(A)} : A \in A \} \notin J \).
Thus \( \bigcap \{ A^{x(A)} : A \in A \} \notin I \).
\( \square \)

**Corollary 2.5.** Let \( I = \text{Fin}(\phi) \) be an \( F_\sigma \) ideal. There is \( x : \mathcal{P}(\omega) \to \{0, 1\} \) such that
\[
\phi \left( \bigcap \{ A^{x(A)} : A \in A \} \right) = \infty
\]
for every finite and nonempty family \( A \subseteq \mathcal{P}(\omega) \).

**Proof.** Apply Lemma 2.4 and note that \( A \notin I \iff \phi(A) = \infty \).
\( \square \)

**Corollary 2.6.** Let \( I = \text{Exh}(\phi) \) be an analytic \( P \)-ideal. There is \( x : \mathcal{P}(\omega) \to \{0, 1\} \) such that
\[
\left\| \bigcap \{ A^{x(A)} : A \in A \} \right\|_\phi > 0
\]
for every finite and nonempty family \( A \subseteq \mathcal{P}(\omega) \).

**Proof.** Apply Lemma 2.4 and note that \( A \notin I \iff \left\| A \right\|_\phi > 0 \).
\( \square \)

Below we show that for ideals with BW property we can obtain a strengthening of the above result.

**Lemma 2.7.** Let \( I = \text{Exh}(\phi) \) be an analytic \( P \)-ideal. The ideal \( I \) has BW property if and only if there are \( \delta > 0 \) and \( x : \mathcal{P}(\omega) \to \{0, 1\} \) such that
\[
\left\| \bigcap \{ A^{x(A)} : A \in A \} \right\|_\phi \geq \delta
\]
for every finite and nonempty family \( A \subseteq \mathcal{P}(\omega) \).

**Proof.** (\( \Rightarrow \)) By [9, Theorem 3.6] there exists \( \delta > 0 \) such that for every finite partition \( A_1 \cup \cdots \cup A_n = \omega \) there exists \( 1 \leq i \leq n \) with \( \left\| A_i \right\|_\phi \geq \delta \). We will show that this \( \delta \) is the required one.

For every finite and nonempty family \( A \subseteq \mathcal{P}(\omega) \) we define
\[
C_A = \left\{ x \in \{0, 1\}^{\mathcal{P}(\omega)} : \left\| \bigcap \{ A^{x(A)} : A \in A \} \right\|_\phi \geq \delta \right\}.
\]
We will show that

1. \( C_A \neq \emptyset \);
2. \( C_A \) is a closed set in \( \{0, 1\}^{\mathcal{P}(\omega)} \);
3. the family \( \{ C_A : A \text{ is finite and nonempty} \} \) is centered.

Then using compactness of the topological space \( \{0, 1\}^{\mathcal{P}(\omega)} \) we get
\[
x \in \bigcap \{ C_A : A \text{ is finite and nonempty} \}.
\]
It is easy to see that this \( x \) is as required. Thus, the proof will be finished as soon as we show properties (1)–(3).

(1). Take any finite and nonempty \( A \subseteq \mathcal{P}(\omega) \). Since the family
\[
\left\{ \bigcap \{ A^{x(A)} : A \in A \} : s \in \{0, 1\}^A \right\}
\]

is a finite partition of \(\omega\), so there is \(s \in \{0,1\}^A\) with \(\left\| \bigcap \{ A^{x(A)} : A \in \mathcal{A} \} \right\|_\phi \geq \delta\). Then any \(x \in \{0,1\}^{P(\omega)}\) such that \(s \subseteq x\) belongs to \(C_A\).

(2). Take any finite and nonempty \(A \subseteq P(\omega)\). Since \(S = \{ x \mid A : x \in C_A \} \subseteq \{0,1\}^A\) is finite and \(C_A = \bigcup_{s \in S} \{ x \in \{0,1\}^{P(\omega)} : s \subseteq x \}\), so \(C_A\) is a finite union of basic clopen sets, hence closed.

(3). Take any finite and nonempty \(A_1, \ldots, A_n \subseteq P(\omega)\). Since \(A = A_1 \cup \cdots \cup A_n\) is finite, so \(C_A \neq \emptyset\) by (1). On the other hand, it is not difficult to see that \(C_A \subseteq C_{A_1} \cap \cdots \cap C_{A_n}\).

\((\Leftarrow)\) Let \(\delta > 0\) and \(x : P(\omega) \to \{0,1\}\) be such that \(\left\| \bigcap \{ A^{x(A)} : A \in \mathcal{A} \} \right\|_\phi \geq \delta\) for every finite and nonempty family \(\mathcal{A} \subseteq P(\omega)\).

