

IDEAL CONVERGENCE VERSUS MATRIX SUMMABILITY

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ABSTRACT. We examine relationship between ideal convergence and matrix summability in the realm of bounded and unbounded sequences.

1. INTRODUCTION

The problem of examining the relationship between ideal convergence and matrix summability dates back to the 30s of the 20th century. In The Scottish Book, Problem 5 stated by Mazur ([24, p. 55] or [23, p. 69]) can be described as “is the notion of statistical convergence of bounded sequences equivalent to some matrix summability method?” There is no clear answer to that problem in the book, but Mazur wrote down in the book two claims, and from the second it follows that a matrix method summing all bounded statistically convergent sequences must also sum other bounded sequences. That corollary would mean that the answer to that problem is negative. However, Buck’s commentary under the problem in [23, 24] claims that this problem remains unsolved.

Khan and Orhan, seemingly unaware of their results relation to The Scottish Book problems, have shown in [18, Theorem 2.2] that for every nonnegative regular matrix summability method A there exists a nonnegative regular matrix method B such that A -statistical convergence and B -summability are equivalent over all bounded sequences. Since statistical convergence is A -statistical convergence when A is the Cesàro matrix, that theorem gives us a positive answer to Problem 5 of The Scottish Book.

It follows that Problem 5 from The Scottish Book is now given a final, positive answer and that the second claim of Mazur written under that problem has to be false.

In this paper we examine relationship between ideal convergence and matrix summability in the realm of bounded and unbounded sequences. In Section 2 we introduce the notions and notations, provide some known results and prove some useful facts about the ideal convergence and matrix summability that are used in the rest of the paper. In Section 3 we show when ideal convergence is equal to some matrix summability method in the case of unbounded sequences, whereas Section 4 is devoted to the case of bounded sequences. In Section 5 we examine ideals for which the false Mazur’s claim about the ideal of density zero sets holds. In Section 6 we show when ideal convergence is equal to the intersection of some matrix summability methods. In particular, we solve a problem posed by Gogola,

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Mačaj and Visnyai [15, Problem 4.6]. In Section 7 we characterize P-ideals for which ideal statistical convergence is stronger than statistical convergence — this partially solves a problem posed by Das [6, Problem 6.1].

In the last section we attached the diagram summarizing relations between classes of considered ideals.

2. SUMMABILITY METHODS

By \mathbb{N} we mean the set of positive natural numbers. By $\mathbb{R}^{\mathbb{N}}$ we mean the family of all real sequences i.e. if $x \in \mathbb{R}^{\mathbb{N}}$ then $x = (x_n)_{n \in \mathbb{N}}$ and $x_n \in \mathbb{R}$ for every $n \in \mathbb{N}$. Let $m = \{x \in \mathbb{R}^{\mathbb{N}} : x \text{ is bounded}\}$ and $c = \{x \in \mathbb{R}^{\mathbb{N}} : x \text{ is convergent}\}$.

Definition 2.1. Let $D \subseteq \mathbb{R}^{\mathbb{N}}$. Any function $\Lambda : D \rightarrow \mathbb{R}$ is called a *summability method*.

Example 2.2. The ordinary limit of sequences $\lim : c \rightarrow \mathbb{R}$ is a summability method.

Definition 2.3. A summability method Λ is *regular* if $\Lambda \upharpoonright c = \lim$ (i.e. $c \subseteq \text{dom}(\Lambda)$ and $\Lambda(x) = \lim x$ for every $x \in c$).

Definition 2.4. Let Λ_1 and Λ_2 be two summability methods. We say that

- Λ_1 and Λ_2 are *equal* if $\text{dom}(\Lambda_1) = \text{dom}(\Lambda_2)$ and $\Lambda_1(x) = \Lambda_2(x)$ for every $x \in \text{dom}(\Lambda_1)$ (i.e. $\Lambda_1 = \Lambda_2$);
- Λ_1 is *contained in* Λ_2 (or Λ_2 *contains* Λ_1) if $\text{dom}(\Lambda_1) \subseteq \text{dom}(\Lambda_2)$ and $\Lambda_1(x) = \Lambda_2(x)$ for every $x \in \text{dom}(\Lambda_1)$ (i.e. $\Lambda_1 \subseteq \Lambda_2$).

2.1. Matrix summability.

Definition 2.5. Let $A = (a_{i,k})_{i,k \in \mathbb{N}}$ be an infinite matrix of reals. We say that $x \in \mathbb{R}^{\mathbb{N}}$ is *A-summable* if

- (1) the series $A_i(x) = \sum_{k \in \mathbb{N}} a_{i,k} x_k$ is convergent for every $i \in \mathbb{N}$, and
- (2) the sequence $(A_i(x))_{i \in \mathbb{N}}$ is convergent.

The real $\lim_{i \rightarrow \infty} A_i(x)$ is called the *A-limit of the sequence x*. By c^A we denote the family of all A-summable sequences. Finally, the *matrix summability generated by a matrix A* (in short *A-summability*) is the function $\lim^A : c^A \rightarrow \mathbb{R}$ given by $\lim^A(x) = \lim_{i \rightarrow \infty} A_i(x)$. We write $\lim^A x$ instead of $\lim^A(x)$.

