

THERE ARE MEASURABLE HAMEL FUNCTIONS

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ABSTRACT. We say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is a *Hamel function* if f , considered as a subset of \mathbb{R}^2 , is a Hamel basis of \mathbb{R}^2 . We show that there is a Marczewski measurable Hamel function. Additionally, we show that there is a Hamel function which is both Lebesgue measurable and with the Baire property.

The symbols \mathbb{R} and \mathbb{Q} stand for the sets of all real and all rational numbers, respectively. A basis of \mathbb{R}^n as a linear space over \mathbb{Q} is called a *Hamel basis*. The cardinality of a set X we denote by $|X|$.

A σ -ideal \mathcal{I} of subsets of \mathbb{R} is family closed under subsets and countably unions. \mathcal{N} denotes the σ -ideal of Lebesgue null sets. \mathcal{M} denotes the σ -ideal of all sets of first category. Recall that a σ -ideal \mathcal{I} is *Borel generated* if there exists a family $\mathcal{J} \subset \mathcal{I}$ of Borel sets such that $\mathcal{I} = \{A : A \subset B, B \in \mathcal{J}\}$. \mathcal{I} is (*ccc*) if every family of disjoint Borel sets which do not belong to the ideal is countable.

A set $A \subset \mathbb{R}$ is *Marczewski measurable* ($A \in (s)$ for short) if for every perfect set $P \subset \mathbb{R}$ either $P \cap A$ or $P \setminus A$ contains a perfect set. (Recall that a perfect set is a non-empty closed set without isolated points.) If every perfect set $P \subset \mathbb{R}$ contains a perfect subset which misses A , then A is called *Marczewski null* ($A \in (s_0)$ for short). It is known that (s) is a σ -field and (s_0) is a σ -ideal of (s) . A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *Marczewski measurable* if it is measurable with respect to the σ -field (s) (i.e. if the preimage of any open set is Marczewski measurable). In [8], Marczewski introduced the σ -field (s) to show that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is Marczewski measurable if and only if every perfect $P \subset \mathbb{R}$ has an uncountable Borel subset Q such that $f \upharpoonright Q$ is continuous. (It is known that we can replace the word “perfect” with “uncountable Borel” in the definition of (s) and (s_0) —we will use this fact in the sequel.)

\mathcal{Bor} stands for the σ -field of Borel subsets of \mathbb{R} . For every σ -ideal \mathcal{I} , $\mathcal{Bor} \Delta \mathcal{I}$ stands the σ -field of all sets of the form $A \Delta B$, where $A \in \mathcal{I}$, $B \in \mathcal{Bor}$, and $A \Delta B$ denotes the symmetric difference between A and B . It is known that the σ -field of all Lebesgue measurable sets is equal to $\mathcal{Bor} \Delta \mathcal{N}$, and the σ -field of all sets with the Baire property equals $\mathcal{Bor} \Delta \mathcal{M}$.

We say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is a *Hamel function* if f , considered as a subset of \mathbb{R}^2 , is a Hamel basis of \mathbb{R}^2 . The class of Hamel functions was introduced by Plotka and researched in [4], [5], [6], [7] and [2]. In [4], the author proved that every function $f : \mathbb{R} \rightarrow \mathbb{R}$ is a sum of two Hamel functions. This implies that there

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is a Hamel function which is not Lebesgue measurable (without the Baire property, which is not Marczewski measurable, respectively).

The aim of this paper is to show that there are Hamel functions which are measurable with respect to some σ -fields. Namely, we show the following theorems—they answer problems posed by T. Natkaniec (oral communication).

Theorem 1. *There exists a Marczewski measurable Hamel function.*

Theorem 2. *Suppose that \mathcal{I} is a σ -ideal of subsets of \mathbb{R} which contains singletons. Suppose that there exists a Borel set $B \in \mathcal{I}$ and a Hamel basis $H \subset B$ with $|B \setminus H| = 2^\omega$. Then there exists a Hamel function which is measurable with respect to the σ -field $\mathcal{Bor} \Delta \mathcal{I}$.*

Corollary 3. *Suppose that \mathcal{I} is a Borel generated (ccc) σ -ideal of subsets of \mathbb{R} which contains singletons. Suppose that there exists a Hamel basis $H \in \mathcal{I}$. Then there exists a Hamel function which is measurable with respect to the σ -field $\mathcal{Bor} \Delta \mathcal{I}$.*

If we use Corollary 3 in the case $\mathcal{I} = \mathcal{N}$ (or $\mathcal{I} = \mathcal{M}$) we get the following corollary. (Recall that the Cantor ternary set contains a Hamel basis, see e.g. [1].)