By [9, Theorem 3.6], \(I\) has BW property if and only if there is \(\varepsilon > 0\) such that for every \(N \in \omega\) and every partition \(A_1, \ldots, A_N\) of \(\omega\) there is \(i \leq N\) with \(\|A_i\|_\phi \geq \varepsilon\).

Let \(\varepsilon = \delta\). Let \(N \in \omega\) and \(A_1, \ldots, A_N\) be a partition of \(\omega\). Let \(\mathcal{A} = \{ A_1, \ldots, A_n \}\). Since \(\mathcal{A}\) is a partition of \(\omega\) so there is \(i \leq N\) with \(x(A_i) = 0\) (otherwise \(\bigcap \{ A^{x(A)} : A \in \mathcal{A} \} \subseteq \emptyset\) hence \(\left\| \bigcap \{ A^{x(A)} : A \in \mathcal{A} \} \right\|_\phi = 0 < \delta\)). Thus \(A_i \supseteq \bigcap \{ A^{x(A)} : A \in \mathcal{A},\}\), hence

\[\|A_i\|_\phi \geq \left\| \bigcap \{ A^{x(A)} : A \in \mathcal{A} \} \right\|_\phi \geq \delta = \varepsilon.\]

\[\square\]

3. \(F_\sigma\) Ideals

Proposition 3.1. Let \(I = \text{Fin}(\phi)\) be an \(F_\sigma\) ideal. Then \(s(I), \tau(I) \geq \omega_1\).

Proof. \((s(I)) \geq \omega_1\).) Let \(S = \{ S_n : n \in \omega \} \subseteq I^+\). We will show that \(S\) is not an \(I\)-splitting family i.e. we will construct an \(A \in I^+\) such that \(A \cap S_0 \in \mathcal{I}\) or \(A \cap S_1 \in \mathcal{I}\) for every \(n \in \omega\).

Let \(\varepsilon \in \{0,1\}^\omega\) be a sequence such that \(\bigcap_{n \leq \varepsilon} S_i \subseteq I^+\) for every \(n \in \omega\). By lsc of \(\phi\), we can find finite sets \(F_n (n \in \omega)\) such that \(F_n \subseteq \bigcap_{n \leq \varepsilon} S_i^n\) and \(\phi(F_n) \geq \varepsilon\).

Let \(\mathcal{A} = \bigcup_n F_n\). Then \(A \in I^+\) and \(A \cap S_i^{\varepsilon - n} \subseteq \bigcup_{n \leq \varepsilon} F_i \in I\) for every \(n \in \omega\).

\((\tau(I)) \geq \omega_1\).) Let \(\mathcal{R} = \{ R_n : n \in \omega \} \subseteq I^+\). We will show that \(\mathcal{R}\) is not an \(I\)-unsplitable family i.e. we will construct a set \(A \subseteq \omega\) such that \(R_n \cap A^0 \in I^+\) and \(R_n \cap A^1 \in I^+\) for every \(n \in \omega\).

By lsc of \(\phi\), we can find pairwise disjoint finite sets \(F_{i,n}^k (i, n \in \omega, k \in \{0,1\})\) such that \(F_{i,n}^k \subseteq R_n\) and \(\phi(F_{i,n}^k) \geq k\) for every \(i, n \in \omega, k \in \{0,1\}\).

Let \(A = \bigcup_{i,n \in \omega} F_{i,n}^k\). If \(n \in \omega\) and \(k \in \{0,1\}\), then \(R_n \cap A^k \supseteq \bigcup_{i \in \omega} F_{i,n}^k\) and hence \(R_n \cap A^k \in I^+\).

\[\square\]

Theorem 3.2. Assume MA. Let \(I = \text{Fin}(\phi)\) be an \(F_\sigma\) ideal. Then \(s(I) = \tau(I) = \omega_1\).

Proof. \((s(I)) = \omega_1\).) Let \(S \subseteq P(\omega)\) be such that \(|S| = \kappa < \omega_1\). We will show that \(S\) is not an \(I\)-splitting family.

Let \(x : P(\omega) \to \{0,1\}\) be as in Corollary 2.5. Let \(\mathcal{F} = \{ S^{x(S)} : S \in S\}\) and \(\mathcal{P} = [\omega]^\omega \times [\mathcal{F}]^{\omega}\). For \((s, A), (t, B) \in \mathcal{P}\) we define \((s, A) \leq (t, B)\) if

\[(1)\] \(s \supseteq t\), and
\[(2)\] \(A \supseteq B\), and
\[(3)\] \(s \setminus t \subseteq \bigcap B\).
Then it is not difficult to show that \( \langle \mathbb{P}, \leq \rangle \) is a ccc poset.

