IDEAL CONVERGENCE

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Contents

The notion of the ideal convergence is dual (equivalent) to the notion of the filter convergence introduced by Cartan in 1937 ([Car37]). The notion of the filter convergence is a generalization of the classical notion of convergence of a sequence and it has been an important tool in general topology and functional analysis since 1940 (when Bourbaki's book [Bou40] appeared). Nowadays many authors prefer to use an equivalent dual notion of the ideal convergence (see e.g. frequently quoted work [KŠW01]).

In this paper we survey ideal convergence of sequences of reals and functions. We focus on three aspects of ideal convergence that are connected with the following well-known results.

(1) Every bounded sequence of reals has the convergent infinite subsequence. (Section 2.)
(2) The limit of a convergent sequence of continuous functions need not be continuous. (Section 3.)

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(3) The set of points where a sequence of continuous functions is convergent forms an $F_{\sigma\delta}$ set. (Section 4.)

1. Preliminaries

An *ideal* on $\mathbb{N}$ is a family of subsets of $\mathbb{N}$ closed under taking finite unions and subsets of its elements. We can speak about ideals on any other infinite countable set by identifying this set with $\mathbb{N}$ via a fixed bijection. If not explicitly said, we assume that an ideal is proper ($\neq \mathcal{P}(\mathbb{N})$) and contains all finite sets. By FIN we denote the ideal of all finite subsets of $\mathbb{N}$.

A *filter* on $\mathbb{N}$ is a family of subsets of $\mathbb{N}$ closed under taking finite intersections and supersets of its elements. For an ideal $\mathcal{I}$ we define $\mathcal{I}^\star = \{A : \mathbb{N} \setminus A \in \mathcal{I}\}$ and call it the *dual filter* to $\mathcal{I}$; and for a filter $\mathcal{F}$ we define $\mathcal{F}^\star = \{A : \mathbb{N} \setminus A \in \mathcal{F}\}$ and call it the *dual ideal* to $\mathcal{F}$. The filter $\mathrm{FIN}^\star$ is called the *Fréchet filter*.

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Let $x_n \in \mathbb{R}$ ($n \in \mathbb{N}$) and $x \in \mathbb{R}$. We say that the sequence (x_n) is *$\mathcal{I}$-convergent* to x if

$$\{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$$

for every $\varepsilon > 0$. We write $\mathcal{I} - \lim x_n = x$ in this case. If $\mathcal{I} = \mathrm{FIN}$, then $\mathcal{I}$-convergence is equivalent to the classical convergence.

By identifying subsets of $\mathbb{N}$ with their characteristic functions, we equip $\mathcal{P}(\mathbb{N})$ with the product topology of $\{0,1\}^{\mathbb{N}}$. It is known that $\mathcal{P}(\mathbb{N})$ with this topology is a compact Polish space without isolated points (it is homeomorphic to the Cantor set). An ideal $\mathcal{I}$ is an *F_σ ideal* (*analytic ideal*, respectively) if $\mathcal{I}$ is an F_σ subset of $\mathcal{P}(\mathbb{N})$ (if it is a continuous image of a G_δ subset of $\mathcal{P}(\mathbb{N})$, respectively.)

A map $\phi\colon \mathcal{P}(\mathbb{N}) \to [0,\infty]$ is a *submeasure* if $\phi(\emptyset) = 0$, ϕ is monotone (i.e. $\phi(A) \leq \phi(B)$ whether $A \subseteq B$) and ϕ is subadditive (i.e. $\phi(A \cup B) \leq \phi(A) + \phi(B)$). We will assume also that $\phi(\mathbb{N}) > 0$.

For a submeasure ϕ we define $\mathcal{Z}(\phi) = \{A \subseteq \mathbb{N} : \phi(A) = 0\}$ and $\mathrm{Fin}(\phi) = \{A \subseteq \mathbb{N} : \phi(A) < \infty\}$. For any $A \subseteq \mathbb{N}$ we define

$$\|A\|_\phi = \lim_{n\to\infty} \phi\left(A \setminus \{0, 1, \ldots, n-1\}\right).$$

$\|\cdot\|_\phi$ is also a submeasure on $\mathbb{N}$.

A submeasure ϕ is *lower semicontinuous* (lsc, in short) if for all $A \subseteq \mathbb{N}$ we have $\phi(A) = \lim_{n\to\infty} \phi(A \cap \{0, 1, \ldots, n-1\})$. For example, $\mathrm{FIN} = \mathrm{FIN}(\phi)$ for $\phi(A) = \mathrm{card}(A)$.

Theorem 1.1 ([Maz91]). *An ideal $\mathcal{I}$ is F_σ if and only if there exists an lsc submeasure ϕ such that $\mathcal{I} = Fin(\phi)$.*

Example 1.2 ([Maz91]). Let $f : \mathbb{N} \to [0,\infty)$ be such that $\sum_n f(n) = \infty$. We define $\mathcal{I}_f = \{A \subseteq \mathbb{N} : \sum_{n\in A} f(n) < \infty\}$. An ideal $\mathcal{I}$ is a *summable ideal* if $\mathcal{I} = \mathcal{I}_f$ for some f. Every summable ideal is an F_σ ideal. ($\mathcal{I}_f = \mathrm{FIN}(\phi)$, $\phi(A) = \sum_{i\in A} f(i)$.)

For an lsc submeasure ϕ we define $\mathrm{Exh}(\phi) = \mathcal{Z}\left(\|\cdot\|_\phi\right)$. An ideal $\mathcal{I}$ is a *P-ideal* if for every family $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{I}$ there exists $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all n.

Theorem 1.3 ([Sol99]). *An ideal $\mathcal{I}$ is an analytic P-ideal if and only if there exists an lsc submeasure ϕ such that $\mathcal{I} = Exh(\phi)$.*

Example 1.4 ([JK84])**.** Let $f : \mathbb{N} \to [0, \infty)$ be a function such that $\sum_n f(n) = \infty$ and $\lim_n f(n)/(\sum_{i\leq n} f(i)) = 0$ We define $\mathcal{EU}_f = \mathcal{Z}\left(\overline{d}_f\right)$, for

$$\overline{d}_f(A) = \limsup_{n\to\infty} \frac{\sum_{i\in A\cap\{0,1,\dots,n-1\}} f(i)}{\sum_{i<n} f(i)}.$$

An ideal $\mathcal{I}$ is called the *Erdős-Ulam ideal* if $\mathcal{I} = \mathcal{EU}_f$ for some f. Every Erdős-Ulam ideal is an analytic P-ideal, i.e. $\mathcal{EU}_f = \mathrm{Exh}(\phi)$, where

$$\phi(A) = \sup\left\{\frac{\sum_{i\in A\cap\{0,1,\dots,n-1\}} f(i)}{\sum_{i<n} f(i)} : n \in \mathbb{N}\right\}.$$

Example 1.5. For $f(n) = 1$ ($n \in \mathbb{N}$) we define the *upper asymptotic density* of a set A as

$$\overline{d}(A) = \overline{d}_f(A) = \limsup_{n\to\infty} \frac{\mathrm{card}(A \cap \{0, 1, \dots, n-1\})}{n}.$$

We define $\mathcal{I}_d = \mathcal{EU}_f$ and call it the *ideal of asymptotic density zero sets.* $\mathcal{I}_d$-convergence appeared to be equivalent to the *statistical convergence* which was introduced by Steinhaus ([Fas51]).

Using Solecki's characterization it is easy to show that every analytic P-ideal is in fact an $F_{\sigma\delta}$ subset of $\mathcal{P}(\mathbb{N})$.

Example 1.6 ([FS03])**.** The ideals

$$\begin{aligned} \mathrm{NWD} &= \{A \subseteq \mathbb{Q} \cap [0, 1] : A \text{ is nowhere dense in } \mathbb{Q} \cap [0, 1]\}, \\ \mathrm{NULL} &= \left\{A \subseteq \mathbb{Q} \cap [0, 1] : \mathrm{cl}_{[0,1]}(A) \text{ has Lebesgue measure zero}\right\} \end{aligned}$$

are $F_{\sigma\delta}$ subsets of $\mathcal{P}(\mathbb{N})$ but they are not P-ideals.

2. Ideal convergence of subsequences of reals

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Let $x_n \in \mathbb{R}$ ($n \in \mathbb{N}$) and $x \in \mathbb{R}$. Let $A \subseteq \mathbb{N}$, $A \notin \mathcal{I}$. We say that the subsequence $(x_n)_{n\in A}$ is *$\mathcal{I}$-convergent* to x if

$$\{n \in A : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$$

for every $\varepsilon > 0$.

We say that an ideal $\mathcal{I}$ on $\mathbb{N}$ has:

- the *Bolzano-Weierstrass property* (shortly BW property, $\mathcal{I} \in \mathrm{BW}$) if for every bounded sequence (x_n) of reals there is $A \notin \mathcal{I}$ such that the subsequence $(x_n)_{n\in A}$ is $\mathcal{I}$-convergent;
- the *finite Bolzano-Weierstrass property* (shortly FinBW property, $\mathcal{I} \in \mathrm{FinBW}$) if for every bounded sequence (x_n) of reals there is $A \notin \mathcal{I}$ such that the subsequence $(x_n)_{n\in A}$ is convergent [FMRS07].

It is easy to see that for any ideal $\mathcal{I}$, FinBW property of $\mathcal{I}$ implies BW property of $\mathcal{I}$, and the reverse implication does not hold. BW property and FinBW property coincide for P-ideals ([FMRS07]).

Example 2.1. By the well-known Bolzano-Weierstrass theorem, the ideal FIN of all finite subsets of $\mathbb{N}$ has FinBW property.

Example 2.2 ([FMRS07])**.** Every F_σ ideal has FinBW property.