Corollary 4. *There exists a Lebesgue measurable Hamel function (a Hamel function with the Baire property, respectively).*

1. PROOFS

We will use the following lemma in our proofs.

Lemma 5. [7, Lemma 2] *Let $H_1, H_2 \subseteq \mathbb{R}$ be a Hamel bases. Suppose that $h : \mathbb{R} \setminus H_1 \rightarrow H_2$ is a bijection. Then a function $H : \mathbb{R} \rightarrow \mathbb{R}$ defined by:*

$$H(x) = \begin{cases} h(x) & \text{if } x \notin H_1 \\ 0 & \text{if } x \in H_1. \end{cases}$$

is a Hamel function.

Proof of Theorem 1. Let H_1 be a Hamel basis which is Marczewski null (see [3]). Let H_2 be a Hamel basis which contains a perfect set (see [1]). Fix a Marczewski null set S of size 2^ω such that $S \cap H_1 = \emptyset$. Choose $\{P_\alpha\}_{\alpha < 2^\omega}$ and P , pairwise disjoint perfect sets contained in H_2 and all homeomorphic to the Cantor set 2^ω . Let $\{Q_\alpha\}_{\alpha < 2^\omega}$ be an enumeration of all perfect subsets of \mathbb{R} .

We will construct by induction a family of sets Q_α^* and functions $f_\alpha : Q_\alpha^* \rightarrow \mathbb{R}$ such that Q_α^* is either the empty set or a perfect set. In case of $Q_\alpha^* = \emptyset$ we have also $f_\alpha = \emptyset$.

Assume that we are in the stage $\gamma < 2^\omega$. There are two possibilities:

- (1) $\forall \alpha < \gamma |Q_\alpha^* \cap Q_\gamma| \leq \aleph_0$.
- (2) $\exists \alpha < \gamma |Q_\alpha^* \cap Q_\gamma| = 2^\omega$.

CASE 1: Choose any perfect $Q_\gamma^* \subseteq Q_\gamma \setminus [\bigcup_{\alpha < \gamma} Q_\alpha^* \cup H_1 \cup S]$ and moreover, such that Q_γ^* is homeomorphic to the Cantor set 2^ω . (This choice is possible since H_1 and S are Marczewski null.) Next, let $f_\gamma : Q_\gamma^* \rightarrow P_\gamma$ be any homeomorphism.

CASE 2: Put $Q_\gamma^* = \emptyset$ and $f_\gamma = \emptyset$.

Now define: $f^* = \bigcup_{\gamma < 2^\omega} f_\gamma$. f^* is a bijection between $\bigcup_{\gamma < 2^\omega} Q_\gamma^*$ and some subset of $\bigcup_{\gamma < 2^\omega} P_\gamma$. Since $|\mathbb{R} \setminus [H_1 \cup \bigcup_{\gamma < 2^\omega} Q_\gamma^*]| = 2^\omega$ and $|H_2 \setminus [\bigcup_{\gamma < 2^\omega} P_\gamma]| = 2^\omega$ we can extend f^* to a bijection $f : \mathbb{R} \setminus H_1 \rightarrow H_2$ arbitrary.

Next we use Lemma 5 to obtain a Hamel function $H : \mathbb{R} \rightarrow \mathbb{R}$.

This function is Marczewski measurable. Indeed, suppose that $Q \subseteq \mathbb{R}$ is any perfect set. Then there exists $\gamma < 2^\omega$ such that $Q_\gamma = Q$.

If $Q_\gamma^* \neq \emptyset$ then $f_\gamma \subseteq H$ is a continuous function from perfect subset $Q_\gamma^* \subseteq Q_\gamma$ into \mathbb{R} .

If $Q_\gamma^* = \emptyset$ then there exists $\alpha < \gamma$ such that $|Q_\alpha^* \cap Q_\gamma| = 2^\omega$ but in this case $f_\alpha^* \upharpoonright (Q_\alpha^* \cap Q_\gamma)$ is a continuous function defined on a Borel subset of Q_γ of size 2^ω \square

Proof of Theorem 2. Let $B \in \mathcal{I}$ be a Borel set and let $H_1 \subset B$ be a Hamel basis with $|B \setminus H_1| = 2^\omega$. Let $H_2 \subseteq \mathbb{R}$ be a Hamel basis which contains some perfect set P . We can also assume that $|H_2 \setminus P| = 2^\omega$.