Example 2.6. For the identity matrix $I = (a_{i,k})$ where $a_{i,i} = 1$ and $a_{i,k} = 0$ for $i \neq k$, the matrix summability is equivalent to the ordinary limit i.e. $\lim^I = \lim$ (i.e. $c^I = c$ and $\lim^I x = \lim x$ for every $x \in c$).

Example 2.7. For the Cesàro matrix $C = (a_{i,k})$ where $a_{i,k} = 1/i$ for $k \leq i$ and $a_{i,k} = 0$ for $k > i$, the matrix summability is regular, and $\lim^C x = \lim \frac{x_1 + \dots + x_n}{n}$ for every $x \in c^C$. In this case, C-summability is called the *Cesàro summability*.

Definition 2.8. We say that a matrix $A = (a_{i,k})$ is *regular* if the matrix summability method generated by a matrix A is regular. It is *nonnegative* if $a_{i,k} \geq 0$ for every $i, k \in \mathbb{N}$.

All regular matrices are characterized by the following theorem.

Theorem 2.9 (Toeplitz [29]). *The matrix summability generated by a matrix A is regular if and only if*

- (1) $\lim_{i \rightarrow \infty} a_{i,k} = 0$ for every $k \in \mathbb{N}$,
- (2) $\sup_i \sum_{k \in \mathbb{N}} |a_{i,k}| < \infty$,
- (3) $\lim_{i \rightarrow \infty} \sum_{k \in \mathbb{N}} a_{i,k} = 1$.

2.2. Ideals on \mathbb{N} .

Definition 2.10. A family \mathcal{I} of subsets of \mathbb{N} is called an *ideal* if

- (1) $\emptyset \in \mathcal{I}$, $\mathbb{N} \notin \mathcal{I}$,
- (2) $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
- (3) $A \subseteq B \wedge B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$,
- (4) \mathcal{I} contains all finite subsets of \mathbb{N} .

An ideal \mathcal{I} is *dense* if for every infinite $A \subseteq \mathbb{N}$ there is an infinite $B \in \mathcal{I}$ such that $B \subseteq A$. An ideal \mathcal{I} is a *P-ideal* if for every countable family $\mathcal{F} \subseteq \mathcal{I}$ there is $A \in \mathcal{I}$ such that $F \setminus A$ is finite for every $F \in \mathcal{F}$. For an ideal \mathcal{I} , we write $\mathcal{I}^* = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \in \mathcal{I}\}$ and call it the *filter dual to \mathcal{I}* , and $\mathcal{I}^+ = \{A \subseteq \mathbb{N} : A \notin \mathcal{I}\}$ and call it the *coideal*. A coideal \mathcal{I}^+ is a *P-coideal* if for every decreasing sequence of sets $A_n \in \mathcal{I}^+$ ($n \in \mathbb{N}$) there is a set $A \in \mathcal{I}^+$ such that $A \setminus A_n$ is finite for every n . Ideals \mathcal{I} and \mathcal{J} are *isomorphic* (in short $\mathcal{I} \approx \mathcal{J}$) if there exists a bijection $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that $A \in \mathcal{I} \iff \phi[A] \in \mathcal{J}$ for every $A \subseteq \mathbb{N}$.

By $e_A : \mathbb{N} \rightarrow A$ we denote the increasing enumeration of a set $A \subseteq \mathbb{N}$.

For an ideal \mathcal{I} we define $\mathcal{I} \upharpoonright A = \{B \subseteq \mathbb{N} : e_A[B] \in \mathcal{I}\}$. It is easy to see that $\mathcal{I} \upharpoonright A$ is an ideal on \mathbb{N} if and only if $A \notin \mathcal{I}$.

By $2\mathbb{N}$ and $2\mathbb{N} + 1$ we denote the sets of all even and odd natural numbers respectively.

For families $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathbb{N})$ we define

$$\mathcal{A} \oplus \mathcal{B} = \{C \subseteq \mathbb{N} : e_{2\mathbb{N}}^{-1}[C \cap 2\mathbb{N}] \in \mathcal{A} \wedge e_{2\mathbb{N}+1}^{-1}[C \cap (2\mathbb{N} + 1)] \in \mathcal{B}\}.$$

It is easy to see that if \mathcal{I}, \mathcal{J} are ideals then $\mathcal{I} \oplus \mathcal{J}$, $\mathcal{I} \oplus \mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{N}) \oplus \mathcal{J}$ are also ideals. Moreover, $\mathcal{I} \oplus \mathcal{J} \upharpoonright 2\mathbb{N} = \mathcal{I}$ and $\mathcal{I} \oplus \mathcal{J} \upharpoonright (2\mathbb{N} + 1) = \mathcal{J}$. Note also that $A \in \text{Fin} \oplus \mathcal{P}(\mathbb{N}) \iff A \cap 2\mathbb{N} \in \text{Fin}$.

