ON HINDMAN SPACES AND THE BOLZANO-WEIERSTRASS PROPERTY

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Abstract. We examine relationships between two classes of topological spaces defined with the aid of the Hindman ideal. We also do the same for another ideal — instead of sums, as in the Hindman ideal, we consider differences.

1. Introduction

In the sequel we assume all our topological spaces to be Hausdorff. The set of natural numbers we denote by the symbol ω. By Y^X we denote the set of all functions from X into Y, and in case of $X \subseteq \omega$ the set Y^X is the set of all sequences $(y_n)_{n\in X}$ with $y_n \in Y$ for $n \in X$. Let $Y^{<\omega} = \bigcup_{n\in\omega} Y^{\{0,1,\ldots,n-1\}}$.

An *ideal* on ω is a family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ (where $\mathcal{P}(\omega)$ denotes the power set of ω) which is closed under taking subsets and finite unions. In the sequel we assume that all considered ideals contain all finite sets and are proper (i.e. $\omega \notin \mathcal{I}$).

Let $\mathcal{I}$ be an ideal on ω. It seems interesting to describe the class of topological spaces X such that for every sequence $(x_n)_{n\in\omega} \in X^\omega$ there exists a converging subsequence $(x_{n_k})_{k\in\omega}$ such that $\{n_k : k \in \omega\} \notin \mathcal{I}$ (i.e. the set of indexes is large in sense of the ideal $\mathcal{I}$). (Note that for the ideal $\mathcal{I}$ of all finite subsets of ω we obtain the class of all sequentially compact spaces.) This question was already considered in the literature. In [7] Kojman examines the above property for the van der Waerden ideal

$$\mathcal{W} = \{A \subseteq \omega : A \text{ does not contain arithmetic progressions of arbitrary length}\},$$

in [3] Flašková examines the ideal $\mathcal{I}_{1/n} = \left\{A \subseteq \omega : \sum_{n\in A} \frac{1}{n} < \infty\right\}$, in [1] and [2] the authors characterize ideals $\mathcal{I}$ for which the above property holds for some subclass of topological spaces.

However, this property seems to be too strong for some ideals. For instance, in [6, Theorem 3], the author proved that there is no uncountable space X which satisfies the above property for the Hindman ideal

$$\mathcal{H} = \{A \subseteq \omega : \text{ there is no infinite set } D \subseteq A \text{ with } \mathrm{FS}(D) \subseteq A\},$$

where $\mathrm{FS}(D) = \left\{\sum_{n\in F} n : F \subseteq D \text{ is nonempty and finite}\right\}$.

One can cope with this difficulty by changing the hypothesis that the subsequence is convergent by other kinds of convergence related to the ideal $\mathcal{I}$. For example, in [4] the authors introduced IP-convergence (see Section 3 for the definition) and proved

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that there are uncountable spaces X such that for every sequence $(x_n)_{n\in\omega} \in X^\omega$ there exists a subsequence which is IP-convergent. In [6], the author calls them Hindman spaces. On the other hand, in [1] and [2] the authors, using the $\mathcal{I}$-convergence (see Section 2 for the definition), examine ideals $\mathcal{I}$ and spaces X such that for every sequence $(x_n)_{n\in\omega} \in X^\omega$ there exists a subsequence which is $\mathcal{I}$-convergent (in this case they say that $(X, \mathcal{I})$ has BW property).

In Section 3 we study the relationship between Hindman spaces and spaces X such that $(X, \mathcal{H})$ has BW property.

In Section 4 we consider an ideal similar to the Hindman ideal and examine topological spaces defined with this ideal in the similar manner as Hindman spaces.

2. Bolzano-Weierstrass property

Let X be a topological space, $\mathcal{I}$ an ideal on ω and $A \subseteq \omega$. We say that a sequence $(x_n)_{n\in A} \in X^A$ is *$\mathcal{I}$-convergent* to $x \in X$ if

$$\{n \in A : x_n \notin U\} \in \mathcal{I}$$

for every open neighborhood $U \subseteq X$, $x \in U$. We say that the pair $(X, \mathcal{I})$ has BW *property* if every sequence $(x_n)_{n\in\omega} \in X^\omega$ has an $\mathcal{I}$-convergent subsequence $(x_n)_{n\in A}$ with $A \notin \mathcal{I}$.

We write $(X, \mathcal{I}) \in BW$ if the pair $(X, \mathcal{I})$ has BW property; we say that an ideal $\mathcal{I}$ has BW *property* ($\mathcal{I} \in BW$, in short) if the pair $([0, 1], \mathcal{I}) \in BW$. For examples and properties of ideals with(out) BW property see [1] where this definition was introduced.