Since the spaces $\mathbb{R} \setminus B$ and P are Borel isomorphic, let $b : B \rightarrow P$ be a Borel bijection.

By virtue of $|B \setminus H_1| = 2^\omega$ and $|H_2 \setminus P| = 2^\omega$ we can extend b to a bijection $b^* : \mathbb{R} \setminus H_1 \rightarrow H_2$. Next we use Lemma 5 to obtain a Hamel function $H : \mathbb{R} \rightarrow \mathbb{R}$.

We will check that H is $\mathcal{Bor} \Delta \mathcal{I}$ measurable. Indeed, suppose that $U \subseteq \mathbb{R}$ is an open set. Then

$$H^{-1}[U] = \begin{cases} (b^*)^{-1}[U] & \text{if } 0 \notin U, \\ (b^*)^{-1}[U] \cup H_1 & \text{if } 0 \in U. \end{cases}$$

But we have $(b^*)^{-1}[U] \Delta b^{-1}[U] \in \mathcal{I}$, therefore $H^{-1}[U] \in \mathcal{Bor} \Delta \mathcal{I}$. \square

Proof of Corollary 3. By Theorem 2 it is enough to show that there is a Borel set $B \in \mathcal{I}$ and a Hamel basis $H \subset B$ with $|B \setminus H| = 2^\omega$.

Let $H \in \mathcal{I}$ be a Hamel basis. Let $E \in \mathcal{I}$ be a Borel set with $H \subset E$.

Since $\mathbb{R} \setminus E$ is a Borel set of cardinality 2^ω , so we can find a pairwise disjoint family \mathcal{B} of cardinality 2^ω of Borel subsets of $\mathbb{R} \setminus E$ each of size 2^ω . Since \mathcal{I} is (ccc) there exists a $B_0 \in \mathcal{B} \cap \mathcal{I}$.

Then $B = E \cup B_0 \in \mathcal{I}$ is a Borel set such that $H \subset B$ and $|B \setminus H| = 2^\omega$. \square

2. ODDS AND ENDS

Given a set $X \subset \mathbb{R}$, the porosity of X at a real $r \in \mathbb{R}$ is defined by

$$p(X, r) = \limsup_{\varepsilon \rightarrow 0^+} \frac{\lambda(X, (r - \varepsilon, r + \varepsilon))}{\varepsilon},$$

where $\lambda(X, I)$ denotes the maximal length of an open subinterval of the interval I which is disjoint from X . A set X is *porous* ($X \in \mathcal{P}$) iff $p(X, a) > 0$ for every $a \in X$. Let $\sigma\mathcal{P}$ denote the sigma-ideal generated by the porous sets. We say that X is a σ -porous set iff $X \in \sigma\mathcal{P}$. (For some properties of σ -porous sets see e.g. [9].)

Let \mathcal{E} be a σ -ideal generated by closed Lebesgue null sets, and let $\mathcal{N} \cap \mathcal{M}$ denote the σ -ideal of sets which are both Lebesgue null and of the first category. It is known that $\sigma\mathcal{P}, \mathcal{E} \subset \mathcal{N} \cap \mathcal{M}$, and $\sigma\mathcal{P} \not\subset \mathcal{E}$, $\mathcal{E} \not\subset \sigma\mathcal{P}$.

If we use Theorem 2 in the case $\mathcal{I} = \sigma\mathcal{P}$, \mathcal{E} or $\mathcal{N} \cap \mathcal{M}$ we get the following corollary.

Corollary 6. (1) *There exists a Hamel function which is measurable with respect to the σ -field $\mathcal{Bor} \Delta \sigma\mathcal{P}$.*

(2) *There exists a Hamel function which is measurable with respect to the σ -field $\mathcal{Bor} \Delta \mathcal{E}$.*

- (3) *There exists a Hamel function which is measurable with respect to the σ -field $\mathcal{Bor}\Delta(\mathcal{N} \cap \mathcal{M})$ (i.e. a Hamel function which is both Lebesgue measurable and with the Baire property).*

Proof. (1). The Cantor ternary set $C \subset [0, 1]$ is σ -porous and contains a Hamel basis $H \subset C$. Let $A \subset \mathbb{R} \setminus C$ be a Borel σ -porous set of cardinality 2^ω . Now, we can use Theorem 2 with $B = C \cup A$.

(2). The Cantor ternary set $C \subset [0, 1]$ belongs to \mathcal{E} and contains a Hamel basis $H \subset C$. Let $A \subset \mathbb{R} \setminus C$ be a Borel set which belongs to \mathcal{E} and is of cardinality 2^ω . Now, we can use Theorem 2 with $B = C \cup A$.