By identifying sets of natural numbers with their characteristic functions, we equip $\mathcal{P}(\mathbb{N})$ with the topology of the Cantor space $\{0, 1\}^\omega$ and therefore we can assign topological complexity to ideals. In particular, an ideal \mathcal{I} is $F_\sigma, F_{\sigma\delta}$, analytic (resp.) if it is an $F_\sigma, F_{\sigma\delta}$, analytic (resp.) subset of the Cantor space.

Example 2.11. The family $\text{Fin} = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$ is an F_σ P-ideal which is not dense.

Definition 2.12. For a set $A \subseteq \mathbb{N}$ we define the *asymptotic density of A* by

$$d(A) = \lim_{i \rightarrow \infty} d_i(A),$$

where $d_i(A) = |A \cap \{1, 2, \dots, n\}|/n$, provided that the considered limit exists.

Example 2.13. The family $\mathcal{I}_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$ of all sets of the asymptotic density zero is a dense $F_{\sigma\delta}$ P-ideal (see e.g. [9, Example 1.2.3(d)]).

Definition 2.14. A map $\Phi : \mathcal{P}(\mathbb{N}) \rightarrow [0, \infty]$ is a *submeasure* on \mathbb{N} if

- (1) $\Phi(\emptyset) = 0$,
- (2) if $A \subseteq B$ then $\Phi(A) \leq \Phi(B)$,
- (3) $\Phi(A \cup B) \leq \Phi(A) + \Phi(B)$.

A submeasure Φ is

- *lower semicontinuous* if $\Phi(A) = \lim_{n \rightarrow \infty} \Phi(A \cap \{1, \dots, n\})$ for every $A \subseteq \mathbb{N}$;
- *nonpathological* if $\Phi(A) = \sup\{\mu(A) : \mu \leq \Phi, \mu \text{ is a finite measure}\}$ for each A .

Definition 2.15. For a submeasure Φ we define $\text{Fin}(\Phi) = \{A \subseteq \mathbb{N} : \Phi(A) < \infty\}$. If $\Phi(\mathbb{N}) = \infty$ and $\Phi(\{n\}) < \infty$ for every $n \in \mathbb{N}$, then $\text{Fin}(\Phi)$ is an ideal.

All F_σ ideals are characterized by the following theorem.

Theorem 2.16 (Mazur [25]). \mathcal{I} is an F_σ ideal $\iff \mathcal{I} = \text{Fin}(\Phi)$ for some lower semicontinuous submeasure Φ on \mathbb{N} such that $\Phi(\mathbb{N}) = \infty$ and $\Phi(\{n\}) < \infty$ for every $n \in \mathbb{N}$.

Definition 2.17. For a submeasure Φ we define $\text{Exh}(\Phi) = \{A \subseteq \mathbb{N} : \lim_{n \rightarrow \infty} \Phi(A \setminus \{1, \dots, n\}) = 0\}$. If $\lim_{n \rightarrow \infty} \Phi(\mathbb{N} \setminus \{1, \dots, n\}) \neq 0$, then $\text{Exh}(\Phi)$ is an ideal (see e.g. [9]).

All F_σ P-ideals are characterized by the following theorem.

Theorem 2.18 (e.g. [9]). \mathcal{I} is an F_σ P-ideal $\iff \mathcal{I} = \text{Fin}(\Phi) = \text{Exh}(\Phi)$ for some lower semicontinuous submeasure Φ on \mathbb{N} such that $\Phi(\mathbb{N}) = \infty$ and $\Phi(\{n\}) < \infty$ for every $n \in \mathbb{N}$.

All $F_{\sigma\delta}$ P-ideals are characterized by the following theorem.

Theorem 2.19 (Solecki [27]). *The following conditions are equivalent.*

- (1) \mathcal{I} is an analytic P-ideal.
- (2) \mathcal{I} is an $F_{\sigma\delta}$ P-ideal.
- (3) $\mathcal{I} = \text{Exh}(\Phi)$ for some lower semicontinuous submeasure Φ on \mathbb{N} such that $\lim_{n \rightarrow \infty} \Phi(\mathbb{N} \setminus \{1, \dots, n\}) \neq 0$.

2.3. Ideal convergence.

Definition 2.20. Let \mathcal{I} be an ideal. A sequence $x \in \mathbb{R}^{\mathbb{N}}$ is \mathcal{I} -convergent if there exists $L \in \mathbb{R}$ such that $\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$. The real L is called the \mathcal{I} -limit of the sequence x . By $c^{\mathcal{I}}$ we denote the family of all \mathcal{I} -convergent sequences. Finally, the *ideal convergence generated by an ideal \mathcal{I}* (in short \mathcal{I} -convergence) is the function $\lim^{\mathcal{I}} : c^{\mathcal{I}} \rightarrow \mathbb{R}$ mapping x into the \mathcal{I} -limit of x .