Example 2.3 ([FMRS07])**.** No Erdős-Ulam ideal has BW-property. (For the ideal $\mathcal{I}_d$ of asymptotic density zero sets it was already shown in [Fri93].)

Example 2.4 ([FMRS07])**.** The ideals NWD and NULL do not have BW property.

For more examples of ideals with(out) BW-like properties we refer the Reader to [FMRS07].

2.1. **Nonatomic submeasures.** An ideal $\mathcal{I}$ of subsets of naturals is called *nonatomic* if there exists a sequence $(\mathcal{P}_n)$ of finite partitions of $\mathbb{N}$ such that each $\mathcal{P}_n$ is refined by $\mathcal{P}_{n+1}$, and whenever (A_n) is a decreasing sequence with $A_n \in \mathcal{P}_n$ for each n, and a set $Z \subseteq \mathbb{N}$ is such that $Z \setminus A_n$ is finite for each n, then $Z \in \mathcal{I}$.

Theorem 2.5 ([BFMS11])**.** *An ideal $\mathcal{I}$ does not have FinBW property if and only if it is nonatomic.*

A submeasure ϕ is *strongly nonatomic* ([DŁ08a]), if for each $\varepsilon > 0$ there exists a partition of $\mathbb{N}$ on finitely many sets $A_0, A_1, \ldots, A_{n-1}$ such that $\phi(A_i) \leq \varepsilon$ for each $i = 0, 1, \ldots, n-1$. In [DŁ08b] the authors showed that the ideal $\mathcal{Z}(\phi)$ is nonatomic for every strongly nonatomic submeasure ϕ. And they also showed that the converse does not hold.

Theorem 2.6 ([BFMS11])**.** *If a submeasure ϕ is strongly nonatomic, then $\mathcal{Z}(\phi)$ does not have BW property.*

It is not possible to prove the converse of Theorem 2.6. The counterexample is the submeasure defined by $\phi(A) = 0$ if $\overline{u}(A) = 0$ and $\phi(A) = 1$ otherwise, for $\overline{u}$ being the upper Banach density ([BFMS11]). However, Theorem 2.5 can be reversed for ϕ being the $\limsup$ of lsc submeasures ([BFMS11]). For an analytic P-ideal one can prove the following equivalence.

Theorem 2.7 ([FMRS07])**.** *An analytic P-ideal $\mathcal{I} = Exh(\phi)$ has BW property if and only if the submeasure $\|\cdot\|_\phi$ is not strongly nonatomic.*

2.2. **Splitting families.** A family $\mathcal{S} \subseteq \mathcal{P}(\mathbb{N})$ is an *$\mathcal{I}$-splitting family* if for every $A \notin \mathcal{I}$ there is $S \in \mathcal{S}$ such that $A \cap S \notin \mathcal{I}$ and $A \setminus S \notin \mathcal{I}$.

Theorem 2.8 ([FMRS07])**.** *An ideal $\mathcal{I}$ has BW property if and only if does not exist a countable $\mathcal{I}$-splitting family.*

Let $\mathfrak{s}(\mathcal{I})$ denote the smallest cardinality of an $\mathcal{I}$-splitting family. For $\mathcal{I} = \mathrm{FIN}$ we write $\mathfrak{s} = \mathfrak{s}(\mathrm{FIN})$ and it is called the *splitting number* (for more about $\mathfrak{s}$, see e.g. [Bla10] or [vD84]).

The above theorem shows that $\mathfrak{s}(\mathcal{I}) = \aleph_0$ for every ideal without BW property. It is known that $\mathfrak{s} = \mathfrak{c}$ if we assume Martin's Axiom. In [Fil] it was shown that if we assume Martin's Axiom then $\mathfrak{s}(\mathcal{I}) = \mathfrak{c}$ for every F_σ ideal, and for every analytic P-ideal with BW property.

2.3. **Katětov order and extendability to F_σ ideals.** Let $\mathcal{I}, \mathcal{J}$ be ideals. We write $\mathcal{I} \leq_K \mathcal{J}$ if there exists a function $h : \mathbb{N} \to \mathbb{N}$ such that $h^{-1}[A] \in \mathcal{J}$ for all $A \in \mathcal{I}$. The relation $\leq_K$ is called *Katětov order* and was introduced in [Kat68] and used in [Kat72] for study of filter convergence of sequences of functions.

By CONV we denote the ideal of all subsets of $\mathbb{Q} \cap [0,1]$ which have only finitely many cluster points.

Theorem 2.9 ([MA09])**.** *An ideal $\mathcal{I}$ has FinBW property if and only if CONV $\not\leq_K$ $\mathcal{I}$.*

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals. We say that:

- $\mathcal{J}$ *extends* an ideal $\mathcal{I}$ if $\mathcal{I} \subseteq \mathcal{J}$;
- $\mathcal{I}$ *contains an isomorphic copy* of an ideal $\mathcal{J}$ if there is a bijection $h : \mathbb{N} \to \mathbb{N}$ such that $h^{-1}[A] \in \mathcal{I}$ for each $A \in \mathcal{J}$.

It is known that $\mathcal{I}$ contains an isomorphic copy of the ideal CONV iff CONV $\leq_K \mathcal{I}$ ([BFMS13]).

In [Hru11] the author asked if for a Borel ideal $\mathcal{I}$, CONV $\not\leq_K \mathcal{I} \iff \mathcal{I}$ can be extended to a proper F_σ ideal. Using Theorem 2.9 this question can be reformulated in the following way.

Problem 1 ([Hru11])**.** Let $\mathcal{I}$ be a Borel ideal. Are the following conditions equivalent?

(1) $\mathcal{I}$ has FinBW property.
(2) $\mathcal{I}$ can be extended to a proper F_σ ideal.

In [FMRS07] the authors proved that the implication (2) ⇒ (1) holds for every ideal.

Since every analytic P-ideal is Borel (in fact $F_{\sigma\delta}$) so the following theorem gives a partial answer to the Hrušák's question. (As far as we know, this question in its general version is still open.)

Theorem 2.10 ([FMRS07], [BFMS13])**.** *Let $\mathcal{I}$ be an analytic P-ideal. The following are equivalent.*

- *$\mathcal{I}$ has FinBW property.*
- *$\mathcal{I}$ can be extended to a proper F_σ ideal.*
- *CONV $\not\leq_K \mathcal{I}$.*
- *$\mathcal{I}$ does not contain an isomorphic copy of the ideal CONV.*

An ideal $\mathcal{I}$ is called *maximal* if there is no proper ideal $\mathcal{J}$ extending $\mathcal{I}$.

Theorem 2.11 ([FMRS07])**.** *Assume the Continuum Hypothesis. Let $\mathcal{I}$ be an analytic P-ideal. The following are equivalent.*

- *$\mathcal{I}$ has FinBW property.*
- *$\mathcal{I}$ can be extended to a maximal P-ideal.*

The dual filter to a maximal ideal is called an *ultrafilter.* Recall that the set of all ultrafilters defined on $\mathbb{N}$ with an appropriate topology is the Čech-Stone compactification $\beta\mathbb{N}$ of the set of natural numbers $\mathbb{N}$. Dual filters to maximal P-ideals are called *P-points* in the Čech-Stone compactification $\beta\mathbb{N}$ realm and it is known that its existence is independent from the axioms of ZFC ([She82]).

2.4. **Combinatorics.** The Ramsey theorem is one of the most known theorems of combinatorics. This theorem has many generalizations. For example Frankl, Graham and Rödl provided its iterated density version for the submeasure $\overline{d}$— i.e. *upper asymptotic density*—defined in Example 1.5. Recall that by $[\mathbb{N}]^2$ we mean a family of all two-element subsets of $\mathbb{N}$, i.e. $[\mathbb{N}]^2 = \{\{x, y\} \ : \ x, y \in \mathbb{N}, x \neq y\}$. In this section we will use the word "coloring" instead of "partition".

Theorem 2.12 ([FGR90])**.** *For every coloring $[\mathbb{N}]^2 = C_0 \cup C_1 \cup \ldots \cup C_r$ there exist $\delta = \delta(r) > 0$ and $i \leq r$ such that*

$$\overline{d}\left(\left\{x \in \mathbb{N} : \overline{d}\left(\left\{y \in \mathbb{N} : \overline{d}\left(\{z \in \mathbb{N} : \{x, y\}, \{x, z\}, \{y, z\} \in C_i\}\right) \geq \delta\right\}\right) \geq \delta\right\}\right) \geq \delta.$$

An analogous result is true for every analytic P-ideal.

Theorem 2.13 ([FS10]). *Let $\mathcal{I} = Exh(\phi)$ be an analytic P-ideal. Then for every coloring $[\mathbb{N}]^2 = C_1 \cup C_2 \cup \ldots \cup C_r$ there exist $\delta = \delta(r)$ and $i \leq r$ with*

$$\left\|\left\{x \in \mathbb{N} : \left\|\left\{y \in \mathbb{N} : \|\{z \in \mathbb{N} : \{x,y\},\{x,z\},\{y,z\} \in C_i\}\|_\phi \geq \delta\right\}\right\|_\phi \geq \delta\right\}\right\|_\phi \geq \delta.$$

We have the following stronger version of the above result for ideals with Bolzano-Weierstrass property.