Define

1. \( D_F = \{(s, A) \in \mathbb{P} : F \subseteq A\} \) for every \( F \in \mathcal{F} \).
2. \( D_n = \{(s, A) \in \mathbb{P} : \phi(s) > n\} \) for every \( n \in \omega \),

It is easy to see that \( D_F \) is dense for every \( F \). We show that \( D_n \) is also dense for every \( n \).

Let \((s, A) \in \mathbb{P} \) and \( A = \{F_0, \ldots, F_{m-1}\} \). Let \( F_i = S^{\varepsilon(S_i)}_i, S_i \in \mathcal{S} \) for \( i<m \).

Since \( \bigcap A = \bigcap_{i < m} F_i = \bigcap_{i < m} S_i^{\varepsilon(S_i)} \), so \( \phi(\bigcap A) = \infty \). By lsc of \( \phi \) there is a finite set \( t \subseteq \bigcap A \) such that \( \phi(t) > n \). Then \((s \cup t, A) \in D_n \) and \((s \cup t, A) \leq (s, A) \).

Applying Martin’s Axiom, there is a filter \( G \subseteq \mathbb{P} \) such that \( G \cap D_n \neq \emptyset \) and \( G \cap D_F \neq \emptyset \) for every \( n \in \omega \) and \( F \in \mathcal{F} \). Let

\[
X = \bigcup \{s : (s, A) \in G\}.
\]

Clearly \( X \in \mathcal{I}^+ \), and \( X \) is not \( \mathcal{I} \)-split by any member of \( S \) because if \( F = S^{\varepsilon(S)} \in \mathcal{F} \) and \((s, A) \in G \cap D_F \), then \( X \cap S^{1-\varepsilon(S)} \subseteq s \) and hence \( X \cap S^{1-\varepsilon(S)} \in \mathcal{I} \).

\((\tau(\mathcal{I}) = \varepsilon)\) Let \( \kappa < \varepsilon \) and \( F = \{F_n : \alpha < \kappa\} \subseteq \mathcal{I}^+ \). We will show that there is a set which \( \mathcal{I} \)-splits all members of \( \mathcal{F} \).

Let \( \mathbb{P} = 2^{\omega_1} \). Then \( (\mathbb{P}, \supseteq) \) is a ccc poset.

Define

\[
D_{\alpha,n} = \{s \in \mathbb{P} : \phi(s^{-1}(0) \cap F_n) > n \land \phi(s^{-1}(1) \cap F_n) > n\}
\]

for every \( n \in \omega \) and \( \alpha < \kappa \). Using lsc of \( \phi \) it is not difficult to show that sets \( D_{\alpha,n} \) are dense in \( \mathbb{P} \).

Applying Martin’s Axiom, there is a filter \( G \subseteq \mathbb{P} \) such that \( G \cap D_{\alpha,n} \neq \emptyset \) for every \( n \in \omega \) and \( \alpha < \kappa \). Let

\[
f = \bigcup G \text{ and } X = f^{-1}(0).
\]

Then it is easy to see that \( X \in \mathcal{I}^+ \). We will show that \( X \mathcal{I} \)-splits all sets in \( \mathcal{F} \).

Let \( \alpha < \kappa \). For any \( n \in \omega \) there is \( s_n \in G \cap D_{\alpha,n} \). Since \( F_n \cap X^1 \supseteq F_n \cap s_n^{-1}(i) \) for \( i = 0, 1 \) and every \( n \), we have \( \phi(F_n \cap X^1) > n \) for \( i = 0, 1 \) and every \( n \), and so \( F_n \cap X^1 \in \mathcal{I}^+ \) (\( i = 0, 1 \)).

4. Analytic P-ideals

**Proposition 4.1.** Let \( \mathcal{I} = \text{Exh}(\phi) \) be an analytic P-ideal. Then \( \tau(\mathcal{I}) \geq \omega_1 \).

**Proof.** Let \( \mathcal{F} = \{F_n \in \mathcal{I}^+ : n \in \omega\} \). We will show that there is a set which \( \mathcal{I} \)-splits all members of \( \mathcal{F} \).

Let \( \delta_n > 0 \) be such that \( \|F_n\|_\phi > \delta_n \) for every \( n \in \omega \). Let \( \{G_n : n \in \omega\} \) be a sequence such that \( \{G_n : n \in \omega\} = \{F_n : n \in \omega\} \) and \( \{k \in \omega : G_k = F_n\} \) is infinite for each \( n \in \omega \). Let \( f : \omega \rightarrow \omega \) be such that \( G_n = F_{f(n)} \) for every \( n \in \omega \). We will construct sequences \( (s_n : n \in \omega) \) and \( (t_n : n \in \omega) \) such that