Proposition 2.21. *The ideal convergence generated by an ideal \mathcal{I} is regular $\iff \text{Fin} \subseteq \mathcal{I}$.*

Proof. (\Rightarrow) Let $B \in \text{Fin}$. Let $x \in \mathbb{R}^{\mathbb{N}}$ be defined by $x_n = 1$ for $n \in B$ and $x_n = 0$ otherwise. Since $\lim x = 0$, $\lim^{\mathcal{I}} x = 0$. Thus, $B = \{n \in \mathbb{N} : |x_n - 0| \geq 1/2\} \in \mathcal{I}$.

(\Leftarrow) Let $x \in c$ with $\lim x = L$. Then for every $\varepsilon > 0$ we have $\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \text{Fin} \subseteq \mathcal{I}$. Thus $x \in c^{\mathcal{I}}$ and $\lim^{\mathcal{I}} x = L$. \square

Example 2.22. The ideal convergence generated by the ideal $\mathcal{I} = \text{Fin}$ is equal to the ordinary limit ($\lim^{\text{Fin}} = \lim$) i.e. $c^{\text{Fin}} = c$ and $\lim^{\text{Fin}} x = \lim x$ for every $x \in c$.

Example 2.23. The ideal convergence generated by the ideal $\mathcal{I} = \mathcal{I}_d$ is regular and strictly contains the ordinary convergence i.e. $\lim \subsetneq \lim^{\mathcal{I}_d}$. It is also called the *statistical convergence*. In ‘‘Scottish Book’’ (e.g. [24, p. 55]), Mazur used the name *asymptotic convergence* in this case. (In fact, Mazur defined it in a different manner, however by [19, Theorem 3.2] both notions coincide).

2.4. Ideals generated by matrices.

Definition 2.24 (Freedman-Sember [12], see also Drewnowski-Paúl [8]). Let A be a nonnegative regular matrix. For a set $B \subseteq \mathbb{N}$ we define *upper A -density of B* by

$$\overline{d^A}(B) = \limsup_{i \rightarrow \infty} d_i^A(B),$$

where $d_i^A(B) = \sum_{k \in B} a_{i,k}$. Moreover we define *A -density of B* by

$$d^A(B) = \lim_{i \rightarrow \infty} d_i^A(B)$$

provided that the considered limit exists. Note that $d_i^A(B) = A_i(\chi_B)$ and $d^A(B) = \lim^A \chi_B$, where χ_B is the characteristic function of the set B .

Definition 2.25. For a nonnegative regular matrix A we define the family

$$\mathcal{I}(A) = \{B \subseteq \mathbb{N} : d^A(B) = 0\}.$$

It is easy to see that $\mathcal{I}(A)$ is an ideal, and we call it *a matrix ideal generated by the matrix A* .

Example 2.26. For the identity matrix I , $d^I(B) = 0$ if B is finite, $d^I(B) = 1$ if $\mathbb{N} \setminus B$ is finite and is not defined in other cases. The matrix ideal $\mathcal{I}(I) = \text{Fin}$.

Example 2.27. For the Cesàro matrix C , the C -density is just the asymptotic density and the matrix ideal $\mathcal{I}(C) = \mathcal{I}_d$.

Lemma 2.28 (Folklore). *If A is a regular matrix, then there is a regular matrix B such that*

- (1) B has only finitely many nonzero elements in each row,
- (2) each row of B sums to 1,
- (3) $\lim^A x = \lim^B x$ for every $x \in m$,
- (4) $\mathcal{I}(A) = \mathcal{I}(B)$.

Proof. Let $C = \{c_1, c_2, c_3, \dots\}$ be the set of rows with infinitely many nonzero elements. Since A is regular, $A_{c_i}(\chi_{\mathbb{N}}) < \infty$ for all $i \in \mathbb{N}$. Thus, for each $i \in \mathbb{N}$, we can find k_i such that $\sum_{k > k_i} a_{c_i,k} < 1/10^i$. Define matrix B' by $b'_{c_i,k} = 0$ for all $i \in \mathbb{N}$ and $k > k_i$ while $b'_{c_i,k} = a_{c_i,k}$ otherwise. Then the matrix B is given by $b_{i,k} = b'_{c_i,k}/B'_{c_i}(\chi_{\mathbb{N}})$ for all natural i and k . It is easy to see that B is regular and that $\mathcal{I}(A) = \mathcal{I}(B)$ since for every set $D \subseteq \mathbb{N}$, $A_n(\chi_D) = B_n(\chi_D) \cdot A_n(\chi_{\mathbb{N}})$ when $n \notin C$ and $A_{c_i}(\chi_D) - 1/10^i \leq B_{c_i}(\chi_D) \cdot B'_{c_i}(\chi_{\mathbb{N}}) \leq A_{c_i}(\chi_D)$ for all $i \in \mathbb{N}$ while $\lim_{n \rightarrow \infty} B'_n(\chi_{\mathbb{N}}) = 1$.