Theorem 2.14 ([FMRS11]). *Let $\mathcal{I} = Exh(\phi)$ be an analytic P-ideal with BW property. Then there exist $\delta = \delta(\phi)$ such that for every finite coloring $[\mathbb{N}]^2 = C_1 \cup C_2 \cup \ldots \cup C_r$ there exist $i \leq r$ and $A \subseteq \mathbb{N}$ with $\|A\|_\phi \geq \delta$ such that for every $x \in A$*

$$\left\|\left\{y \in A : \|\{z \in A : \{x,y\},\{x,z\},\{y,z\} \in C_i\}\|_\phi \geq \delta\right\}\right\|_\phi \geq \delta.$$

Corollary 2.15. *An analytic P-ideal has the Bolzano-Weierstrass property if and only if the constant δ in Theorem 2.13 can be found independently on the number of colors r.*

Proof. The implication "⇒" follows from Theorem 2.14. Now we show the implication "⇐". Let $\mathcal{I} = \mathrm{Exh}(\phi)$ be an analytic P-ideal without BW property and suppose that there is $\delta > 0$ such that for every coloring $[\mathbb{N}]^2 = C_1 \cup C_2 \cup \ldots \cup C_r$ there is $i \leq r$ with

$$\left\|\left\{x \in \mathbb{N} : \left\|\left\{y \in \mathbb{N} : \|\{z \in \mathbb{N} : \{x,y\},\{x,z\},\{y,z\} \in C_i\}\|_\phi \geq \delta\right\}\right\|_\phi \geq \delta\right\}\right\|_\phi \geq \delta.$$

Since $\mathcal{I}$ does not have BW property so by Theorem 2.7, there is a partition $\mathbb{N} = A_1 \cup \cdots \cup A_N$ such that $\|A_i\|_\phi \leq \delta/2$ for every $i \leq N$.

For $k, l \leq N$ let

$$C_{k,l} = \left\{\{x_0, x_1\} \in [\mathbb{N}]^2 : \exists i \in \{0,1\}\ (x_i \in A_k \text{ and } x_{1-i} \in A_l)\right\}.$$

Since $[\mathbb{N}]^2 = \bigcup_{k,l \leq N} C_{k,l}$, there are $k_0, l_0 \leq N$ with

$$\left\|\left\{x \in \mathbb{N} : \left\|\left\{y \in \mathbb{N} : \|\{z \in \mathbb{N} : \{x,y\},\{x,z\},\{y,z\} \in C_{k_0,l_0}\}\|_\phi \geq \delta\right\}\right\|_\phi \geq \delta\right\}\right\|_\phi \geq \delta.$$

It is easy to check that if $\{x,y\},\{x,z\},\{y,z\} \in C_{k_0,l_0}$ then $A_{k_0} \cap A_{l_0} \neq \emptyset$. Since sets $A_1, \ldots, A_N$ are pairwise disjoint, so $k_0 = l_0$. Thus

$$\{z \in \mathbb{N} : \{x,y\},\{x,z\},\{y,z\} \in C_{k_0,l_0}\} \subseteq A_{k_0}$$

so

$$\|\{z \in \mathbb{N} : \{x,y\},\{x,z\},\{y,z\} \in C_{k_0,l_0}\}\|_\phi \leq \|A_{k_0}\|_\phi \leq \delta/2.$$

Then

$$\left\{y \in \mathbb{N} : \|\{z \in \mathbb{N} : \{x,y\},\{x,z\},\{y,z\} \in C_{k_0,l_0}\}\|_\phi \geq \delta\right\} = \emptyset$$

so

$$\left\|\left\{x \in \mathbb{N} : \left\|\left\{y \in \mathbb{N} : \|\{z \in \mathbb{N} : \{x,y\},\{x,z\},\{y,z\} \in C_{k_0,l_0}\}\|_\phi \geq \delta\right\}\right\|_\phi \geq \delta\right\}\right\|_\phi = 0,$$

a contraction. □ □

Another well-known theorem from infinite combinatorics is the Schur theorem which says that for every coloring of the set of natural numbers $\mathbb{N} = C_0 \cup \ldots \cup C_r$, there exist $i \le r$ and $x, y, z \in C_i$ with $x + y = z$. This theorem also has many generalizations. Bergelson and Hindman provided density version of the Schur theorem for the submeasure $\overline{d}$.

Theorem 2.16 ([BH88]). *For every coloring* $\mathbb{N} = C_0 \cup C_1 \cup \ldots \cup C_r$ *there exist* $\delta = \delta(r) > 0$ *and* $i \le r$ *such that*

$$\overline{d}\left(\{x \in \mathbb{N} \ : \ \overline{d}\left(\{y \in \mathbb{N} \ : \ x, y, x + y \in C_i\}\right) \ge \delta\}\right) \ge \delta.$$

We will say that a submeasure ϕ is *invariant under translations* if $\phi(A + t) = \phi(A)$ for each $A \subseteq \mathbb{N}$ and $t \in \mathbb{N}$ (where $A + t = \{a + t : \ a \in A\}$). In [FS10] it was shown that it is possible to generalize Theorem 2.16 on any submeasure of the form $\|\cdot\|_\phi$, for $\|\cdot\|_\phi$ being invariant under translations (in particular, $\overline{d}$ is of this form).

Theorem 2.17 ([FS10]). *Let* $\mathcal{I} = Exh(\phi)$ *be an analytic P-ideal with* $\|\cdot\|_\phi$ *invariant under translations. Then for every coloring* $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_r$ *there exists* $\delta = \delta(r)$ *and* $i \le r$ *with*

$$\left\|\left\{x \in \mathbb{N} \ : \ \|\{y \in \mathbb{N} \ : \ x, y, x + y \in C_i\}\|_\phi \ge \delta\right\}\right\|_\phi \ge \delta.$$

Theorem 2.18 ([FS10]). *Let* $\mathcal{I} = Exh(\phi)$ *be an analytic P-ideal such that* $\|\cdot\|_\phi$ *is invariant under translations. The ideal* $\mathcal{I}$ *has the BW property if and only if there exists* $\delta > 0$ *such that for every* $r \in \mathbb{N}$ *and every coloring* $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_r$ *there is* $i \le r$ *with*

$$\left\|\left\{x \in \mathbb{N} \ : \ \|\{y \in \mathbb{N} \ : \ x, y, x + y \in C_i\}\|_\phi \ge \delta\right\}\right\|_\phi \ge \delta.$$

Note that the constant δ in Theorems 2.16 2.17 depends on the number of colors r. Theorem 2.18 yields the following corollary.

Corollary 2.19. *An analytic P-ideal* $\mathcal{I} = Exh(\phi)$ *with* $\|\cdot\|_\phi$ *invariant under translations has the Bolzano-Weierstrass property if and only if the constant* δ *in Theorem 2.17 can be found independently on the number of colors* r.

In this context it seems to be interesting to find out in which density theorems (see e.g. [FGR90, Th. 3.1, 5.2, 6.1]) the condition "the constant δ does not depend on the number of colors" characterize non strongly nonatomic submeasures.

3. Ideal convergence of sequences of functions

Let Φ be a fixed kind of convergence of sequences of real-valued functions (e.g. pointwise convergence, equal convergence or discrete convergence, or their ideal counterpart). For a family $\mathcal{F}$ of real-valued functions defined on X there is the smallest family $\mathcal{B}^\Phi(\mathcal{F})$ of all real-valued functions defined on X which contains $\mathcal{F}$ and which is closed under the process of taking Φ-limits of sequences. This family is called the *Baire system* with respect to Φ generated by $\mathcal{F}$. One method of generating $\mathcal{B}^\Phi(\mathcal{F})$ from $\mathcal{F}$ is by iteration of Φ-limits:

- $\mathcal{B}^\Phi_0(\mathcal{F}) = \mathcal{F}$;
- $\mathcal{B}^\Phi_\alpha(\mathcal{F}) = \mathrm{LIM}^\Phi\left(\bigcup_{\beta<\alpha} \mathcal{B}^\Phi_\beta(\mathcal{F})\right)$ for $\alpha > 0$,

where $\mathrm{LIM}^{\Phi}(\mathcal{G})$ denotes the family of all Φ-limits of sequences from $\mathcal{G}$. This system was described in 1899 by Baire in the case when Φ is the pointwise convergence and $\mathcal{F}$ is the family of all continuous functions defined on a topological space X. In this case we write $\mathcal{B}_\alpha(X)$ instead of $\mathcal{B}^{\Phi}(C(X))$, $\alpha < \omega_1$. In particular, $\mathcal{B}_1(X)$ denotes the class of all *Baire class one* real-valued function defined on X, i.e. $f \in \mathcal{B}_1(X) \iff f = \lim_n f_n$ for some sequence (f_n) of continuous real-valued functions defined on X.

3.1. **Pointwise convergence.** Let f_n $(n \in \mathbb{N})$ and f be real-valued functions defined on a set X. We say that the sequence (f_n) is *pointwise $\mathcal{I}$-convergent* to f if $\mathcal{I} - \lim f_n(x) = f(x)$ for every $x \in X$. We write $f = \mathcal{I} - \lim_n f_n$ in this case.

By $\mathcal{B}_1^{\mathcal{I}}(X)$ we will denote the *$\mathcal{I}$-Baire class one* of real-valued function defined on X i.e. $f \in \mathcal{B}_1^{\mathcal{I}}(X) \iff f = \mathcal{I} - \lim_n f_n$ for some sequence (f_n) of continuous real-valued functions defined on X. It is well-known that each Baire class one function is Borel measurable. One can expect that the same holds for the class $\mathcal{B}_1^{\mathcal{I}}(X)$ for all ideals $\mathcal{I}$. Unfortunately, this is not true. The following example shows that the ideal limit of a sequence of continuous functions need not to be measurable nor have the Baire property.

Example 3.1. Let $X = \mathcal{P}(\mathbb{N})$ be identified with the Cantor space $\{0,1\}^{\mathbb{N}}$ with the product topology. Let $\mathcal{I}$ be a maximal ideal on $\mathbb{N}$. We define $h_n : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ by

$$h_n(A) = \begin{cases} 0 & \text{if } n \notin A, \\ 1 & \text{otherwise.} \end{cases}$$

Then each h_n is continuous and $\mathcal{I} - \lim_n h_n = \chi_{\mathcal{I}}$ (the characteristic function of $\mathcal{I}$). It is well-known that for a maximal ideal $\mathcal{I}$, $\chi_{\mathcal{I}}$ is not measurable and does not have the Baire property (see e.g. [BJ95]).