In the sequel we will use the following characterization of BW property.

Proposition 2.1 ([1, Proposition 3.3])**.** *An ideal $\mathcal{I}$ has BW property if and only if for every family $\{A_s : s \in \{0, 1\}^{<\omega}\}$ fulfilling the following conditions*

(T1) $A_\emptyset = \omega$,
(T2) $A_s = A_{s^\frown 0} \cup A_{s^\frown 1}$,
(T3) $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$,

there exist $x \in \{0, 1\}^\omega$ and $B \subseteq \omega$, $B \notin \mathcal{I}$ such that $B \setminus A_{x\restriction n} \in \mathcal{I}$ for all n.

We say that sets A, B are *almost disjoint* if $A \cap B$ is finite.

Let $\mathcal{A}$ be a pairwise almost disjoint family of infinite subsets of ω. Define a topological space $\Psi(\mathcal{A})$ as follows: the underlying set of $\Psi(\mathcal{A})$ is $\omega \cup \mathcal{A}$, the points of ω are isolated and a basic neighborhood of $A \in \mathcal{A}$ has the form $\{A\} \cup (A \setminus F)$, with F finite. (The space $\Psi(\mathcal{A})$ was introduced in [8].) It is known that the space $\Psi(\mathcal{A})$ is Hausdorff, regular, locally compact, first countably and separable; if $\mathcal{A}$ is infinite then $\Psi(\mathcal{A})$ is not compact (see [8] or [10, Section 11]). Moreover, it is easy to see that

- $A \cup \{A\}$ is compact in $\Psi(\mathcal{A})$ for every $A \in \mathcal{A}$;
- $K \cap \mathcal{A}$ is finite for every compact $K \subseteq \Psi(\mathcal{A})$;
- $(K \setminus \bigcup(K \cap \mathcal{A})) \cap \omega$ is finite for every compact $K \subseteq \Psi(\mathcal{A})$.

Let $\Phi(\mathcal{A}) = \Psi(\mathcal{A}) \cup \{\infty\}$ be the one-point compactification of $\Psi(\mathcal{A})$. (Recall that open neighborhoods of ∞ are of the form $\Phi(\mathcal{A}) \setminus K$ for compact sets $K \subseteq \Psi(\mathcal{A})$.) Thus, $\Phi(\mathcal{A})$ is Hausdorff and compact. It is not difficult to show that $\Phi(\mathcal{A})$ is separable and first countable at every point of $\Phi(\mathcal{A}) \setminus \{\infty\}$; if $\mathcal{A}$ is infinite then $\Phi(\mathcal{A})$ is not first countable at the point ∞. If $\mathcal{A}$ is a *mad family on* ω (i.e. infinite

maximal pairwise almost disjoint family of infinite subsets of ω), then $\Phi(\mathcal{A})$ is sequentially compact (see [7, Theorem 6]).

Theorem 2.2. *Let $\mathcal{I}$ be an ideal on ω. If $\mathcal{A} \subseteq \mathcal{I}$ is a mad family on ω, then $(\Phi(\mathcal{A}), \mathcal{I}) \in BW$.*

Proof. Let $Y = \Psi(\mathcal{A})$, $X = \Phi(\mathcal{A})$, $(x_n)_{n\in\omega} \in X^\omega$. We have 3 cases:

(1) $B = \{n \in \omega : x_n = \infty\} \notin \mathcal{I}$;
(2) $B = \{n \in \omega : x_n \in \mathcal{A}\} \notin \mathcal{I}$;
(3) $B = \{n \in \omega : x_n \in \omega\} \notin \mathcal{I}$.

In case (1) the subsequence $(x_n)_{n\in B}$ is $\mathcal{I}$-convergent to ∞ and $B \notin \mathcal{I}$.

In case (2) we have two subcases:

(2a) there is $A \in \mathcal{A}$ with $C = \{n \in B : x_n = A\} \notin \mathcal{I}$;
(2b) $\{n \in B : x_n = A\} \in \mathcal{I}$ for every $A \in \mathcal{A}$.

In case (2a) the subsequence $(x_n)_{n\in C}$ is $\mathcal{I}$-convergent to A and $C \notin \mathcal{I}$.

In case (2b) the sequence $(x_n)_{n\in B}$ is $\mathcal{I}$-convergent to ∞ and $B \notin \mathcal{I}$. Indeed, let $K \subseteq Y$ be a compact subset of Y. We will show that $\{n \in B : x_n \notin X \setminus K\} \in \mathcal{I}$. Let $K \cap \mathcal{A} = \{A_1, \dots, A_m\}$. Let $F_i = \{n \in B : x_n = A_i\}$ $(i = 1, \dots, m)$. Since $F_i \in \mathcal{I}$ for every $i = 1, \dots, m$, so $\{n \in B : x_n \notin X \setminus K\} = F_1 \cup \dots \cup F_m \in \mathcal{I}$.