Now, we only need to show that $\lim^A x = \lim^B x$ for every $x \in m$. Take any $x \in m$. Then $A_n(x) = A_n(\chi_{\mathbb{N}}) \cdot B_n(x)$ for $n \notin C$. On the other hand, $A_{c_i}(x) \leq B_{c_i}(x) \cdot B'_{c_i}(\chi_{\mathbb{N}}) + \sup_{n \in \mathbb{N}} |x_n|/10^i$ and $A_{c_i}(x) \geq B_{c_i}(x) \cdot B'_{c_i}(\chi_{\mathbb{N}}) - \sup_{n \in \mathbb{N}} |x_n|/10^i$ for all $i \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} B'_n(\chi_{\mathbb{N}}) = \lim_{n \rightarrow \infty} A_n(\chi_{\mathbb{N}}) = 1$, x is bounded and $1/10^i$ tends to 0, $\lim_{n \rightarrow \infty} B_n(x) = \lim_{n \rightarrow \infty} A_n(x)$ if any of these two limits exists. \square

Remark. In general, Lemma 2.28(3) cannot be extended for unbounded sequences x . Indeed, take a partition of \mathbb{N} into infinitely many infinite sets A_1, A_2, \dots and let A be any matrix such that $a_{i,k} \neq 0 \Leftrightarrow k \in A_i$. Let $B = (b_{i,k})$ be any regular matrix with finitely many nonzero elements in each row and define k_i as the smallest element such that $b_{i,k} = 0$ for all $k > k_i$ and let K_1 be the smallest element greater than k_1 belonging to A_1 . Suppose we have defined K_1, \dots, K_n . We search for L_{n+1}

such that for all $j \geq L_{n+1}$ we have $b_{j,K_1}/a_{1,K_1} + \dots + b_{j,K_n}/a_{n,K_n} < 1/2$ and define K_{n+1} as the smallest element greater than $k_{L_{n+1}}$ belonging to A_{n+1} . Define x by $x_{K_i} = 1/a_{i,K_i}$ and $x_n = 0$ otherwise. It is easy to see that $A_i(x) = 1$ for all $i \in \mathbb{N}$, thus $\lim^A(x) = 1$. On the other hand, $B_{L_n}(x) < 1/2$, hence $\lim^B(x)$ either does not exist or is not greater than $1/2$. In both cases, $\lim^A(x) \neq \lim^B(x)$.

Proposition 2.29 (Freedman-Sember [12, Propositions 3.1 and 3.2]). *If A is a nonnegative regular matrix, then $\mathcal{I}(A)$ is a P -ideal.*

Proposition 2.30 (Bartoszewicz-Das-Głab [1, Proposition 13]). *If A is a regular matrix, then $\mathcal{I}(A)$ is an $F_{\sigma\delta}$ set.*

Proof. In [1, Proposition 13] the authors find a lower semicontinuous submeasure Φ such that $\mathcal{I}(A) = \text{Exh}(\Phi)$, hence, by Theorem 2.19, they obtain that $\mathcal{I}(A)$ is an $F_{\sigma\delta}$ P -ideal. Below we present a direct proof.

By Lemma 2.28 we may assume that A has only finitely many nonzero elements in each row. Now, we observe that

$$\mathcal{I}(A) = \bigcap_{k \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{n > N} G_{n,k},$$

where $G_{n,k} = \{C \subseteq \mathbb{N} : A_n(\chi_C) < 1/k\}$.

Notice that since A has only finitely many nonzero elements in each row, $G_{n,k}$ is a closed set in the Cantor space, which finishes the proof. \square

Proposition 2.31 (Drewnowski-Paúl [8, Proposition 7.2]). *Let $A = (a_{i,k})$ be a nonnegative regular matrix. The ideal $\mathcal{I}(A)$ is dense if and only if $\lim_{i,k \rightarrow \infty} a_{i,k} = 0$ (i.e. for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $a_{i,k} < \varepsilon$ for all $i, k > n$).*

Remark. The notion of ideal convergence with respect to the matrix ideal $\mathcal{I}(A)$ was introduced by Connor [5, Definition 7] who coined the name *A -statistical convergence* for this kind of convergence.

3. IDEAL CONVERGENCE VERSUS MATRIX SUMMABILITY

In this section we examine the relationship between matrix summability and ideal convergence in the realm of all sequences (bounded and unbounded). A comparison of these two methods in the realm of bounded sequences is done in Section 4.

Theorem 3.1. *The ideal convergence generated by an ideal \mathcal{I} is contained in some matrix summability if and only if \mathcal{I} is not dense.*

Proof. (\Leftarrow) Let $B \subseteq \mathbb{N}$ be an infinite set such that for every $C \subseteq B$, if $C \in \mathcal{I}$ then C is finite. Let $A = (a_{i,k})$ be a matrix given by $a_{i,e_B(i)} = 1$ for $i \in \mathbb{N}$ and $a_{i,k} = 0$ otherwise. We show that $\lim^{\mathcal{I}} \subseteq \lim^A$. Let $x \in c^{\mathcal{I}}$ with $\lim^{\mathcal{I}} x = L$. Let $\varepsilon > 0$. Since $C_\varepsilon = \{n \in \mathbb{N} : |x_n - L| > \varepsilon\} \in \mathcal{I}$, $C_\varepsilon \cap B \in \text{Fin}$. Thus $\{i \in \mathbb{N} : |x_{e_B(i)} - L| > \varepsilon\} \in \text{Fin}$ and that means that $\lim_{i \rightarrow \infty} x_{e_B(i)} = L$. On the other hand $A_i(x) = x_{e_B(i)}$ for every $i \in \mathbb{N}$, so $\lim^A x = L$.