It is easy to see that $\mathcal{B}_1(X) \subseteq \mathcal{B}_1^{\mathcal{I}}(X)$ for every ideal $\mathcal{I}$ which contains all finite sets. Laczkovich and Recław [LR09] (for Borel ideals), and independently Debs and Saint Raymond [DSR09] (for analytic ideals), investigated for which ideals $\mathcal{I}$ and topological spaces X the equality $\mathcal{B}_1(X) = \mathcal{B}_1^{\mathcal{I}}(X)$ holds. Recently Bouziad [Bou12] solved this problem in a general case (for all ideals).

For a given class $\mathbf{\Gamma} \subseteq \mathcal{P}(X)$ and disjoint sets $A, B \subseteq X$ we say that A can be *$\mathbf{\Gamma}$-separated* from B if there exists $E \in \mathbf{\Gamma}$ with $A \subseteq E \subseteq X \setminus B$[1].

Theorem 3.2 ([DSR09], [LR09], [Bou12])**.** *For every ideal $\mathcal{I}$ and an uncountable perfectly normal topological space X the following conditions are equivalent:*

(1) $\mathcal{B}_1(X) = \mathcal{B}_1^{\mathcal{I}}(X)$;
(2) *$\mathcal{I}$ and $\mathcal{I}^\star$ can be F_σ-separated.*

Solecki [Sol00] proved that every $F_{\sigma\delta}$ ideal can be F_σ-separated from its dual filter. Thus, $\mathcal{B}_1(X) = \mathcal{B}_1^{\mathcal{I}}(X)$ for NWD, NULL, for all F_σ ideals and for all analytic P-ideals.

By FIN $\times$ FIN we denote the ideal of all subsets $A \subseteq \mathbb{N} \times \mathbb{N}$ such that

$$\exists N \in \mathbb{N}\, \forall n > N\ (\{k \in \mathbb{N} : (n,k) \in A\} \text{ is finite}).$$

[1] In this section we use only F_σ-separability of $\mathcal{I}$ and $\mathcal{I}^\star$. In Section 4.2 we introduce the definition of the rank of an ideal $\mathcal{I}$; the rank of $\mathcal{I}$ is equal to 1 if and only if $\mathcal{I}$ and $\mathcal{I}^\star$ can be F_σ-separated.

Theorem 3.3 ([DSR09], [LR09], [BFMS13]). *Let X be an uncountable Polish space and $\mathcal{I}$ be an analytic ideal. The following conditions are equivalent:*

(1) $\mathcal{B}_1(X) = \mathcal{B}_1^{\mathcal{I}}(X)$;
(2) *$\mathcal{I}$ does not contain an isomorphic copy of the ideal FIN $\times$ FIN;*
(3) *FIN $\times$ FIN $\not\leq_K \mathcal{I}$.*

It is not possible to generalize Theorem 3.3 on the class of all ideals. This is a consequence of the fact that a maximal ideal is a P-ideal if and only if it does not contain an isomorphic copy of the ideal FIN $\times$ FIN. Thus, using Example 3.1 for $\mathcal{I}$ being a maximal P-ideal we get $\chi_{\mathcal{I}} \in \mathcal{B}_1^{\mathcal{I}}(X) \setminus \mathcal{B}_1(X)$. However, the following weak version of Theorem 3.3 holds for any ideal $\mathcal{I}$.

Theorem 3.4 ([Rec12]). *For any ideal $\mathcal{I}$ and an uncountable Polish space X the following conditions are equivalent:*

(1) $\mathcal{B}_1(X) = \mathcal{B}_1^{\mathcal{I}}(X) \cap \mathcal{B}or(X)$[2] ;
(2) *$\mathcal{I}$ does not contain an isomorphic copy of the ideal FIN $\times$ FIN.*

Recall also that $\mathcal{B}_1^{\mathcal{I}}(X) \subseteq \mathcal{B}or(X)$ for every analytic $\mathcal{I}$ and a Polish space X [DSR09].

3.2. **Ideal Baire classes.** Let X be a topological space and α be a countable ordinal.

For an ideal $\mathcal{I}$ and a topological space X we define $\mathcal{I}$-Baire classes:

- $\mathcal{B}_0^{\mathcal{I}}(X) = \mathcal{C}(X)$;
- $\mathcal{B}_\alpha^{\mathcal{I}}(X) = \mathcal{I} - \mathrm{LIM}\left(\bigcup_{\beta<\alpha} \mathcal{B}_\beta^{\mathcal{I}}(X)\right)$, for $\alpha > 0$,

where for any family $\mathcal{E} \subseteq \mathbb{R}^X$, $\mathcal{I} - \mathrm{LIM}\,(\mathcal{E})$ is the family of all $\mathcal{I}$-limits of pointwise $\mathcal{I}$-convergent sequences of functions belonging to the family $\mathcal{E}$.

Theorem 3.5 ([FS12]). *Let X be a perfectly normal topological space. Let $\mathcal{I}$ be an ideal such that $\mathcal{I}$ and $\mathcal{I}^\star$ can be F_σ-separated. Then $\mathcal{B}_\alpha(X) = \mathcal{B}_\alpha^{\mathcal{I}}(X)$ for every countable α.*

In [LR09] the authors proved the above theorem for Polish spaces, Borel ideals which do not contain FIN $\times$ FIN and finite ordinals α.

Theorem 3.6 ([FS12]). *Let X be an uncountable Polish space and $\mathcal{I}$ be an analytic ideal. Let $\alpha \geq 1$ be a countable ordinal. The ideal $\mathcal{I}$ does not contain an isomorphic copy of the ideal FIN $\times$ FIN if and only if $\mathcal{B}_\alpha(X) = \mathcal{B}_\alpha^{\mathcal{I}}(X)$.*

3.3. **Equal and discrete convergence.** Let f_n ($n \in \mathbb{N}$) and f be real-valued functions defined on a set X. The sequence (f_n) is:

- *equally convergent* to f (e-lim$f_n = f$) if there exists a sequence of positive reals (ε_n) such that $\lim_n \varepsilon_n = 0$ and for every $x \in X$ there is N with $|f_n(x) - f(x)| < \varepsilon_n$ for every $n > N$;
- *discretely convergent* to f (d-lim$f_n = f$) if for every $x \in X$ there is N with $f_n(x) = f(x)$ for every $n > N$.

The notions of discrete and equal convergence were introduced by Császár and Laczkovich [CL75]. It is known that if (f_n) is uniformly convergent to f then (f_n) is equally convergent to f; and if (f_n) is equally convergent to f then (f_n)

[2]Here $\mathcal{B}or(X)$ denotes the class of all Borel functions on X.

is pointwise convergent to f; and if (f_n) is discretely convergent to f then (f_n) is equally convergent to f.

For a family of functions $\mathcal{E} \subseteq \mathbb{R}^X$ by the symbol d-LIM $(\mathcal{E})$ (respectively: e-LIM $(\mathcal{E})$) we denote the family of all discrete limits (equal limits, respectively) of discretely convergent (equally-convergent, respectively) sequences of functions from the family $\mathcal{E}$.

Analogously to the definition of Baire classes (with respect to pointwise convergence) one can define *discrete Baire classes* $\mathcal{B}_\alpha^{(d)}(X)$ and *equal Baire classes* $\mathcal{B}_\alpha^{(e)}(X)$ ([CL75]).

- $\mathcal{B}_0^{(d)}(X) = \mathcal{B}_0^{(e)}(X) = \mathcal{C}(X)$;
- $\mathcal{B}_\alpha^{(d)}(X) = \text{d-LIM}\left(\bigcup_{\beta<\alpha} \mathcal{B}_\beta^{(d)}(X)\right)$, for every $\alpha > 0$;
- $\mathcal{B}_\alpha^{(e)}(X) = \text{e-LIM}\left(\bigcup_{\beta<\alpha} \mathcal{B}_\beta^{(e)}(X)\right)$, for every $\alpha > 0$.

The ideal versions of discrete and equal convergence and Baire classes were introduced in [FS12] in the following manner. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. A sequence (f_n) is:

- *equally** $\mathcal{I}$*-convergent* to f ($\mathcal{I} - \text{e}^\star\text{-lim} f_n = f$) if there exists a sequence of positive reals (ε_n) such that $\lim_n \varepsilon_n = 0$ and for every $x \in X$ the set $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$.[3]
- *discretely* $\mathcal{I}$*-convergent* to f ($\mathcal{I} - \text{d-lim} f_n = f$) if for every $x \in X$ the set $\{n \in \mathbb{N} : f_n(x) \neq f(x)\} \in \mathcal{I}$.

For a family $\mathcal{E} \subseteq \mathbb{R}^X$ by the symbol $\mathcal{I} - \text{d-LIM}(\mathcal{E})$ ($\mathcal{I} - \text{e}^\star\text{-LIM}(\mathcal{E})$, respectively) we denote the family of all discrete $\mathcal{I}$-limits (equal $\mathcal{I}$-limits, respectively) of all discretely $\mathcal{I}$-convergent (equally* $\mathcal{I}$-convergent, respectively) sequences of functions belonging to $\mathcal{E}$. And finally, ideal discrete and equal* Baire classes are defined in the following way.