In case (3) we have 3 subcases:

(3a) there is $r \in \omega$ with $C = \{n \in B : x_n = r\} \notin \mathcal{I}$;
(3b) $\{n \in B : x_n = r\} \in \mathcal{I}$ for every $r \in \omega$ and $C = \{n \in B : x_n \in A\} \notin \mathcal{I}$ for some $A \in \mathcal{A}$;
(3c) $\{n \in B : x_n = r\} \in \mathcal{I}$ for every $r \in \omega$ and $\{n \in B : x_n \in A\} \in \mathcal{I}$ for every $A \in \mathcal{A}$.

In case (3a) the subsequence $(x_n)_{n\in C}$ is $\mathcal{I}$-convergent to r and $C \notin \mathcal{I}$.

In case (3b) the subsequence $(x_n)_{n\in C}$ is $\mathcal{I}$-convergent to A and $C \notin \mathcal{I}$. Indeed, let $F \subseteq A$ be a finite set. We will show that $\{n \in C : x_n \notin \{A\} \cup (A \setminus F)\} \in \mathcal{I}$. Let $F = \{r_1, \dots, r_m\}$. Let $F_i = \{n \in B : x_n = r_i\}$ $(i = 1, \dots, m)$. Since $F_i \in \mathcal{I}$ for every $i = 1, \dots, m$, so $\{n \in C : x_n \notin \{A\} \cup (A \setminus F)\} \subseteq F_1 \cup \dots \cup F_m \in \mathcal{I}$.

In case (3c) the subsequence $(x_n)_{n\in B}$ is $\mathcal{I}$-convergent to ∞ and $B \notin \mathcal{I}$. Indeed, let $K \subseteq Y$ be a compact subset of Y. We will show that $\{n \in B : x_n \notin X \setminus K\} \in \mathcal{I}$. Let $K \cap \mathcal{A} = \{A_1, \dots, A_m\}$. Let $F_i = \{n \in B : x_n \in A_i\}$ $(i = 1, \dots, m)$. Let $F = (K \setminus (A_1 \cup \dots \cup A_m)) \cap \omega = \{r_1, \dots, r_k\}$. Let $G_j = \{n \in B : x_n = r_j\}$ $(j = 1, \dots, k)$. Since $F_i, G_j \in \mathcal{I}$ for every $i = 1, \dots, m$, $j = 1, \dots, k$, so $\{n \in B : x_n \notin X \setminus K\} \subseteq F_1 \cup \dots \cup F_m \cup G_1 \cup \dots \cup G_k \in \mathcal{I}$. □

3. Hindman ideal

For $A \subseteq \omega$ we write

$$\mathrm{FS}(A) = \left\{ \sum_{n\in F} n : F \subseteq A \text{ is nonempty and finite} \right\},$$

i.e. $\mathrm{FS}(A)$ is the set of all finite non-repeating sums of members of A. A set $A \subseteq \omega$ is called an *IP-set* if there is an infinite $D \subseteq A$ with $\mathrm{FS}(D) \subseteq A$.

Theorem 3.1 (Hindman [5])**.** *If an IP-set is partitioned into finitely many parts, then one of the parts is an IP-set.*

Let $\mathcal{H}$ denote the family of all non-IP-sets $A \subseteq \omega$. Then by Hindman's theorem $\mathcal{H}$ is an ideal, and we call it the *Hindman ideal*.

An infinite set $D \subseteq \omega$ is called *sparse* if for every $x \in \mathrm{FS}(D)$ there is a unique nonempty finite set $F \subseteq D$ with $x = \sum_{n \in F} n$. It is not difficult to show that for every infinite D there is an infinite sparse $E \subseteq D$.

For $A, B \subseteq \omega$ we write $A + B = \{a + b : a \in A \wedge b \in B\}$. If $n \in \omega$ then $A + n = A + \{n\}$.