1. \( s_n, t_n \) are finite,
2. \( s_n, t_n \subseteq G_n \setminus \{0, 1, \ldots, n-1\} \) for every \( n \in \omega \),
3. \( s_n \cap s_k = \emptyset, t_n \cap t_k = \emptyset \) and \( s_n \cap t_k = \emptyset \) for every \( n, k \in \omega \),
4. \( \phi(s_n) > \delta_{f(n)}, \phi(t_n) > \delta_{f(n)} \).
Suppose that we have already constructed $s_i, t_i$ for $i \leq n$. Let $s = s_0 \cup \cdots \cup s_n$ and $t = t_0 \cup \cdots \cup t_n$. Let $G = G_{n+1} \setminus (s \cup t)$. Since $s \cup t$ is finite so $\|G\|_\phi > \delta_{f(n+1)}$.

By the definition of $\|\cdot\|_\phi$ and lsc of $\phi$ there is a finite set $s_{n+1} \subseteq G \setminus \{0,1,\ldots,n\}$ with $\phi(s_{n+1}) > \delta_{f(n+1)}$. Applying the definition of $\|\cdot\|_\phi$ and lsc of $\phi$ again, there is a finite set $t_{n+1} \subseteq G \setminus s_{n+1}$ with $\phi(t_{n+1}) > \delta_{f(n+1)}$.

Let $X = \bigcup_{n \in \omega} s_n$. Then $s_n \subseteq G_n \setminus \{0,1,\ldots,n-1\} = F_0 \setminus \{0,1,\ldots,n-1\}$ for every $n \in f^{-1}(0)$. Thus $\phi(X \setminus \{0,1,\ldots,n-1\}) \geq \phi(s_n) > \delta_0 > 0$ for every $n \in f^{-1}(0)$, hence $\|X\|_\phi \geq \delta_0 > 0$. We will show that $X$ $\mathbb{I}$-splits all sets in the family $F$.

First of all, we will show that $F_k \cap X \in \mathbb{I}^+$. Let $i \in \omega$. Then there is $n \in f^{-1}(k)$ with $n > i$. Then $\phi((F_k \cap X) \setminus \{0,1,\ldots,i-1\}) = \phi((G_n \cap X) \setminus \{0,1,\ldots,i-1\}) \geq \phi((G_n \cap X) \setminus \{0,1,\ldots,n-1\}) \geq \phi(s_n) > \delta_k$. Thus $\|F_k \cap X\|_\phi \geq \delta_k > 0$.

Using the same argument as above one can show that $F_k \setminus X \in \mathbb{I}^+$. □

**Theorem 4.2.** Assume MA. Let $\mathcal{I} = \text{Exh}(\phi)$ be an analytic $P$-ideal. Then $\kappa(\mathcal{I}) = \kappa$.

**Proof.** Let $\kappa < \kappa$ and $\mathcal{F} = \{F_\alpha : \alpha < \kappa\} \subseteq \mathbb{I}^+$. Let $\delta_\alpha > 0$ be such that $\|F_\alpha\|_\phi > \delta_\alpha$ for every $\alpha < \kappa$.

Let $\mathbb{P} = 2^{<\omega}$. Then $(\mathbb{P}, \supseteq)$ is a ccc poset.

Define

$$D_{\alpha,n} = \{s \in \mathbb{P} : \phi((F_\alpha \cap s^{-1}(i)) \setminus \{0,1,\ldots,n-1\}) > \delta_n \text{ for } i = 0, 1\}$$

for every $n \in \omega$ and $\alpha < \kappa$. It is not difficult to show that $D_{\alpha,n}$ is dense in $\mathbb{P}$.

Applying Martin’s Axiom, there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_{\alpha,n} \neq \emptyset$ for every $n \in \omega$ and $\alpha < \kappa$. Let

$$f = \bigcup G$$

and $X = f^{-1}(0)$.

Then $X \in \mathbb{I}^+$ and $X$ $\mathbb{I}$-splits all sets in $F$.

□

**Theorem 4.3.** Assume MA. Let $\mathcal{I} = \text{Exh}(\phi)$ be an analytic $P$-ideal with BW property. Then $s(\mathcal{I}) = \kappa$.

**Proof.** Let $S \subseteq \mathcal{P}(\omega)$ be such that $|S| = \kappa < \omega$. We will show that $S$ is not an $\mathbb{I}$-splitting family.

Let $\delta > 0$ and $x : \mathcal{P}(\omega) \to \{0,1\}$ be as in Lemma 2.7.

Let $\mathcal{F} = \{S^{x(S)} : S \in S\}$ and $\mathbb{P} = [\omega]^{<\omega} \times [\mathcal{F}]^{<\omega}$. For $(s, A), (t, B) \in \mathbb{P}$ we define $(s, A) \leq (t, B)$ if

1. $s \supseteq t$, and
2. $A \supseteq B$, and
3. $s \setminus t \subseteq \bigcap B$.