(\Rightarrow) Let $A = (a_{i,k})$ be a matrix such that $\lim^{\mathcal{I}} \subseteq \lim^A$. If $\mathcal{I} = \text{Fin}$ we are done, so assume $\mathcal{I} \neq \text{Fin}$ (i.e. \mathcal{I} contains an infinite set). We have 3 cases.

- (1) $\forall B \in \mathcal{I}, |B| = \aleph_0 ((\forall i \in \mathbb{N} (k_i(B) = \sup\{k \in B : a_{i,k} \neq 0\}) < \infty)) \wedge k(B) = \sup\{k_i(B) : i \in \mathbb{N}\} < \infty)$.
- (2) $\exists B \in \mathcal{I}, |B| = \aleph_0 \exists i_1 < i_2 < \dots ((\forall n \in \mathbb{N} (k_n = \sup\{k \in B : a_{i_n,k} \neq 0\}) < \infty)) \wedge \sup\{k_n : n \in \mathbb{N}\} = \infty)$.

(3) $\exists B \in \mathcal{I}, |B| = \aleph_0 \exists i_0 \forall k \in B (a_{i_0, k} \neq 0)$.

Below we show that in case (1) the ideal \mathcal{I} is not dense, and the remaining cases are not possible.

Case (1). Let

$$B_1 = \bigcup \{B \setminus \{1, \dots, k(B)\} : B \in \mathcal{I} \wedge |B| = \aleph_0\}.$$

Note that $a_{i, k} = 0$ for every $k \in B_1$ and $i \in \mathbb{N}$.

Since \mathcal{I} contains an infinite set, B_1 is infinite. Let $B_0 = \mathbb{N} \setminus B_1$. We claim that B_0 is also infinite. Suppose to the contrary that B_0 is finite and $\max B_0 = k_0$. We define the sequence $x \in \mathbb{R}^{\mathbb{N}}$ by $x_k = 0$ for $k \leq k_0$ and $x_k = 1$ otherwise. Then $x \in c^{\mathcal{I}}$ and $\lim^{\mathcal{I}} x = 1$, so $\lim^A x = 1$. On the other hand $A_i(x) = 0$ for every $i \in \mathbb{N}$. Thus, $\lim^A x = 0$, a contradiction.

We show that $C \cap B_0$ is finite for every $C \in \mathcal{I}$ (i.e. \mathcal{I} is not dense).

Let $C \in \mathcal{I}$. If C is finite we are done. If C is infinite then $C \setminus \{1, \dots, k(C)\} \subseteq B_1$, hence $C \cap B_0 \subseteq \{1, \dots, k(C)\}$.

Case (2). We can assume that the sequence (k_n) is increasing (otherwise consider a subsequence (i_{j_n}) such that (k_{j_n}) is increasing). Let $B_0 = \{k_n : n \in \mathbb{N}\}$. (Note that $B_0 \subseteq B$, so $B_0 \in \mathcal{I}$.) We define the sequence $x \in \mathbb{R}^{\mathbb{N}}$ by $x_k = 0$ for $k \notin B_0$ and $x_{k_1} = 1/a_{i_1, k_1}$, $x_{k_2} = (2 - a_{i_2, k_1} \cdot x_{k_1})/a_{i_2, k_2}$, and in general for any $n > 1$ we put

$$x_{k_n} = \frac{1}{a_{i_n, k_n}} \cdot \left(n - \sum_{j=1}^{n-1} a_{i_n, k_j} \cdot x_{k_j} \right).$$

Since $B_0 \in \mathcal{I}$, $x \in c^{\mathcal{I}}$. On the other hand, it is not difficult to see that $A_{i_n}(x) = n$ for every $n \in \mathbb{N}$, hence $x \notin c^A$, a contradiction.

Case (3). We define the sequence $x \in \mathbb{R}^{\mathbb{N}}$ by $x_k = 1/a_{i_0, k}$ for $k \in B$ and $x_k = 0$ otherwise. Since $B \in \mathcal{I}$, $x \in c^{\mathcal{I}}$. On the other hand $A_{i_0}(x) = \infty$, so $x \notin c^A$, a contradiction. \square

Remark. Since the ideal \mathcal{I}_d is dense, so the ideal convergence generated by \mathcal{I}_d is not contained in any matrix summability. This result was proved by Fridy [13, Theorem 2].