Lemma 3.2. *If $E \subseteq \omega$ is an infinite sparse set then $FS(E \cap \{0, 1, \ldots, m-1\}) + FS(E \setminus \{0, 1, \ldots, m-1\}) \in \mathcal{H}$ for every $m \in \omega$.*

Proof. Let $m \in \omega$. If $E \cap \{0, 1, \ldots, m-1\} = \emptyset$, then $\mathrm{FS}(E \cap \{0, 1, \ldots, m-1\}) + \mathrm{FS}(E \setminus \{0, 1, \ldots, m-1\}) = \emptyset \in \mathcal{H}$. Now suppose that $E \cap \{0, 1, \ldots, m-1\} \neq \emptyset$. Since $\mathrm{FS}(E \cap \{0, 1, \ldots, m-1\})$ is finite so $\mathrm{FS}(E \cap \{0, 1, \ldots, m-1\}) + \mathrm{FS}(E \setminus \{0, 1, \ldots, m-1\})$ is a finite union of sets of the form

$$(x_1 + \cdots + x_n) + \mathrm{FS}(E \setminus \{0, 1, \ldots, m-1\})$$

for distinct $x_1, \ldots, x_n \in E \cap \{0, 1, \ldots, m-1\}$ and $n \geq 1$. If we show that each of the above sets is in $\mathcal{H}$ then $\mathrm{FS}(E \cap \{0, 1, \ldots, m-1\}) + \mathrm{FS}(E \setminus \{0, 1, \ldots, m-1\}) \in \mathcal{H}$.

Let $A = (x_1 + \cdots + x_n) + \mathrm{FS}(E \setminus \{0, 1, \ldots, m-1\})$ where $x_1, \ldots, x_n \in E \cap \{0, 1, \ldots, m-1\}$ are pairwise distinct. We will show that the sum of any two different elements from A is not in A. Thus there is no infinite subset of A all of whose finite sums belong to it.

Suppose it is not the case. Then there are $e_1, \ldots, e_k \in E \setminus \{0, 1, \ldots, m-1\}$ (pairwise distinct), $f_1, \ldots, f_p \in E \setminus \{0, 1, \ldots, m-1\}$ (pairwise distinct) and $g_1, \ldots, g_l \in E \setminus \{0, 1, \ldots, m-1\}$ (pairwise distinct) such that

$$\begin{aligned}(x_1 + \cdots + x_n + e_1 + \cdots + e_k) + (x_1 + \cdots + x_n + f_1 + \cdots + f_p)\\ = x_1 + \cdots + x_n + g_1 + \cdots + g_l.\end{aligned}$$

Then

$$(x_1 + \cdots + x_n) + (e_1 + \cdots + e_k) + (f_1 + \cdots + f_p) = g_1 + \cdots + g_l \in \mathrm{FS}(E).$$

We know that every element of $\mathrm{FS}(E)$ has a unique representation as a sum of some elements from E. But $g_i \neq x_j$ for every i, j, a contradiction. □

A *filter* on ω is a family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ which is closed under taking supersets and finite intersections. A maximal filter is called *ultrafilter*.

For $A \subseteq \omega$ we define $A - n = \{k \in \omega : k + n \in A\}$.

An ultrafilter $\mathcal{U}$ on ω is called *idempotent* if $\mathcal{U} = \mathcal{U} + \mathcal{U}$, where

$$\mathcal{U} + \mathcal{U} = \{A \subseteq \omega : \{n \in \omega : A - n \in \mathcal{U}\} \in \mathcal{U}\}.$$

It is known that there are idempotent ultrafilters (see e.g. [9, p. 68]).

Theorem 3.3. *The Hindman ideal $\mathcal{H}$ has BW property.*

Proof. Let $\{A_s : s \in \{0, 1\}^{<\omega}\}$ satisfy $(T1) - (T3)$ from Proposition 2.1. Let $\mathcal{U}$ be an idempotent ultrafilter on ω.

Since $\mathcal{U}$ is an ultrafilter, so if $A \cup B \in \mathcal{U}$ and $A \cap B = \emptyset$, then either $A \in \mathcal{U}$ or $B \in \mathcal{U}$. Thus there is (a unique) $x \in \{0, 1\}^\omega$ such that $A_{x \restriction n} \in \mathcal{U}$ for every $n \in \omega$.

We define a sequence $(b_n : n \in \omega)$ such that

(1) for $i \in \{0, 1, \ldots, n\}$ and $a \in \mathrm{FS}(\{b_i, b_{i+1}, \ldots, b_n\})$ one has that $a \in A_{x \restriction i}$ and $(A_{x \restriction i} - a) \cap \omega \in \mathcal{U}$ and

(2) if $i < n$, then $b_{i+1} > b_0 + \cdots + b_i$.