Then it is not difficult to show that $(\mathbb{P}, \leq)$ is a ccc poset.

Define

1. $D_n = \{(s, A) \in \mathbb{P} : \phi(s \setminus \{0,1,\ldots,n-1\}) > \frac{n}{2}\}$ for every $n \in \omega$,
2. $D_F = \{(s, A) \in \mathbb{P} : F \in A\}$ for every $F \in \mathcal{F}$.

Clearly $D_F$ is dense for every $F \in \mathcal{F}$. We will show that sets $D_n$ are dense.

Let $(s, A) \in \mathbb{P}$ and $A = \{F_0, \ldots, F_{m-1}\}$. Let $F_i = S^{x(S_i)}, S_i \in S$ for $i < m$. Since

$$\bigcap A = \bigcap_{i<m} F_i = \bigcap_{i<m} S^{x(S_i)},$$

so $\bigcap A \setminus \phi \geq \delta$. Since $\|\bigcap A\|_\phi = \lim_{k \to \infty} \phi(\bigcap A \setminus $
{0, 1, ..., k − 1}) so φ(∩A \ {0, 1, ..., n − 1}) > \frac{3}{2}. By lsc of φ there is a finite set \( t \subseteq \bigcap A \setminus \{0, 1, ..., n - 1\} \) such that φ(t) > \frac{3}{2}. Then (s ∪ t, A) ∈ Dn and (s ∪ t, A) ≤ (s, A).

Applying Martin’s Axiom, there is a filter \( G \subseteq \mathcal{P} \) such that \( G \cap D_n \neq \emptyset \) and \( G \cap D_F \neq \emptyset \) for every \( n \in \omega \) and \( F \in \mathcal{F} \). Let

\[
X = \bigcup \{s : (s, A) \in G\}.
\]

Clearly, \( \|X\|_\varphi \geq \frac{3}{2} \) so \( X \in \mathcal{I}^+ \), and \( X \) is not \( \mathcal{I} \)-split by any member of \( \mathcal{S} \) because if \( S \in \mathcal{S} \), \( F = S^{\omega}(S) \), and \( (s, A) \in G \cap D_F \), then \( X \cap S^{\omega - x(S)} \subseteq s \).

5. Applications

It is not difficult to prove that the Bolzano-Weierstrass theorem (that every bounded sequences of reals has a convergent subsequence) fails if we consider sequences of functions instead of reals (i.e. there exists a uniformly bounded sequence \((f_n)_{n \in \omega}\) of real-valued functions defined on \( \mathbb{R} \) such that no subsequence of \((f_n)_{n \in \omega}\) is pointwise convergent). The ideal versions of this result is presented below (in this case we have to consider two cases: either \( \mathcal{I} \) is a “somewhere” maximal ideal or not).

Let \( \mathcal{I} \) be an ideal on \( \omega \) and \( A \subseteq \omega \). We say that a sequence \((f_n)_{n \in A}\) of real-valued functions defined on a set \( X \) is pointwise \( \mathcal{I} \)-convergent to \( f : X \to \mathbb{R} \) if for every \( x \in X \) the sequence of reals \((f_n(x))_{n \in A}\) is \( \mathcal{I} \)-convergent to \( f(x) \). (See [18], [20] and [6] for description of pointwise \( \mathcal{I} \)-limits of continuous functions; in [12], [5] and [11] the authors consider also ideal version of discrete and equal convergence of sequences of functions.)

For an ideal \( \mathcal{I} \) on \( \omega \) and \( A \subseteq \omega \) we define the ideal \( \mathcal{I} \upharpoonright A = \{B \subseteq \omega : B \cap A \in \mathcal{I}\} \).

Proposition 5.1. Let \( \mathcal{I} \) be an ideal on \( \omega \). Let \( f_n : \mathbb{R} \to \mathbb{R} \ (n \in \omega) \) be a uniformly bounded sequence of functions.

1. If \( \mathcal{I} \) is a maximal ideal then \((f_n)_{n \in \omega}\) is pointwise \( \mathcal{I} \)-convergent.
2. If there is \( A \in \mathcal{I}^+ \) such that \( \mathcal{I} \upharpoonright A \) is a maximal ideal then the subsequence \((f_n)_{n \in A}\) is pointwise \( \mathcal{I} \)-convergent.

Proof. (1). Follows from the fact that every bounded sequence of reals is \( \mathcal{I} \)-convergent for a maximal ideal \( \mathcal{I} \).

(2). Follows from (1).