- $\mathcal{B}_0^{(\mathcal{I}-d)}(X) = \mathcal{B}_0^{(\mathcal{I}-e^\star)}(X) = \mathcal{C}(X)$;
- $\mathcal{B}_\alpha^{(\mathcal{I}-d)}(X) = \mathcal{I} - \text{d-LIM}\left(\bigcup_{\beta<\alpha} \mathcal{B}_\beta^{(\mathcal{I}-d)}(X)\right)$, for every $\alpha > 0$;
- $\mathcal{B}_\alpha^{(\mathcal{I}-e^\star)}(X) = \mathcal{I} - \text{e}^\star\text{-LIM}\left(\bigcup_{\beta<\alpha} \mathcal{B}_\beta^{(\mathcal{I}-e^\star)}(X)\right)$, for every $\alpha > 0$.

Theorem 3.7 ([FS12])**.** *Let X be a perfectly normal topological space. Let $\mathcal{I}$ be an ideal such that $\mathcal{I}$ and $\mathcal{I}^\star$ can be F_σ-separated. Then*

- $\mathcal{B}_\alpha^{(\mathcal{I}-d)}(X) = \mathcal{B}_\alpha^{(d)}(X)$ *for every countable ordinal α.*
- $\mathcal{B}_\alpha^{(\mathcal{I}-e^\star)}(X) = \mathcal{B}_\alpha^{(e)}(X)$ *for every finite ordinal α.*

Theorem 3.8 ([FS12])**.** *Let X be an uncountable Polish space and $\mathcal{I}$ be a Borel ideal. Let $\alpha \geq 1$ be a countable (finite, respectively) ordinal. The ideal $\mathcal{I}$ does not contain an isomorphic copy of the ideal FIN × FIN if and only if $\mathcal{B}_\alpha^{(\mathcal{I}-d)}(X) = \mathcal{B}_\alpha^{(d)}(X)$ ($\mathcal{B}_\alpha^{(\mathcal{I}-e^\star)}(X) = \mathcal{B}_\alpha^{(e)}(X)$, respectively).*

As far as we know, the answer to the question if $\mathcal{B}_\alpha^{(\mathcal{I}-e^\star)}(X) = \mathcal{B}_\alpha^{(e)}(X)$ for every Borel ideal which does not contain an isomorphic copy of the ideal FIN × FIN and for all $1 \leq \alpha < \omega_1$ is unknown.

---

[3]We use here the "star" notation, since in this section we will also introduce another variant of the definition of "equal $\mathcal{I}$-convergence" which seems to be more adequate to the "without star" notation.

The ideal version of equal convergence of sequences of functions was also introduced in [DDP]. However, the authors did it in a different way. Namely, they say that a sequence (f_n) is *equally $\mathcal{I}$-convergent* to f if there exists a sequence of positive reals (ε_n) such that $\mathcal{I} - \lim_n \varepsilon_n = 0$ and for every $x \in X$ the set $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$. We will write $\mathcal{I} - \text{e-lim} f_n = f$ in this case. The only difference between the definitions in [FS12] and [DDP] is that in the latter paper the authors only require that the sequence (ε_n) is $\mathcal{I}$-convergent (not necessary convergent) to zero. (So if $\mathcal{I} - \text{e}^\star\text{-lim} f_n = f$ then $\mathcal{I} - \text{e-lim} f_n = f$.)

In [FS] the authors compare both definitions of ideal equal convergence and among others proved the following characterization.

Theorem 3.9 ([FS]). *Let X be a nonempty set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, $\mathcal{I} - \text{e}^\star\text{-lim} f_n = f \iff \mathcal{I} - \text{e-lim} f_n = f$.*
(2) *$\mathcal{I}$ is a P-ideal.*

Problem 2. Describe Baire systems with respect to ideal convergences generated by other families of functions: Borel functions; functions possessing the Baire property or quasi-continuous functions.[4]

4. Sets of ideal convergence of sequences of functions

Let (f_n) be a sequence of continuous real-valued functions defined on a metric space X. It is not difficult to show that the set $\{x \in X : (f_n(x))_n \text{ is convergent}\}$ is $F_{\sigma\delta}$. On the other hand, Hahn ([Hah19]) and Sierpiński ([Sie21]) proved independently that for every $F_{\sigma\delta}$ set $A \subseteq X$ there exists a sequence (f_n) of continuous real-valued functions defined on a metric space X such that $A = \{x \in X : (f_n(x))_n \text{ is convergent}\}$.

Further research (see e.g. Kornfel′d [Kor63] and Lipiński [Lip61] [Lip63]) involved also sets of points where the sequence is divergent to infinity and the like. The full description of these sets was given by Lunina [Lun75] (see Theorem 4.1 below).

Let $\vec{f} = (f_n)$ be a sequence of real-valued functions defined on a set X. We define seven types of sets of convergence and divergence of the sequence $\vec{f}$.

$$
\begin{aligned}
E^1(\vec{f}) &= \{x\colon (f_n(x)) \text{ is convergent}\},\\
E^2(\vec{f}) &= \{x\colon \lim f_n(x) = -\infty\},\\
E^3(\vec{f}) &= \{x\colon \lim f_n(x) = +\infty\},\\
E^4(\vec{f}) &= \{x\colon -\infty < \underline{\lim}\, f_n(x) < \overline{\lim}\, f_n(x) < +\infty\},\\
E^5(\vec{f}) &= \{x\colon -\infty = \underline{\lim}\, f_n(x) < \overline{\lim}\, f_n(x) < +\infty\},\\
E^6(\vec{f}) &= \{x\colon -\infty < \underline{\lim}\, f_n(x) < \overline{\lim}\, f_n(x) = +\infty\},\\
E^7(\vec{f}) &= \{x\colon -\infty = \underline{\lim}\, f_n(x) \text{ and } \overline{\lim}\, f_n(x) = +\infty\}.
\end{aligned}
$$

It is easy to see that $\{E^1(\vec{f}), \ldots, E^7(\vec{f})\}$ is a partition of X.

Let $\mathcal{F} \subseteq \mathbb{R}^X$ be a family of real-valued functions defined on a set X. Let $E_i \subseteq X$ $(i = 1, \ldots, 7)$. The sequence $(E^1, \ldots, E^7)$ is called a *Lunina's 7-tuple for $\mathcal{F}$* if there is a sequence $\vec{f} = (f_n)$, $f_n \in \mathcal{F}$ $(n \in \mathbb{N})$ such that $E^i = E^i(\vec{f})$ for $i = 1, \ldots, 7$. The family of all Lunina's 7-tuples for $\mathcal{F}$ is denoted by $\Lambda(\mathcal{F})$.

[4] The definition of quasi-continuous functions can be found in the next section.

Theorem 4.1 ([Lun75]). *Let X be a metric space. Let $E^i \subseteq X$ $(i = 1, \ldots, 7)$. Then* $(E^1, \ldots, E^7) \in \Lambda(\mathcal{C}(X)) \iff$

(1) $\{E^1, \ldots, E^7\}$ *is a partition of* X,
(2) E^1, E^2, E^3 *are* $F_{\sigma\delta}$ *in* X,
(3) $E^2 \cup E^5 \cup E^7, E^3 \cup E^6 \cup E^7$ *are* G_δ *in* X.

The following theorem is an extension of Lunina's theorem on all Baire classes (and it answers a question posed by Wesołowska in [Wes01]).

Theorem 4.2 ([Bor]). *Let X be a separable metric space. Let α be a countable ordinal. Then* $(E^1, \ldots, E^7) \in \Lambda(\mathcal{B}_\alpha(X)) \iff$

(1) $\{E^1, \ldots E^7\}$ *is a partition of* X,
(2) E^1, E^2, E^3 *are* $\mathbf{\Pi}^0_{\alpha+3}$,
(3) $E^2 \cup E^5 \cup E^7$, $E^3 \cup E^6 \cup E^7$ *are* $\mathbf{\Pi}^0_{\alpha+2}$.

Let X be a topological space. A function $f\colon X \to \mathbb{R}$ is *quasi-continuous* if $f^{-1}[V] \subseteq \mathrm{cl}(\mathrm{int}(f^{-1}[V]))$ for any open set $V \subseteq \mathbb{R}$ (i.e. the set $f^{-1}[V]$ is semi-open). Let $\mathcal{QC}(X)$ denote the family of all quasi-continuous real-valued functions defined on X.

Theorem 4.3 ([NW]). *Let X be a dense in itself separable metric Baire space. Then* $(E^1, \ldots, E^7) \in \Lambda(\mathcal{QC}(X)) \iff$

(1) $\{E^1, \ldots E^7\}$ *is a partition of* X,
(2) $E^i = (G_i \setminus P_i) \cup Q_i$, *where G_i is regular open, P_i, Q_i are meager in X, $P_i \subseteq G_i$, $G_i \cap Q_i = \emptyset$ for all $i = 1, \ldots, 7$, and moreover $P_i \cap Q_j$ are nowhere dense for* $(i, j) \in (\{1, 4\} \times \{2, 3, 5, 6, 7\}) \cup (\{2, 5\} \times \{3, 6, 7\}) \cup (\{3, 6\} \times \{2, 5, 7\})$.

Let $\mathcal{M}$ be a family of subsets of X. By $\mathcal{F}_\mathcal{M}$ we denote the class of all $\mathcal{M}$-measurable real-valued functions defined on X, i.e. $f \in \mathcal{F}_\mathcal{M}$ iff $f^{-1}[G] \in \mathcal{M}$ for each open $G \subseteq \mathbb{R}$. It is easy to observe that if $\mathcal{M}$ is a σ-algebra of subsets of X, then $(E^1, \ldots, E^7) \in \Lambda(\mathcal{F}_\mathcal{M}) \iff \{E^1, \ldots E^7\}$ is a partition of X onto $\mathcal{M}$-measurable sets.

The above results can be generalized by considering ideal convergence of sequences of functions.