Theorem 3.2. *The ideal convergence generated by an ideal \mathcal{I} is equal to some matrix summability if and only if $\mathcal{I} = \text{Fin}$ or $\mathcal{I} \approx \text{Fin} \oplus \mathcal{P}(\mathbb{N})$.*

Proof. (\Leftarrow) If $\mathcal{I} = \text{Fin}$, we are done by Examples 2.6 and 2.22. Assume that $\mathcal{I} \approx \text{Fin} \oplus \mathcal{P}(\mathbb{N})$. Let $\phi : \mathbb{N} \rightarrow \mathbb{N}$ be a bijection such that $B \in \mathcal{I} \iff \phi[B] \in \text{Fin} \oplus \mathcal{P}(\mathbb{N})$.

Let $B = \phi^{-1}[2\mathbb{N}]$. Let $A = (a_{i, k})$ be a matrix given by $a_{i, e_B(i)} = 1$ for $i \in \mathbb{N}$ and $a_{i, k} = 0$ otherwise. Proceeding as in the proof of Theorem 3.1(\Leftarrow) we can show that $\lim^{\mathcal{I}} \subseteq \lim^A$. Now we show that $\lim^A \subseteq \lim^{\mathcal{I}}$.

Let $x \in c^A$ with $\lim^A x = L$. Let $\varepsilon > 0$. Since $A_i(x) = x_{e_B(i)}$ for every $i \in \mathbb{N}$, $C_\varepsilon = \{e_B(i) \in \mathbb{N} : |x_{e_B(i)} - L| > \varepsilon\} \in \text{Fin}$. Let $D_\varepsilon = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$. Then $\phi[D_\varepsilon] \cap 2\mathbb{N} = \phi[D_\varepsilon \cap B] = \phi[C_\varepsilon] \in \text{Fin}$. Hence $\phi[D_\varepsilon] \in \text{Fin} \oplus \mathcal{P}(\mathbb{N})$, so $D_\varepsilon \in \mathcal{I}$.

(\Rightarrow) Proceeding as in the proof of Theorem 3.1(\Rightarrow) we see that only in case (1) we need to prove that \mathcal{I} is isomorphic to $\text{Fin} \oplus \mathcal{P}(\mathbb{N})$.

Since B_0, B_1 are infinite, disjoint and $B_0 \cup B_1 = \mathbb{N}$, if we show that

$$B \in \mathcal{I} \iff B \cap B_0 \in \text{Fin}$$

then \mathcal{I} will be isomorphic to $\text{Fin} \oplus \mathcal{P}(\mathbb{N})$.

Let $B \in \mathcal{I}$. If B is finite we are done. If B is infinite then $B \setminus \{1, \dots, k(B)\} \subseteq B_1$, hence $B \cap B_0 \in \text{Fin}$.

Now take $B \subseteq \mathbb{N}$ such that $B \cap B_0 \in \text{Fin}$. We define the sequence $x \in \mathbb{R}^{\mathbb{N}}$ by $x_k = 1$ for $k \in B \setminus (B \cap B_0)$ and $x_k = 0$ otherwise. Then $A_i(x) = 0$ for every $i \in \mathbb{N}$, so $\lim^A x = 0$. Then also $\lim^{\mathcal{I}} x = 0$, so $B \setminus (B \cap B_0) \subseteq \{k \in \mathbb{N} : |x_k - 0| > 1/2\} \in \mathcal{I}$. Thus $B \in \mathcal{I}$. \square

Remark. Since \mathcal{I}_d is dense, it is not isomorphic to Fin nor $\text{Fin} \oplus \mathcal{P}(\mathbb{N})$. Thus, the ideal convergence generated by \mathcal{I}_d is not equal to any matrix summability. This result was announced (without a proof) by Mazur in the Scottish Book (e.g. [24, p. 56]).

4. SUMMABILITY METHODS IN THE REALM OF BOUNDED SEQUENCES

In this section we consider the relationship between matrix summability and ideal convergence considered only for bounded sequences.

Theorem 4.1 (Khan-Orhan [18, Theorem 2.2]). *For every nonnegative regular matrix A there exists a nonnegative regular matrix B such that $\mathcal{I}(A) = \mathcal{I}(B)$ and the ideal convergence generated by the ideal $\mathcal{I}(A)$ is equal to the matrix summability generated by the matrix B in the realm of bounded sequences (i.e. $\lim^{\mathcal{I}(A)} \upharpoonright m = \lim^B \upharpoonright m$).*

Remark. If C is the Cesàro matrix then it is not difficult to see that $\mathcal{I}(C) = \mathcal{I}_d$. So by Theorem 4.1 the ideal convergence generated by the ideal \mathcal{I}_d (i.e. statistical convergence) is equal to the matrix summability generated by some nonnegative matrix. This result of Khan and Orhan is the answer to Mazur's Problem 5 of the "Scottish Book" (e.g. [24, p. 55] or [23, p. 69]). It is worth to note that the fact that the result of Khan and Orhan answers the question of Mazur has not been noticed neither by the authors in their paper nor in the commentary to Problem 5 in the latest edition of the "Scottish Book" [24, p. 55].