Proposition 5.2. Let \( \mathcal{I} \) be an ideal on \( \omega \) such that \( \mathcal{I} \upharpoonright A \) is not maximal for any \( A \in \mathcal{I}^+ \). There exists a uniformly bounded sequence of functions \( f_n : \mathbb{R} \to \mathbb{R} \ (n \in \omega) \) such that \((f_n)_{n \in A}\) is not pointwise \( \mathcal{I} \)-convergent for any \( A \in \mathcal{I}^+ \).

Proof. Let \( \{0, 1\}^\omega = \{s_\alpha : \alpha < \mathfrak{c}\} \) and \( \mathbb{R} = \{x_\alpha : \alpha < \mathfrak{c}\} \). We define \( f_n : \mathbb{R} \to \mathbb{R} \) by \( f_n(x_\alpha) = s_\alpha(n) \ (n \in \omega, \alpha < \mathfrak{c}) \).

Let \( A \in \mathcal{I}^+ \). Then there are \( B, C \subseteq \omega \) such that \( A = B \cup C \), \( B \cap C = \emptyset \) and \( B, C \in \mathcal{I}^+ \).

Let \( \alpha \) be such that \( s_\alpha(n) = 0 \) for \( n \in B \) and \( s_\alpha(n) = 1 \) for \( n \in C \).

Since \( \mathcal{I}^+ \supseteq C \subseteq \{n : f_n(x_\alpha) \neq 0\} \) and \( \mathcal{I}^+ \supseteq B \subseteq \{n : f_n(x_\alpha) \neq 1\} \), so \((f_n)_{n \in A}\) is not \( \mathcal{I} \)-convergent.

Saks asked the question (see [24]) if for every uniformly bounded sequence \((f_n)_{n \in \omega}\) of real-valued functions defined on \( \mathbb{R} \) there exists an infinite set \( A \subseteq \omega \)
such that the subsequence \((f_n(x))_{n \in A}\) is convergent for uncountably many \(x \in \mathbb{R}\). This question was answered in the negative by Sierpiński ([24]) under the assumption of the Continuum Hypothesis (CH). Later, Fuchino and Plewik proved ([14]) that if \(\mathfrak{s} > \omega_1\) then the answer to the question is positive. In fact, they proved that for every uniformly bounded sequence \(f_n : \mathbb{R} \to \mathbb{R}\) and every \(X \subseteq \mathbb{R}, |X| < \mathfrak{s}\) there exists an infinite \(A \subseteq \omega\) such that \((f_n \upharpoonright X)_{n \in A}\) is pointwise convergent. The ideal versions of these results are presented below.

First, if \(\mathcal{I}\) is a “somewhere” maximal ideal then the answer to ideal version of Saks question is positive (by Proposition 5.1).

Second, if an ideal \(\mathcal{I} \not\in \text{BW}\) then there exists (in ZFC) a uniformly bounded sequence \((f_n)_{n \in \omega}\) of real-valued functions defined on \(\mathbb{R}\) such that for every \(A \in \mathcal{I}^+\) the subsequence \((f_n(x))_{n \in A}\) is \(\mathcal{I}\)-convergent for less than \(\mathfrak{c}\) many \(x \in \mathbb{R}\). (Indeed, let \((x_n)_{n \in \omega}\) be a bounded sequence such that \((x_n)_{n \in A}\) is not \(\mathcal{I}\)-convergent for any \(x \in \mathbb{R}\). Then the functions \(f_\alpha(x) = x_n (n \in \omega, x \in \mathbb{R})\) are as required.) Thus, the answer to ideal version of Saks question is negative.

Below (Corollaries 5.4 and 5.6) we prove that in the third case (i.e. \(\mathcal{I} \in \text{BW}\) and \(\mathcal{I} \upharpoonright A\) is not a maximal ideal) the answer to ideal version of Saks question is independent of ZFC for \(F_\sigma\) ideals and analytic P-ideals.

**Proposition 5.3.** Let \(\mathcal{I}\) be an ideal on \(\omega\). If \(\tau(\mathcal{I}) = \mathfrak{c}\) then there exists a uniformly bounded sequence \((f_n)_{n \in \omega}\) of real-valued functions defined on \(\mathbb{R}\) such that for every \(A \in \mathcal{I}^+\) the subsequence \((f_n(x))_{n \in A}\) is \(\mathcal{I}\)-convergent for less than \(\mathfrak{c}\) many \(x \in \mathbb{R}\).

**Proof.** Let \(\mathcal{R} = \{x_\alpha : \alpha < \mathfrak{c}\}\) and \(\mathcal{I}^+ = \{A_\alpha : \alpha < \mathfrak{c}\}\). We defined \(f_n : \mathbb{R} \to \mathbb{R}\) by

\[
 f_n(x_\alpha) = \begin{cases} 
 0 & \text{for } n \in S_\alpha, \\
 1 & \text{for } n \in \omega \setminus S_\alpha,
\end{cases}
\]

where \(S_\alpha \in \mathcal{I}^+\) is a set that \(\mathcal{I}\)-splits the family \(\{A_\beta : \beta < \alpha\}\) (there is one since \(|\alpha| < \tau(\mathcal{I})\)).