Take any $b_0 \in A_{x\restriction 0} = \omega$. Suppose we have already constructed $b_0, \ldots, b_n$ as required. Let

$$A = \bigcap_{i \leq n} \bigcap_{a \in \mathrm{FS}(\{b_i, b_{i+1}, \ldots, b_n\})} (A_{x\restriction i} - a) \cap \omega.$$

Since A is an intersection of finitely many sets from $\mathcal{U}$ so $A \in \mathcal{U}$. Since $\mathcal{U} = \mathcal{U} + \mathcal{U}$ so $\{k \in \omega : (A_{x\restriction n+1} - k) \cap \omega \in \mathcal{U}\} \in \mathcal{U}$ and $\{k \in \omega : (A - k) \cap \omega \in \mathcal{U}\} \in \mathcal{U}$. Let $b_{n+1} \in A_{x\restriction n+1}$ such that $(A_{x\restriction n+1} - b_{n+1}) \cap \omega \in \mathcal{U}$, $(A - b_{n+1}) \cap \omega \in \mathcal{U}$ and $b_{n+1} > b_0 + \cdots + b_n$.

It is not difficult to check that $(b_n : n \in \omega)$ satisfies the required conditions.

Let $B = \mathrm{FS}(\{b_n : n \in \omega\}) \notin \mathcal{H}$. We will show that $B \setminus A_{x\restriction n} \in \mathcal{H}$ for every $n \in \omega$. Let $n \in \omega$. Since

$$B = \mathrm{FS}(\{b_0, \ldots, b_{n-1}\}) \cup \mathrm{FS}(\{b_i : i \geq n\}) \cup (\mathrm{FS}(\{b_0, \ldots, b_{n-1}\}) + \mathrm{FS}(\{b_i : i \geq n\}))\,.$$

and $\mathrm{FS}(\{b_i : i \geq n\}) \subseteq A_{x\restriction n}$ so

$$B \setminus A_{x\restriction n} \subseteq \mathrm{FS}(\{b_0, \ldots, b_{n-1}\}) \cup (\mathrm{FS}(\{b_0, \ldots, b_{n-1}\}) + \mathrm{FS}(\{b_i : i \geq n\}))\,.$$

But $\{b_n : n \in \omega\}$ is a sparse set so $\mathrm{FS}(\{b_0, \ldots, b_{n-1}\}) + \mathrm{FS}(\{b_i : i \geq n\}) \in \mathcal{H}$ by Lemma 3.2. Thus finally $B \setminus A_{x\restriction n} \in \mathcal{H}$. □

It was shown in [1, p. 505] that if $\mathcal{I} \in BW$ then $(X, \mathcal{I}) \in BW$ for every uncountable compact metric space.

Corollary 3.4. *If X is an uncountable compact metric space, then $(X, \mathcal{H}) \in BW$.*

Let $D \subseteq \omega$ be an infinite set and X a topological space. A sequence $(x_n)_{n \in \mathrm{FS}(D)}$, $x_n \in X$ is *IP-convergent* to $x \in X$ if for every open neighborhood $U \subseteq X$, $x \in U$ there is $m \in \omega$ such that

$$\{x_n : n \in \mathrm{FS}(D \setminus \{0, 1, \ldots, m-1\})\} \subseteq U.$$

A topological space X is *Hindman* if for every sequence $(x_n)_{n \in \omega}$, $x_n \in X$ there is an infinite set $D \subseteq \omega$ such that $(x_n)_{n \in \mathrm{FS}(D)}$ is IP-convergent to some $x \in X$ ([6, Definition 4]).

Proposition 3.5. *If X is a Hindman space then $(X, \mathcal{H}) \in BW$.*

Proof. Let $(x_n)_{n \in \omega} \in X^\omega$. Let $D \subseteq \omega$ be an infinite set such that $(x_n)_{n \in \mathrm{FS}(D)}$ is IP-convergent to some $x \in X$. We can assume that D is sparse. Let $B = \mathrm{FS}(D)$. We claim that $(x_n)_{n \in B}$ is $\mathcal{H}$-convergent.

Obviously $B \notin \mathcal{H}$. Let $U \subseteq X$ be an open neighborhood of x. Since $(x_n)_{n \in B}$ is IP-convergent to x, so there is $m \in \omega$ with $\{x_n : n \in \mathrm{FS}(D \setminus \{0, 1, \ldots, m-1\})\} \subseteq U$. Then

$$\{n \in B : x_n \notin U\} \subseteq \mathrm{FS}(D \cap \{0, 1, \ldots, m-1\}) + \mathrm{FS}(D \setminus \{0, 1, \ldots, m-1\}).$$

By Lemma 3.2, $\mathrm{FS}(D \cap \{0, 1, \ldots, m-1\}) + \mathrm{FS}(D \setminus \{0, 1, \ldots, m-1\}) \in \mathcal{H}$, so $\{n \in B : x_n \notin U\} \in \mathcal{H}$. □

In [6, Theorem 11] it is shown that every topological space X satisfying the following condition:

($\star$) closure of every countable set in X is compact and first countable

is Hindman.

Corollary 3.6. *If a space X satisfies condition ($\star$) then $(X, \mathcal{H}) \in BW$.*

Below we show that BW property for a space is weaker that being Hindman.