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Let $\vec{f} = (f_n)$ be a sequence of real-valued functions defined on a set X. We define seven types of sets of $\mathcal{I}$-convergence and divergence of the sequence $\vec{f}$.

$$
\begin{aligned}
E^1_\mathcal{I}(\vec{f}) &= \{x\colon (f_n(x)) \text{ is } \mathcal{I}\text{-convergent}\},\\
E^2_\mathcal{I}(\vec{f}) &= \{x\colon \mathcal{I}-\lim f_n(x) = -\infty\},\\
E^3_\mathcal{I}(\vec{f}) &= \{x\colon \mathcal{I}-\lim f_n(x) = +\infty\},\\
E^4_\mathcal{I}(\vec{f}) &= \{x\colon -\infty < \mathcal{I}-\underline{\lim}\, f_n(x) < \mathcal{I}-\overline{\lim}\, f_n(x) < +\infty\},\\
E^5_\mathcal{I}(\vec{f}) &= \{x\colon -\infty = \mathcal{I}-\underline{\lim}\, f_n(x) < \mathcal{I}-\overline{\lim}\, f_n(x) < +\infty\},\\
E^6_\mathcal{I}(\vec{f}) &= \{x\colon -\infty < \mathcal{I}-\underline{\lim}\, f_n(x) < \mathcal{I}-\overline{\lim}\, f_n(x) = +\infty\},\\
E^7_\mathcal{I}(\vec{f}) &= \{x\colon -\infty = \mathcal{I}-\underline{\lim}\, f_n(x) \text{ and } \mathcal{I}-\overline{\lim}\, f_n(x) = +\infty\},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}-\lim x_n = +\infty \quad &\text{if} \quad \{n \in \mathbb{N} : x_n < M\} \in \mathcal{I} \text{ for any } M > 0,\\
\mathcal{I}-\lim x_n = -\infty \quad &\text{if} \quad \{n \in \mathbb{N} : x_n > M\} \in \mathcal{I} \text{ for any } M < 0,\\
\mathcal{I}-\overline{\lim}\, x_n &= \inf\{\alpha\colon \{n\colon x_n > \alpha\} \in \mathcal{I}\},\\
\mathcal{I}-\underline{\lim}\, x_n &= \sup\{\alpha\colon \{n\colon x_n < \alpha\} \in \mathcal{I}\}.
\end{aligned}
$$

It is easy to see that $\{E^1_{\mathcal{I}}(\vec{f}), \dots, E^7_{\mathcal{I}}(\vec{f})\}$ is a partition of X. Moreover, we have $E^i(\vec{f}) = E^i_{\mathcal{I}}(\vec{f})$ for $i = 1, 2, \dots, 7$ and $\mathcal{I} = \mathrm{FIN}$.

Let $\mathcal{F} \subseteq \mathbb{R}^X$ be a family of real-valued functions defined on a set X. Let $E^i \subseteq X$ $(i = 1, \dots, 7)$. The sequence $(E^1, \dots, E^7)$ is called an *$\mathcal{I}$-Lunina's 7-tuple for $\mathcal{F}$* if there is a sequence $\vec{f} = (f_n)$, $f_n \in \mathcal{F}$ $(n \in \mathbb{N})$ such that $E^i = E^i_{\mathcal{I}}(\vec{f})$ for $i = 1, \dots, 7$. The family of all $\mathcal{I}$-Lunina's 7-tuples for $\mathcal{F}$ is denoted by $\Lambda_{\mathcal{I}}(\mathcal{F})$.

Let $\mathcal{I}, \mathcal{J}$ be ideals. We write $\mathcal{I} \leq_{RK} \mathcal{J}$ if there exists a function $h : \mathbb{N} \to \mathbb{N}$ such that $A \in \mathcal{I} \iff h^{-1}[A] \in \mathcal{J}$. (The relation $\leq_{RK}$ is called the *Rudin-Keisler order*.)

Proposition 4.4 ([BR10]). *Let X be a set. Let $\mathcal{I}, \mathcal{J}$ be ideals. Let $\mathcal{F}$ be a family of real-valued functions defined on X. If $\mathcal{I} \leq_{RK} \mathcal{J}$ then $\Lambda_{\mathcal{I}}(\mathcal{F}) \subseteq \Lambda_{\mathcal{J}}(\mathcal{F})$.*

Since FIN $\leq_{RK} \mathcal{I}$ for every ideal $\mathcal{I}$ with the Baire property ([Tal80]) so we get the following corollary.

Corollary 4.5 ([BR10]). *Let X be a set. Let $\mathcal{I}$ be an ideal with the Baire property. Let $\mathcal{F}$ a family of real-valued functions defined on X. Then $\Lambda(\mathcal{F}) \subseteq \Lambda_{\mathcal{I}}(\mathcal{F})$.*

For ideals without the Baire property it is possible that $\Lambda_{\mathcal{I}}(\mathcal{F}) \neq \Lambda(\mathcal{F})$ even for $\mathcal{F} = \mathcal{C}(X)$, which is shown by Example 4.6.

Example 4.6. Let $\vec{f} = (n \cdot h_n)_n$ (where h_n are defined as in Example 3.1). Then $E^1_{\mathcal{I}}(\vec{f}) = \mathcal{I}$, and for a maximal ideal $\mathcal{I}$, $E^1_{\mathcal{I}}(\vec{f})$ does not have the Baire property. Hence by Theorem 4.1, $E^1_{\mathcal{I}}(\vec{f}) \neq E^1(\vec{g})$ for any sequence $\vec{g}$ of continuous functions. Thus $\Lambda_{\mathcal{I}}(\mathcal{F}) \neq \Lambda(\mathcal{F})$.

4.1. F_σ ideals.

Theorem 4.7 ([BR10]). *Let X be a metric space. Let $\mathcal{I}$ be an F_σ ideal. Then $\Lambda_{\mathcal{I}}(\mathcal{C}(X)) = \Lambda(\mathcal{C}(X))$.*

Theorem 4.8 ([BR10]). *Let X be a metric space which contains a subspace homeomorphic to the Cantor space. If $\Lambda(\mathcal{C}(X)) = \Lambda_{\mathcal{I}}(\mathcal{C}(X))$ then $\mathcal{I}$ is an F_σ ideal.*

Theorem 4.9 ([Bor]). *Let $\mathcal{M}$ be a σ-additive and (finitely) multiplicative family of subsets of a set X. Let $\mathcal{M}^c = \{X \setminus M : M \in \mathcal{M}\}$ and $\mathcal{F}_{\mathcal{M}}$ be the family of $\mathcal{M}$-measurable functions. Let $\mathcal{I}$ be an F_σ ideal. If $\{E^1, \dots, E^7\} \in \Lambda_{\mathcal{I}}(\mathcal{F}_{\mathcal{M}})$ then*

1. *$E^1, E^2, E^3 \in (\mathcal{M}^c)_{\sigma\delta}$,*
2. *$E^2 \cup E^5 \cup E^7$ and $E^3 \cup E^6 \cup E^7$ are $\mathcal{M}_\delta$ in X.*

Corollary 4.10 ([Bor]). *Let X be a separable metric space. Let $\mathcal{I}$ be an F_σ ideal. Then $\Lambda_{\mathcal{I}}(\mathcal{B}_\alpha(X)) = \Lambda(\mathcal{B}_\alpha(X))$ for every countable ordinal α.*

Corollary 4.11 ([Bor]). *Let $\mathcal{I}$ be an F_σ ideal and $\mathcal{M}$ be a σ-algebra of subsets of X. Then $\Lambda_{\mathcal{I}}(\mathcal{F}_{\mathcal{M}})) = \Lambda(\mathcal{F}_{\mathcal{M}})$.*

Theorem 4.12 ([NW]). *Let X be a dense in itself separable metric Baire space. Let $\mathcal{I}$ be an F_σ ideal. Then $\Lambda_{\mathcal{I}}(\mathcal{QC}(X)) = \Lambda(\mathcal{QC}(X))$.*

4.2. Borel ideals and continuous functions.

Let $\mathcal{F}$ be a family of real-valued functions defined on a set X. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. We define the following families of subsets of X:

$$\begin{aligned}\mathcal{E}^1_{\mathcal{I}}(\mathcal{F}) &= \{E^1_{\mathcal{I}}(\vec{f}) : \vec{f} \in \mathcal{F}\},\\ \mathcal{E}^2_{\mathcal{I}}(\mathcal{F}) &= \{E^2_{\mathcal{I}}(\vec{f}) : \vec{f} \in \mathcal{F}\},\\ \mathcal{E}^3_{\mathcal{I}}(\mathcal{F}) &= \{E^3_{\mathcal{I}}(\vec{f}) : \vec{f} \in \mathcal{F}\}.\end{aligned}$$

The sequence $(E^1, E^2, E^3) \in X^3$ is called an *$\mathcal{I}$-Lipinski's triple for $\mathcal{F}$* if there is a sequence $\vec{f} = (f_n)$, $f_n \in \mathcal{F}$ $(n \in \mathbb{N})$ such that $E^i = E^i_{\mathcal{I}}(\vec{f})$ for $i = 1, 2, 3$. The family of all $\mathcal{I}$-Lipinski's triples for $\mathcal{F}$ is denoted by $\Lambda^3_{\mathcal{I}}(\mathcal{F})$. We write $\Lambda^3(\mathcal{F})$ instead of $\Lambda^3_{\mathrm{FIN}}(\mathcal{F})$.