(3). Since $\mathcal{E} \subset \mathcal{N} \cap \mathcal{M}$, so every function which is $\mathcal{Bor}\Delta\mathcal{E}$ measurable is also $\mathcal{Bor}\Delta(\mathcal{N} \cap \mathcal{M})$ measurable. \square

Remark. In case of $\mathcal{N} \cap \mathcal{M}$ we can also use Corollary 3 (since this ideal is Borel generated and (ccc)). However, in case of $\sigma\mathcal{P}$ and \mathcal{E} we cannot use Corollary 3 since it is known that these ideals are not (ccc).

We can also construct a Hamel function which is measurable in one sense and non-measurable in another.

- Proposition 7.** (1) *There exists a Lebesgue measurable Hamel function without the Baire property.*
 (2) *There exists a Lebesgue nonmeasurable Hamel function with the Baire property.*

Proof. We will show the first case and the second one can be shown similarly.

Let H_1 be a Hamel basis which is Lebesgue null and does not have the Baire property. Let $B \in \mathcal{N}$ be a Borel set with $H \subset B$ and $|B \setminus H| = 2^\omega$. Now, we proceed as in the proof of Theorem 2 and construct a Hamel function $H : \mathbb{R} \rightarrow \mathbb{R}$. Then H is Lebesgue measurable. On the other hand, $H^{-1}(\{0\}) = H_1$, so H does not have the Baire property. \square

Finally, we show that Theorem 1 does not follow from Theorem 2.

Proposition 8 (folklore). *$(s) \neq \mathcal{Bor}\Delta\mathcal{I}$ for every σ -ideal \mathcal{I} .*

Proof. We provide a proof for the completeness. Suppose, for the sake of contradiction, that there is a σ -ideal \mathcal{I} with $(s) = \mathcal{Bor}\Delta\mathcal{I}$. We have two cases.

- (1) $\mathcal{I} \setminus (s_0) \neq \emptyset$.
 (2) $\mathcal{I} \subset (s_0)$.

In the first case, take an $A \in \mathcal{I} \setminus (s_0)$. Since $A \in (s)$ so there is a perfect set $P \subset A$. Now, take a set $B \subset P$ with $B \notin (s)$. On the other hand, $B \in \mathcal{I} \subset \mathcal{Bor}\Delta\mathcal{I}$, a contradiction.

Now, we consider the second case. Let $f : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ be a Borel isomorphism. It is not difficult to show, that $A \in (s)$ iff $f(A) \in (s^2)$ and $A \in (s_0)$ iff $f(A) \in (s_0^2)$ (here (s^2) and (s_0^2) stands for the σ -field of Marczewski measurable subsets of the plane and σ -ideal of Marczewski null subsets of the plane, which are defined similarly to (s) and (s_0) .)

Let $\mathcal{K} = \{f(A) : A \in \mathcal{I}\}$. Then \mathcal{K} is a σ -ideal and $(s^2) = \mathcal{Bor}^2\Delta\mathcal{K}$ (here \mathcal{Bor}^2 stands for the σ -field of Borel subsets of the plane.)

Let $Z \in (s_0)$ be a set of cardinality 2^ω . Since $\{X \times \mathbb{R} : X \subset Z\} \subset (s^2)$ is of cardinality 2^{2^ω} , $|\mathcal{Bor}| = 2^\omega$, and for each $X \subset Z$ there is a Borel set $B_X \subset$

\mathbb{R} and an $A_X \subset \mathbb{R}$, $A_X \in \mathcal{I}$ with $X \times \mathbb{R} = f(B_X) \Delta f(A_X)$, so there are two distinct sets $X_1, X_2 \subset Z$, a Borel set $B \subset \mathbb{R}$ and two sets $A_1, A_2 \in \mathcal{I}$ such that $X_1 \times \mathbb{R} = f(B) \Delta f(A_1)$ and $X_2 \times \mathbb{R} = f(B) \Delta f(A_2)$. Let $W = (X_1 \times \mathbb{R}) \Delta (X_2 \times \mathbb{R}) = f(A_1) \Delta f(A_2) = f(A_1 \Delta A_2)$. Then $f^{-1}(W) = A_1 \Delta A_2 \in \mathcal{I} \subset (s_0)$. On the other hand, since $X_1 \neq X_2$, so $W \notin (s_0^2)$. Thus $f^{-1}(W) \notin (s_0)$, a contradiction. \square

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