Theorem 3.7. *There exists a Hausdorff, compact, sequentially compact, separable space X which is first-countable at all points but one such that $(X, \mathcal{H}) \in BW$ but X is not Hindman.*

Proof. Let $\mathcal{A}$ be a mad family such that $\mathcal{A} \subseteq \mathcal{H}$. Let $X = \Phi(\mathcal{A})$. By Theorem 2.2 $(X, \mathcal{H}) \in BW$ and by [6, Theorem 10] X is not Hindman. □

4. Differences instead of sums

For $A, B \subseteq \omega$ we write $A - B = \{a - b : a \in A, b \in B, a > b\}$ and $\mathrm{D}(A) = A - A$. A set $A \subseteq \omega$ is called *D-set* if there is an infinite $E \subseteq \omega$ with $\mathrm{D}(E) \subseteq A$. Let $\mathcal{D}$ denote the family of all non-D-sets $A \subseteq \omega$.

Proposition 4.1. *$\mathcal{D}$ is a proper ideal on ω.*

Proof. Clearly $\mathcal{D}$ is proper and closed under subsets. We will show that it is closed under finite unions. Let $A, B \in \mathcal{D}$ and suppose that $A \cup B \notin \mathcal{D}$. Thus there is an infinite $E \subseteq \omega$ with $\mathrm{D}(E) \subseteq A \cup B$. We define a coloring $c : [E]^2 \to \{0, 1\}$ by $c(n, m) = 0 \iff |m - n| \in A$. By Ramsey's theorem there is an infinite $F \subseteq E$ which is homogeneous for c. If it is 0-homogeneous then $\mathrm{D}(F) \subseteq A$ and if it is 1-homogeneous then $\mathrm{D}(F) \subseteq B$. In both cases we get a contradiction. □

Proposition 4.2. *$\mathcal{D} \subseteq \mathcal{H}$ and $\mathcal{D} \neq \mathcal{H}$.*

Proof. Let $A \notin \mathcal{H}$. Then there is an infinite $B = \{b_n : n \in \omega\}$ with $\mathrm{FS}(B) \subseteq A$. Let $E = \{\sum_{i=0}^{n} b_i : n \in \omega\}$. Then E is infinite and $\mathrm{D}(E) \subseteq \mathrm{FS}(B) \subseteq A$. Thus $A \notin \mathcal{D}$.

Now we show that $\mathcal{D} \neq \mathcal{H}$. Let $A = \left\{\sum_{i=k}^{n} 4^i : k, n \in \omega \wedge k \leq n\right\}$. Then $A \notin \mathcal{D}$ since $\mathrm{D}(B) \subseteq A$, where $B = \left\{\sum_{i=0}^{n} 4^i : n \in \omega\right\}$. Suppose that $A \notin \mathcal{H}$. Then there is an infinite $C \subseteq A$ with $\mathrm{FS}(C) \subseteq A$. We have two cases:

(1) There are $x = \sum_{i=k_1}^{n_1} 4^i, y = \sum_{i=k_2}^{n_2} 4^i \in C$ such that $x < y$ and $k_2 \leq n_1$;
(2) For every $x = \sum_{i=k_1}^{n_1} 4^i, y = \sum_{i=k_2}^{n_2} 4^i \in C$, if $x < y$ then $n_1 < k_2$.

In the first case, the base 4 representation of $x + y$ has a block of 2's, whereas the members of A are those numbers whose base 4 representation consists of a block of 1's followed by a (possibly empty) block of 0's, a contradiction.

Now, consider the second case. Let $x = \sum_{i=k_1}^{n_1} 4^i, y = \sum_{i=k_2}^{n_2} 4^i, z = \sum_{i=k_3}^{n_3} 4^i \in C$ such that $x < y < z$. Then $n_1 < k_2 < k_3$. Since $x + z \in A$ so there are $n < k$ such that $x + z = \sum_{i=k}^{n} 4^i$. Thus $x + z = \sum_{i=k_1}^{n_1} 4^i + \sum_{i=k_3}^{n_3} 4^i = \sum_{i=k}^{n} 4^i$, but A is a sparse set, a contradiction. □

A set $A \subseteq \omega$ is *$\mathcal{D}$-sparse* if for every $k \in \mathrm{D}(A)$ there is only one pair $n, m \in A$ with $k = m - n$.