Theorem 4.13 ([Rec12])**.** *Let X be a metric space, $\mathcal{F} = \mathcal{C}(X)$ and $\mathcal{I}$ be an ideal. If $\mathcal{I} \in \mathbf{\Pi}^0_\alpha \setminus \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta$ for some countable ordinal α then*

(1) $\mathcal{E}^1_{\mathcal{I}}(\mathcal{F}) \cup \mathcal{E}^2_{\mathcal{I}}(\mathcal{F}) \cup \mathcal{E}^3_{\mathcal{I}}(\mathcal{F}) \subseteq \mathbf{\Pi}^0_\alpha$.
(2) *If X is a separable zero-dimensional metric space, then* $\mathcal{E}^1_{\mathcal{I}}(\mathcal{F}) = \mathcal{E}^2_{\mathcal{I}}(\mathcal{F}) = \mathcal{E}^3_{\mathcal{I}}(\mathcal{F}) = \mathbf{\Pi}^0_\alpha$.
(3) *If $\alpha = 3$ then* $\mathcal{E}^1_{\mathcal{I}}(\mathcal{F}) = \mathcal{E}^2_{\mathcal{I}}(\mathcal{F}) = \mathcal{E}^3_{\mathcal{I}}(\mathcal{F}) = \mathbf{\Pi}^0_\alpha$.
(4) *If $X = \mathbb{R}$ then* $\mathcal{E}^1_{\mathcal{I}}(\mathcal{F}) = \mathbf{\Pi}^0_\alpha$.

Problem 3 ([Rec12])**.** Is Theorem 4.13(4) true for the plane? (Is it true, in general, for all Polish spaces?)

Theorem 4.14 ([Lip63])**.** *Let $E^i \subseteq \mathbb{R}$ $(i = 1, 2, 3)$. Then $(E^1, E^2, E^3) \in \Lambda^3(\mathcal{C}(\mathbb{R}))$ $\iff$*

(1) *E^1, E^2, E^3 are $F_{\sigma\delta}$,*
(2) *$E^1 \cup E^i$ can be F_σ-separated from E^j for $(i, j) = (2, 3), (3, 2)$.*

Using Theorem 4.1 one can show that the above result characterizes also $\Lambda^3(\mathcal{C}(X))$ for every metric space X.

In [DSR09] the authors defined the rank of an ideal $\mathcal{I}$:

$$rk(\mathcal{I}) = \min\{\xi < \omega_1 : \mathcal{I} \text{ is } \mathbf{\Sigma}^0_{1+\xi}\text{-separated from } \mathcal{I}^\star\},$$

It is known that analytic ideals have countable rank; if $\mathcal{I} \in \mathbf{\Pi}^0_\alpha$ then $1 + rk(\mathcal{I}) < \alpha$, and $rk(\mathcal{I}) = 1$ for every $\mathbf{\Pi}^0_4$ ideal $\mathcal{I}$ [DSR09].

Theorem 4.15 ([Rec12])**.** *Let X be a metric space and $\mathcal{F} = \mathcal{C}(X)$ and $\mathcal{I}$ be a $\mathbf{\Pi}^0_\alpha$-ideal. If $(E^1, E^2, E^3) \in \Lambda^3_{\mathcal{I}}(\mathcal{F})$ then*

(1) *$E^1, E^2, E^3 \in \mathbf{\Pi}^0_\alpha$;*
(2) *$E^1 \cup E^i$ can be $\mathbf{\Sigma}^0_{1+rk(\mathcal{I})}$-separated from E^j for $(i, j) = (2, 3), (3, 2)$.*

For $\alpha = 4$, the $\mathbf{\Sigma}^0_{1+rk(\mathcal{I})}$-separation simply means F_σ-separation, and the previous result can be reversed.

Theorem 4.16 ([Rec12][5])**.** *Let X be a metric space, $\mathcal{F} = \mathcal{C}(X)$ and $\mathcal{I}$ be a $\mathbf{\Pi}^0_4$-ideal. Then $\Lambda^3_{\mathcal{I}}(\mathcal{F}) = \Lambda^3(\mathcal{F})$.*

In particular, Theorem 4.16 characterizes $\mathcal{I}$-Lipinski's triples for all analytic P-ideals (in particular: for the statistical convergence, i.e. for the ideal $\mathcal{I} = \mathcal{Z}\left(\overline{d}\right)$ of sets of asymptotic density 0). It cannot be generalized on $\alpha = 5$, since FIN $\times$ FIN is a $\mathbf{\Pi}^0_5$ ideal of rank 2.

[5]In fact, Recław proved this theorem for $\mathbf{\Pi}^0_3$-ideals. By [DSR09, Th. D], this result can be easily extended on all $\mathbf{\Pi}^0_4$-ideals.

Problem 4 ([Rec12])**.** Can Theorem 4.15 be reversed for $\alpha > 4$ (at least for Polish spaces or zero-dimensional spaces)?

The following theorem gives some partial answer to the problem.

Proposition 4.17 ([Rec12])**.** *Let X be a separable, zero-dimensional metric space, $\mathcal{F} = \mathcal{C}(X)$ and let $\mathcal{I} \in \mathbf{\Pi}^0_\alpha \setminus \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta$ be an ideal. Assume that A, B are $\mathbf{\Pi}^0_\alpha$ subsets of X such that A can be $\mathbf{\Sigma}_{1+rk(\mathcal{I})}$-separated from B. Then $(A, B, \emptyset) \in \Lambda^3_\mathcal{I}(\mathcal{F})$ and $(A, \emptyset, B) \in \Lambda^3_\mathcal{I}(\mathcal{F})$.*

4.3. **Non-Borel ideals and continuous functions.**

Proposition 4.18 ([Rec12])**.** *Let X be a Polish space and $\mathcal{F} = \mathcal{C}(X)$. For each triple of pairwise disjoint sets $A, B, C \subseteq X$ there is an ideal $\mathcal{I}$ such that $(A, B, C) \in \Lambda^3_\mathcal{I}(\mathcal{F})$.*

It is not difficult to show that if $\mathcal{I}$ is a maximal ideal then every sequence (x_n) of reals is either $\mathcal{I}$-convergent or $\mathcal{I} - \lim x_n = \pm\infty$. Hence $E^1_\mathcal{I}(\vec{f}) \cup E^2_\mathcal{I}(\vec{f}) \cup E^3_\mathcal{I}(\vec{f}) = X$ for any sequence $\vec{f} = (f_n)$, $f_n \in \mathcal{C}(X)$ $(n \in \mathbb{N})$.

Proposition 4.19 ([Rec12])**.** *Let X be a Polish space and $\mathcal{F} = \mathcal{C}(X)$. For each partition $\{A, B, C\}$ of X there is a maximal ideal $\mathcal{I}$ such that $(A, B, C) \in \Lambda^3_\mathcal{I}(\mathcal{F})$.*

Proposition 4.20 ([Rec12])**.** *Let X be a Polish space and $\mathcal{F} = \mathcal{C}(X)$.*

(1) *There is a coanalytic ideal $\mathcal{I}$ such that $(A, B, C) \in \Lambda^3_\mathcal{I}(\mathcal{F})$ for each triple of pairwise disjoint Borel sets A, B, C.*
(2) *There is a maximal ideal $\mathcal{I}$ such that $(A, B, C) \in \Lambda^3_\mathcal{I}(\mathcal{F})$ for each partition of X on three Borel sets A, B, C.*

Observe that for any Polish space X and $\mathcal{F} = \mathcal{C}(X)$ there is no ideal $\mathcal{I}$ such that the set $\Lambda^3_\mathcal{I}(\mathcal{F})$ is equal to the set of all triples of pairwise disjoint Borel sets. Indeed, if $\mathcal{I}$ is Borel then, by Theorem 4.13(1), all sets in the triples from $\Lambda^3_\mathcal{I}(\mathcal{F})$ are of limited class. Now assume that $\mathcal{I}$ is not Borel. Let $\vec{f}$ be the sequence from Example 4.6. Then $E^1_\mathcal{I}(\vec{f}) = \mathcal{I}$ is not Borel.

4.4. **Discrete convergence.** Let $\mathcal{I}$ be an ideal. Let $\vec{f} = (f_n)$ be a sequence of real-valued functions defined on a set X. By $D_\mathcal{I}(\vec{f})$ we denote the set of all $x \in X$ for which the sequence $(f_n(x))$ is discretely $\mathcal{I}$-convergent. Clearly, $D_\mathcal{I}(\vec{f}) \subseteq E^1_\mathcal{I}(\vec{f})$. For a family $\mathcal{F}$ of real-valued functions defined on X we define

$$\begin{aligned} \mathcal{D}_\mathcal{I}(\mathcal{F}) &= \left\{ D_\mathcal{I}(\vec{f})\colon \vec{f} = (f_n), f_n \in \mathcal{F}, n \in \mathbb{N} \right\}, \\ \Delta_\mathcal{I}(\mathcal{F}) &= \left\{ (D_\mathcal{I}(\vec{f}), E^1_\mathcal{I}(\vec{f}))\colon \vec{f} = (f_n), f_n \in \mathcal{F}, n \in \mathbb{N} \right\}. \end{aligned}$$

For $\mathcal{I} = \mathrm{FIN}$ we write $\mathcal{D}(\mathcal{F})$ instead of $\mathcal{D}_\mathcal{I}(\mathcal{F})$ and $\Delta(\mathcal{F})$ instead of $\Delta_\mathcal{I}(\mathcal{F})$.

Classes $\mathcal{D}(\mathcal{F})$ for various families of functions $\mathcal{F}$ have been considered by Wesołowska in [Wes01], [Wes04]. In particular, she showed that $\mathcal{D}(\mathcal{B}_\alpha(X)) = \mathbf{\Sigma}^0_{\alpha+2}$ for any $\alpha < \omega_1$. The same arguments as in the case of pointwise convergence imply the following results.