Let \(A = A_\beta \in \mathcal{I}^+\). We will show that the subsequence \((f_n(x_\alpha))_{n \in A}\) is not \(\mathcal{I}\)-convergent for every \(\alpha > \beta\) and that will finish the proof.

Let \(\alpha > \beta\). Then \(\{n \in A : f_n(x_\alpha) = 0\} = A_\beta \cap S_\alpha \in \mathcal{I}^+\) and \(\{n \in A : f_n(x_\alpha) = 1\} = A_\beta \setminus S_\alpha \in \mathcal{I}^+\). Thus \((f_n(x_\alpha))_{n \in A}\) is not \(\mathcal{I}\)-convergent. \(\square\)

**Corollary 5.4.** Assume CH. Let \(\mathcal{I}\) be an \(F_\sigma\) ideal or analytic P-ideal on \(\omega\). There exists a uniformly bounded sequence \((f_n)_{n \in \omega}\) of real-valued functions defined on \(\mathbb{R}\) such that \(\{x : (f_n(x))_{n \in \omega}\) is \(\mathcal{I}\)-convergent\} is countable for every \(A \in \mathcal{I}^+\).

**Proof.** Apply Proposition 5.3 and Proposition 3.1 or 4.1 respectively. \(\square\)

**Proposition 5.5.** Let \(\mathcal{I}\) be an ideal on \(\omega\) with BW property. Let \(f_n : \mathbb{R} \to \mathbb{R}\) \((n \in \omega)\) be a uniformly bounded sequence of functions. Let \(X \subseteq \mathbb{R}\) be such that \(|X| < \mathfrak{s}(\mathcal{I})\). There exists \(A \subseteq \mathcal{I}^+\) such that \((f_n \upharpoonright X)_{n \in A}\) is pointwise \(\mathcal{I}\)-convergent.

**Proof.** The proof is a slight modification of the proof of [14, Lemma 4]. We provide it for the completeness.

Let \(|X| = \kappa < \mathfrak{s}(\mathcal{I})\). For every \(x, y \in \mathbb{R}\) let \(C^y_x = \{n \in \omega : f_n(x) < y\}\). Let \(\mathcal{C} = \{C^y_x : q \in \mathcal{Q}, x \in X\}\). Since \(|\mathcal{C}| < \mathfrak{s}(\mathcal{I})\), so there exists \(A \in \mathcal{I}^+\) such that \(A \cap \mathcal{C} \in \mathcal{I}\) or \(A \setminus \mathcal{C} \in \mathcal{I}\) for every \(C \in \mathcal{C}\).

We claim that \((f_n \upharpoonright X)_{n \in A}\) is \(\mathcal{I}\)-convergent to the function \(f : X \to \mathbb{R}\) given by \(f(x) = \inf \{y \in \mathbb{R} : \{n \in A : f_n(x) < y\} \in \mathcal{I}^+\} = \inf \{y \in \mathbb{R} : A \cap C^y_x \subseteq \mathcal{I}^+\}\).
Let $x \in X$ and $\varepsilon > 0$. Let $B_1 = \{ n \in A : f_n(x) < f(x) - \varepsilon \}$ and $B_2 = \{ n \in A : f_n(x) > f(x) + \varepsilon \}$.

Since $\{ n \in A : |f_n(x) - f(x)| > \varepsilon \} = B_1 \cup B_2$, so it is enough to show that $B_1, B_2 \in \mathcal{I}$.

Suppose that $B_1 \in \mathcal{I}^+$. Since $A \cap C^{f(x) - \varepsilon}_x = B_1 \in \mathcal{I}^+$, so $f(x) = \inf \{ y \in \mathbb{R} : A \cap C^y_x \in \mathcal{I}^+ \} \leq f(x) - \varepsilon$, a contradiction.