Proposition 4.3. (1) *If $A \subseteq \omega$ is $\mathcal{D}$-sparse and $n \in \omega$, then $A + n \in \mathcal{D}$.*
(2) *For every infinite $A \subseteq \omega$ there is an infinite $\mathcal{D}$-sparse $B \subseteq A$.*
(3) *The ideal $\mathcal{D}$ is dense (i.e. every infinite $A \subseteq \omega$ contains an infinite $B \subseteq A$ with $B \in \mathcal{I}$).*

Proof. (1). Suppose that there is an infinite $E \subseteq \omega$ with $\mathrm{D}(E) \subseteq A + n$. Let $a < b < c < d \in E$. Then $d - a - n, d - b - n, c - b - n, c - a - n \in A$ and $d - b - n \neq d - a - n$. On the other hand,

$$(d - b - n) - (c - b - n) = (d - a - n) - (c - a - n) \in \mathrm{D}(A),$$

a contradiction.

(2). Straightforward.

(3). Follows from (2). □

Theorem 4.4. *The ideal $\mathcal{D}$ has BW property.*

Proof. Let $\{A_s : s \in \{0,1\}^{<\omega}\}$ satisfy $(T1)-(T3)$ from Proposition 2.1. We will construct $E_n \subseteq \omega$ and $j_n \in \{0,1\}$ $(n \in \omega)$ such that

(1) E_n is an infinite set,
(2) $E_n \supseteq E_{n+1}$,
(3) $\mathrm{D}(E_n) \subseteq A_{(j_0,j_1,\dots,j_{n-1})}$.

Let $E_0 = \omega$. Suppose that we have already constructed E_k and j_k for $k \leq n$. We define a coloring $c : [E_n]^2 \to \{0,1\}$ by

$$c(a,b) = j \iff |a-b| \in A_{(j_0,j_1,\dots,j_{n-1},j)}.$$

By Ramsey's theorem there is an infinite $E_{n+1} \subseteq E_n$ and $j_n \in \{0,1\}$ with $c(a,b) = j_n$ for every $a,b \in E_{n+1}$. That finishes the construction.

Let $E \subseteq \omega$ be an infinite set such that $F_n = E \setminus E_n$ is finite for every $n \in \omega$. By Proposition 4.3 we can assume that E is $\mathcal{D}$-sparse.

Let $B = \mathrm{D}(E)$ and $x \in \{0,1\}^\omega$ such that $x(n) = j_n$ for each $n \in \omega$. Then $B \notin \mathcal{D}$ and we claim that $B \setminus A_{x\restriction n} \in \mathcal{D}$ for every $n \in \omega$ (and that will finish the proof by Proposition 2.1).

Let $n \in \omega$. Then $B \setminus A_{x\restriction n} \subseteq \mathrm{D}(F_n \cup (E \cap E_n)) \setminus A_{x\restriction n} = [\mathrm{D}(F_n) \cup \mathrm{D}(E \cap E_n) \cup (F_n - (E \cap E_n)] \setminus A_{x\restriction n} \subseteq \mathrm{D}(F_n) \cup (F_n - (E \cap E_n))$.

The set $\mathrm{D}(F_n) \in \mathcal{D}$ as a finite set. Moreover, $F_n - (E \cap E_n) = \bigcup_{a \in F_n} (E \cap E_n - a)$ is a finite union of translations of the $\mathcal{D}$-sparse set, hence, by Proposition 4.3, $F_n - (E \cap E_n) \in \mathcal{D}$. Thus $B \setminus A_{x\restriction n} \in \mathcal{D}$. □

It was shown in [1, p. 505] that if $\mathcal{I} \in BW$ then $(X, \mathcal{I}) \in BW$ for every uncountable compact metric space.

Corollary 4.5. *If X is an uncountable compact metric space, then $(X, \mathcal{D}) \in BW$.*

Let $A \subseteq \omega$ be an infinite set and X a topological space. A sequence $(x_n)_{n \in \mathrm{D}(A)}$, $x_n \in X$ is *R-convergent* to $x \in X$ if for every open neighborhood $U \subseteq X$, $x \in U$ there is $m \in \omega$ with

$$\{x_n : n \in \mathrm{D}(A \setminus \{0,1,\dots,m-1\})\} \subseteq U.$$

A topological space X is an *R-space* if for every sequence $(x_n)_{n\in\omega}$, $x_n \in X$ there is an infinite set $A \subseteq \omega$ such that $(x_n)_{n \in \mathrm{D}(A)}$ is R-convergent to some $x \in X$.