Proposition 4.21. *If an ideal $\mathcal{I}$ has the Baire property then $\mathcal{D}(\mathcal{F}) \subseteq \mathcal{D}_\mathcal{I}(\mathcal{F})$ and $\Delta(\mathcal{F}) \subseteq \Delta_\mathcal{I}(\mathcal{F})$.*

Proposition 4.22. *If $\mathcal{M}$ is a σ-additive and (finitely) multiplicative family on X and $\mathcal{I}$ is an F_σ ideal then $\mathcal{D}_\mathcal{I}(\mathcal{F}_\mathcal{M}) \subseteq (\mathcal{M}^c)_\sigma$.*

Theorem 4.23. *Assume that $\mathcal{I}$ is an F_σ ideal and X is a separable metric space. For any $\alpha < \omega_1$, $\mathcal{D}_\mathcal{I}(\mathcal{B}_\alpha(X)) = \mathcal{D}(\mathcal{B}_\alpha(X)) = \mathbf{\Sigma}^0_{\alpha+2}$. In particular: $\mathcal{D}_\mathcal{I}(\mathcal{C}(X)) = \mathcal{D}(\mathcal{C}(X)) = F_\sigma$.*

Proposition 4.24. *Assume that $\mathcal{I}$ is an F_σ ideal. If $\mathcal{M}$ is a σ-algebra of subsets of X then $\Delta_\mathcal{I}(\mathcal{F}_\mathcal{M}) = \Delta(\mathcal{F}_\mathcal{M}) = \{(A, B) \in \mathcal{M} \times \mathcal{M} \colon A \subseteq B\}$.*

Problem 5. Characterize pairs $\Delta_\mathcal{I}(\mathcal{F})$ for other classes of functions: $B_\alpha(X)$ for $\alpha \geq 0$, or quasi-continuous functions.

References

[BFMS11] Paweł Barbarski, Rafał Filipów, Nikodem Mrożek, and Piotr Szuca. Uniform density u and $\mathcal{I}_u$-convergence on a big set. *Math. Commun.*, 16(1):125–130, 2011.

[BFMS13] Paweł Barbarski, Rafał Filipów, Nikodem Mrożek, and Piotr Szuca. When does the Katětov order imply that one ideal extends the other? *Colloq. Math.*, 130(1):91–102, 2013.

[BH88] Vitaly Bergelson and Neil Hindman. Density versions of two generalizations of Schur's theorem. *J. Combin. Theory Ser. A*, 48(1):32–38, 1988.

[BJ95] Tomek Bartoszyński and Haim Judah. *Set theory.* A K Peters Ltd., Wellesley, MA, 1995. On the structure of the real line.

[Bla10] Andreas Blass. Combinatorial cardinal characteristics of the continuum. In *Handbook of set theory. Vols. 1, 2, 3*, pages 395–489. Springer, Dordrecht, 2010.

[Bor] Dariusz Borzestowski. Oral communication.

[Bou40] Nicolas Bourbaki. *Éléments de mathématique. Part I. Les structures fondamentales de l'analyse. Livre III. Topologie générale. Chapitres I et II.* Actual. Sci. Ind., no. 858. Hermann & Cie., Paris, 1940.

[Bou12] Ahmed Bouziad. The point of continuity property, neighbourhood assignments and filter convergences. *Fund. Math.*, 218(3):225–242, 2012.

[BR10] Dariusz Borzestowski and Ireneusz Recław. On Lunina's 7-tuples for ideal convergence. *Real Anal. Exchange*, 35(2):479–485, 2010.

[Car37] Henri Cartan. Filtres et ultrafiltres. *C. R. Acad. Sci. Paris*, 205:777–779, 1937.

[CL75] Á. Császár and M. Laczkovich. Discrete and equal convergence. *Studia Sci. Math. Hungar.*, 10(3-4):463–472 (1978), 1975.

[DDP] Pratulananda Das, Sudipta Dutta, and Sudip Kumar Pal. On $\mathcal{I}$ and $\mathcal{I}^*$-equal convergence and an Egoroff-type theorem. *Matematički Vesnik.* To appear.

[DŁ08a] Lech Drewnowski and Tomasz Łuczak. On nonatomic submeasures on $\mathbb{N}$. *Arch. Math. (Basel)*, 91(1):76–85, 2008.

[DŁ08b] Lech Drewnowski and Tomasz Łuczak. On nonatomic submeasures on $\mathbb{N}$. II. *J. Math. Anal. Appl.*, 347(2):442–449, 2008.

[DSR09] Gabriel Debs and Jean Saint Raymond. Filter descriptive classes of Borel functions. *Fund. Math.*, 204(3):189–213, 2009.

[Fas51] H. Fast. Sur la convergence statistique. *Colloquium Math.*, 2:241–244 (1952), 1951.

[FGR90] P. Frankl, R. L. Graham, and V. Rödl. Iterated combinatorial density theorems. *J. Combin. Theory Ser. A*, 54(1):95–111, 1990.

[Fil] Rafał Filipów. Cardinal characteristics of quotient boolean algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and ideal convergence of sequences of functions. *Preprint.*

[FMRS07] Rafał Filipów, Nikodem Mrożek, Ireneusz Recław, and Piotr Szuca. Ideal convergence of bounded sequences. *J. Symbolic Logic*, 72(2):501–512, 2007.

[FMRS11] Rafał Filipów, Nikodem Mrożek, Ireneusz Recław, and Piotr Szuca. Ideal version of Ramsey's theorem. *Czechoslovak Math. J.*, 61(136)(2):289–308, 2011.

[Fri93] J. A. Fridy. Statistical limit points. *Proc. Amer. Math. Soc.*, 118(4):1187–1192, 1993.

[FS] Rafał Filipów and Marcin Staniszewski. On ideal equal convergence. *Preprint.*

[FS03] Ilijas Farah and Sławomir Solecki. Two $F_{\sigma\delta}$ ideals. *Proc. Amer. Math. Soc.*, 131(6):1971–1975 (electronic), 2003.

[FS10] Rafał Filipów and Piotr Szuca. Density versions of Schur's theorem for ideals generated by submeasures. *J. Combin. Theory Ser. A*, 117(7):943–956, 2010.

[FS12] Rafał Filipów and Piotr Szuca. Three kinds of convergence and the associated $\mathcal{I}$-Baire classes. *J. Math. Anal. Appl.*, 391(1):1–9, 2012.

[Hah19] H. Hahn. Uber die Menge der Konvergenzpunkte einer Funktionfolge. *Arch. der Math. und Physik*, 28:34–45, 1919.

[Hru11] Michael Hrušák. Combinatorics of filters and ideals. In *Set theory and its applications*, volume 533 of *Contemp. Math.*, pages 29–69. Amer. Math. Soc., Providence, RI, 2011.

[JK84] Winfried Just and Adam Krawczyk. On certain Boolean algebras $\mathcal{P}(\omega)/I$. *Trans. Amer. Math. Soc.*, 285(1):411–429, 1984.

[Kat68] M. Katětov. Products of filters. *Comment. Math. Univ. Carolinae*, 9:173–189, 1968.

[Kat72] M. Katětov. On descriptive classes of functions. In *Theory of sets and topology (in honour of Felix Hausdorff, 1868–1942)*, pages 265–278. VEB Deutsch. Verlag Wissensch., Berlin, 1972.

[Kor63] I. P. Kornfel′d. Sets of convergence and divergence of functional sequences. *Izv. Vysš. Učebn. Zaved. Matematika*, 1963(4 (35)):79–88, 1963. In Russian.

[KŠW01] Pavel Kostyrko, Tibor Šalát, and Władysław Wilczyński. $\mathcal{I}$-convergence. *Real Anal. Exchange*, 26(2):669–685, 2000/01.

[Lip61] Jan Lipiński. Convergence to infinity of a sequence of continuous functions. *Dokl. Akad. Nauk SSSR*, 140:752–754, 1961. In Russian.

[Lip63] J. S. Lipiński. Sets of points of convergence to infinity of a sequence of continuous functions. *Fund. Math.*, 51:35–43, 1962/1963. In Russian.

[LR09] Miklós Laczkovich and Ireneusz Recław. Ideal limits of sequences of continuous functions. *Fund. Math.*, 203(1):39–46, 2009.

[Lun75] M. A. Lunina. Sets of convergence and divergence of sequences of real-valued continuous functions on a metric space. *Mat. Zametki*, 17:205–217, 1975. In Russian.

[MA09] D. Meza-Alcántara. *Ideals and filters on countable set.* PhD thesis, Universidad Nacional Autónoma de México, 2009.

[Maz91] Krzysztof Mazur. F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebras $P(\omega)/I$. *Fund. Math.*, 138(2):103–111, 1991.

[NW] T. Natkaniec and J. Wesołowska. Sets of ideal convergence of sequences of quasicontinuous functions. Preprint.

[Rec12] Ireneusz Recław. Sets of filter convergence of sequences of continuous functions. *J. Math. Anal. Appl.*, 394(2):475–480, 2012.

[She82] Saharon Shelah. *Proper forcing*, volume 940 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982.

[Sie21] W. Sierpiński. Sur l'ensemble des points de convergence d'une suite de fonctions continues. *Fund. Math.*, 2:41–49, 1921.

[Sol99] Sławomir Solecki. Analytic ideals and their applications. *Ann. Pure Appl. Logic*, 99(1-3):51–72, 1999.

[Sol00] Sławomir Solecki. Filters and sequences. *Fund. Math.*, 163(3):215–228, 2000.

[Tal80] Michel Talagrand. Compacts de fonctions mesurables et filtres non mesurables. *Studia Math.*, 67(1):13–43, 1980.

[vD84] Eric K. van Douwen. The integers and topology. In *Handbook of set-theoretic topology*, pages 111–167. North-Holland, Amsterdam, 1984.

[Wes04] J. Wesołowska. On sets of discrete convergence points of sequences of real functions. *Real Anal. Exchange*, 29(1):107–120, 2003/04.

[Wes01] J. Wesołowska. *Sets of points of convergence of sequences of real functions.* PhD thesis, University of Gdańsk, 2001. In Polish.

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