Suppose that $B_2 \in \mathcal{I}^+$. Let $q \in \mathbb{Q}$ be such that $f(x) < q < f(x) + \varepsilon$. Since $B_2 \subseteq A \setminus C^q_x$, so $A \cap C^q_x \notin \mathcal{I}$. But $C^q_x \in \mathcal{C}$ and $\mathcal{C}$ does not $\mathcal{I}$-split $A$, so $A \cap C^q_x \in \mathcal{I}$. So $f(x) = \inf \{ y \in \mathbb{R} : A \cap C^q_x \in \mathcal{I}^+ \} \geq q$, a contradiction. \qed

Remark. The assumption that $\mathcal{I}$ has BW property is necessary in Proposition 5.5. Indeed, let $\mathcal{I}$ be an ideal without BW. By Theorem 2.3, $\mathfrak{s}(\mathcal{I}) = \omega$. If $(f_n)_{n \in \omega}$ is the sequence defined above Proposition 5.3, and $X = \{ 0 \}$, then $|X| < \mathfrak{s}(\mathcal{I})$ but $(f_n | X)_{n \in \omega} = (x_n)_{n \in \omega}$ is not $\mathcal{I}$-convergent for any $A \in \mathcal{I}^+$.

**Corollary 5.6.** Assume MA and $\neg$CH. Let $\mathcal{I}$ be an $F_\sigma$ ideal or analytic P-ideal with BW property on $\omega$. For every uniformly bounded sequence $(f_n)_{n \in \omega}$ of real-valued functions defined on $\mathbb{R}$ there exists $A \in \mathcal{I}^+$ such that the subsequence $(f_n(x))_{n \in A}$ is $\mathcal{I}$-convergent for uncountably many $x \in \mathbb{R}$.

**Proof.** Apply Proposition 5.5 and Theorems 3.2 and 4.3 respectively. \qed

Mazurkiewicz proved [22] that if one takes a uniformly bounded sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ ($n \in \omega$) then there always exists a perfect set $P \subseteq \mathbb{R}$ and an infinite set $A \subseteq \omega$ such that $(f_n(x))_{n \in A}$ is convergent for every $x \in P$. (Since perfect sets are uncountable so his result yields a positive answer to Saks question in the realm of continuous functions.) In [10] the authors proved that ideal version of Mazurkiewicz’s result holds for $F_\sigma$ ideals and analytic P-ideals with BW property.

Mazurkiewicz’s result shows (taking into account that perfect sets are of cardinality $\omega$) that for a uniformly bounded sequence of continuous functions $(f_n)_{n \in \omega}$ one always finds an infinite $A \subseteq \omega$ such that the subsequence $(f_n(x))_{n \in A}$ is convergent for $\varepsilon$ many $x \in \mathbb{R}$. Of course, Sierpiński’s result shows that under CH there is a uniformly bounded sequence $(f_n)_{n \in \omega}$ such that there is no infinite $A \subseteq \omega$ such that $(f_n(x))_{n \in A}$ is convergent for $\varepsilon$ many $x \in \mathbb{R}$. Ciesielski and Pawlikowski [4] proved that it is consistent with the axioms of ZFC that for every uniformly bounded sequence $(f_n)_{n \in \omega}$ of real-valued functions defined on $\mathbb{R}$ there exists an infinite $A \subseteq \omega$ such that the subsequence $(f_n(x))_{n \in A}$ is convergent for $\varepsilon$ many $x \in \mathbb{R}$. We do not know if the result of Ciesielski and Pawlikowski can be generalized for ideal convergence.

It is known (see e.g. [4] or [19]) that assuming MA for every uniformly bounded sequence $(f_n)_{n \in \omega}$ of real-valued functions defined on $\mathbb{R}$ and every $|X| < \varepsilon$ there exists an infinite $A \subseteq \omega$ such that the subsequence $(f_n | X)_{n \in A}$ is pointwise convergent, and on the other hand, there exists a uniformly bounded sequence $(f_n)_{n \in \omega}$ of real-valued functions defined on $\mathbb{R}$ such that for every infinite $A \subseteq \omega$ the subsequence $(f_n(x))_{n \in A}$ is convergent for less than $\varepsilon$ many $x \in \mathbb{R}$.

**Corollary 5.7.** Assume MA. Let $\mathcal{I}$ be an $F_\sigma$ ideal or analytic P-ideal with BW property on $\omega$. For every uniformly bounded sequence $(f_n)_{n \in \omega}$ of real-valued functions defined on $\mathbb{R}$ and every $|X| < \varepsilon$ there exists $A \in \mathcal{I}^+$ such that the subsequence $(f_n | X)_{n \in A}$ is pointwise $\mathcal{I}$-convergent.
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Proof. Apply Proposition 5.5 and Theorems 3.2 or 4.3 respectively.

Corollary 5.8. Assume MA. Let $I$ be an $F_\sigma$ ideal or analytic $P$-ideal on $\omega$. There exists a uniformly bounded sequence $(f_n)_{n \in \omega}$ of real-valued functions defined on $\mathbb{R}$ such that for every $A \in I^+$ the subsequence $(f_n(x))_{n \in A}$ is $I$-convergent for less than $\epsilon$ many $x \in \mathbb{R}$.

Proof. Apply Proposition 5.3 and Theorems 3.2 or 4.2 respectively.

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