Proposition 4.6. *Every Hindman space is an R-space.*

Proof. Let X be a Hindman space and $(x_n)_{n\in\omega} \in X^\omega$. Then there is an infinite $A = \{a_n : n \in \omega\}$ $(a_n < a_{n+1}$ for every $n \in \omega)$ such that $(x_n)_{n \in \mathrm{FS}(A)}$ is IP-convergent to some $x \in X$. Let $B = \{\sum_{i=0}^n a_i : n \in \omega\}$. We will show that $(x_n)_{n \in \mathrm{D}(B)}$ is D-convergent to x. Let $U \subseteq X$ be an open neighborhood of x. Then there is $m \in \omega$ such that $\{x_n : n \in \mathrm{FS}(A \setminus \{0,1,\dots,m-1\})\} \subseteq U$. Let $N \in \omega$ such that $a_N > m$. Let $k = \sum_{i=0}^{N-1} a_i$. Then $\{x_n : n \in \mathrm{D}(B \setminus \{0,1,\dots,k-1\})\} \subseteq U$. □

Proposition 4.7. *If X is an R-space then $(X, \mathcal{D}) \in BW$.*

Proof. Let X be an R-space and $(x_n)_{n\in\omega} \in X^\omega$. Then there is an infinite $E \subseteq \omega$ such that $(x_n)_{n\in \mathrm{D}(E)}$ R-converges to some $x \in X$. Let $F \subseteq E$ be an infinite $\mathcal{D}$-sparse set. Let $B = D(F)$. Then $B \notin \mathcal{D}$ and we claim that $(x_n)_{n\in B}$ $\mathcal{D}$-converges to x. Indeed, let $U \subseteq X$ be an open neighborhood of x. There is $m \in \omega$ such that $\{x_n : n \in \mathrm{D}(F \setminus \{0, 1, \dots, m-1\})\} \subseteq U$. Thus

$$\{n \in B : x_n \notin U\} \subseteq \mathrm{D}(\{0, 1, \dots, m-1\}) \cup ((F \setminus \{0, 1, \dots, m-1\}) - \{0, 1, \dots, m-1\}).$$

But $\mathrm{D}(\{0, 1, \dots, m-1\}) \in \mathcal{D}$ as a finite set. Moreover, $(F \setminus \{0, 1, \dots, m-1\}) - \{0, 1, \dots, m-1\} \in \mathcal{D}$ as a finite union of translations of $\mathcal{D}$-sparse sets (by Proposition 4.3). □

By Proposition 4.6 and [6, Theorem 11] we get the following corollary.

Corollary 4.8. *If X satisfies $(\star)$-property then X is an R-space and $(X, \mathcal{D}) \in BW$.*

Finally, we show that BW property for a space is weaker than being R-space.

Theorem 4.9. *There exists a Hausdorff, compact, sequentially compact, separable space X which is first-countable at all points but one such that $(X, \mathcal{D}) \in BW$ but X is not an R-space.*

Proof. Let $\mathcal{A}$ be a mad family such that $\mathcal{A} \subseteq \mathcal{D}$. Let $X = \Phi(\mathcal{F})$. By Theorem 2.2 $(X, \mathcal{D}) \in BW$. Below we will show that X is not an R-space.

Let $x_n = n$ for every $n \in \omega$. We claim that this sequence does not have an R-convergent subsequence. Suppose that there is an infinite $E \subseteq \omega$ such that $(x_n)_{n\in \mathrm{D}(E)}$ is R-convergent to $x \in X$. Obviously $x \notin \omega$, hence we have two cases:

(1) $x \in \mathcal{A}$ or
(2) $x = \infty$.

In the first case, let $x = A \in \mathcal{A}$. Then for $U = A \cup \{A\}$ there is $m \in \omega$ with $\{x_n : n \in \mathrm{D}(E \setminus \{0, 1, \dots, m-1\})\} \subseteq U$. Thus $\mathrm{D}(E \setminus \{0, 1, \dots, m-1\}) \subseteq A$, so $A \notin \mathcal{D}$, a contradiction.

Now consider the second case. Let $B_m = \mathrm{D}(E \setminus \{0, 1, \dots, m-1\})$. Let $B \subseteq \omega$ be an infinite set such that $B \setminus B_m$ is finite for every $m \in \omega$. Since $\mathcal{D}$ is dense (by Proposition 4.3) so there is $C \subseteq B$ with $C \in \mathcal{D}$. Since $\mathcal{A}$ is maximal so there is $A \in \mathcal{A}$ with $|A \cap C| = \omega$. Let $U = X \setminus (A \cup \{A\})$. Then there is $m \in \omega$ with $\{x_n : n \in \mathrm{D}(E \setminus \{0, 1, \dots, m-1\})\} \subseteq U$. Thus $B_m \cap A = \emptyset$ so $C \cap A$ is finite, a contradiction. □

5. Problems

Problem 1. Does there exist an R-space which is not a Hindman space?

Problem 2. Is there a relationship between "$(X, \mathcal{H}) \in BW$" and "$(X, \mathcal{D}) \in BW$"?

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