Corollary 4.2. *The ideal convergence generated by an ideal \mathcal{I} is equal to the matrix summability generated by a nonnegative regular matrix in the realm of bounded sequences if and only if \mathcal{I} is a matrix ideal generated by a nonnegative regular matrix.*

Proof. Since (\Leftarrow) is proved by Khan-Orhan (see Theorem 4.1), we only have to show (\Rightarrow) . Let $\lim^{\mathcal{I}} \upharpoonright m = \lim^B \upharpoonright m$ where B is a nonnegative regular matrix. Once we show $\mathcal{I} = \mathcal{I}(B)$, the proof is finished

(\subseteq) Let $C \in \mathcal{I}$. Since $\chi_C \in m$ and $\lim^{\mathcal{I}} \chi_C = 0$, $\lim^B \chi_C = 0$. Thus $C \in \mathcal{I}(B)$.

(\supseteq) Let $C \in \mathcal{I}(B)$. Since $\chi_C \in m$ and $\lim^B \chi_C = 0$, $\lim^{\mathcal{I}} \chi_C = 0$. Hence $C = \{k \in \mathbb{N} : |\chi_C(k) - 0| > 1/2\} \in \mathcal{I}$. \square

Corollary 4.3. *The ideal convergence generated by an ideal \mathcal{I} is contained in the matrix summability generated by a nonnegative regular matrix in the realm of bounded sequences if and only if the ideal \mathcal{I} can be extended to the matrix summability ideal generated by a nonnegative regular matrix.*

Proof. (\Rightarrow) Let $\lim^{\mathcal{I}} \upharpoonright m \subseteq \lim^B \upharpoonright m$ where B is a nonnegative regular matrix. Once we show $\mathcal{I} \subseteq \mathcal{I}(B)$, the proof is finished. Let $C \in \mathcal{I}$. Since $\chi_C \in m$ and $\lim^{\mathcal{I}} \chi_C = 0$, $\lim^B \chi_C = 0$. Thus $C \in \mathcal{I}(B)$.

(\Leftarrow) Let A be a nonnegative regular matrix A with $\mathcal{I} \subseteq \mathcal{I}(A)$. Then $\lim^{\mathcal{I}} \subseteq \lim^{\mathcal{I}(A)}$. By Theorem 4.1 there is a nonnegative regular matrix B with $\lim^{\mathcal{I}(A)} = \lim^B$. Thus $\lim^{\mathcal{I}} \subseteq \lim^B$. \square

4.1. Some special summability methods. Corollary 4.2 says that the ideal convergence generated by a matrix ideal $\mathcal{I}(A)$ is equal to the matrix summability generated by some nonnegative regular matrix B in the realm of bounded sequences. However, in general it is not the case that $B = A$. Below (Proposition 4.4(1)) we show that the matrix summability generated by a matrix A contains the ideal convergence generated by the matrix ideal $\mathcal{I}(A)$. Moreover, we provide (Proposition 4.4(2)) a sufficient condition to guarantee that the matrix summability generated by A strictly contains the ideal convergence generated by the matrix ideal $\mathcal{I}(A)$.

Proposition 4.4. *Let A be a nonnegative regular matrix.*

- (1) *The ideal convergence generated by the ideal $\mathcal{I}(A)$ is contained in the matrix summability generated by the matrix A in the realm of bounded sequences (i.e. $c^{\mathcal{I}(A)} \cap m \subseteq c^A \cap m$ and $\lim^{\mathcal{I}(A)} x = \lim^A x$ for every $x \in c^{\mathcal{I}(A)} \cap m$).*
- (2) *If there exists $B \subseteq \mathbb{N}$ such that $d^A(B)$ exists and is not equal to 0 nor 1 then $c^A \cap m \setminus c^{\mathcal{I}(A)} \cap m \neq \emptyset$.*

Proof. (1) Let $x \in c^{\mathcal{I}(A)} \cap m$ with $\lim^{\mathcal{I}(A)} x = L$. Since $\mathcal{I}(A)$ is a P-ideal (see Proposition 2.29), there is a set $F \in \mathcal{I}(A)^*$ such that $\lim_{n \in F} x_n = L$ (see [19, Thm. 3.2]). Consider the sequences y and z given by $y_n = x_n$ when $n \in F$ while $y_n = L$ otherwise and $z_n = x_n - L$ when $n \notin F$ and $z_n = 0$ otherwise. Note that $x = y + z$. First we note that $\lim^A z = 0$. Indeed, since x is bounded and $\mathbb{N} \setminus F \in \mathcal{I}(A)$ we have

$$\lim_{i \rightarrow \infty} |A_i(z)| \leq \lim_{i \rightarrow \infty} \sum_{k \in \mathbb{N} \setminus F} a_{i,k} |z_k| \leq d^A(\mathbb{N} \setminus F) \cdot \sup\{|x_k - L| : k \in \mathbb{N} \setminus F\} = 0.$$

Second we note that the sequence y is ordinarily convergent to L , so $\lim^A y = L$, because A is regular. The only thing left is to see that $\lim^A x = \lim^A y + \lim^A z = L$.

(2) Let $x = \chi_B$. Since $B \notin \mathcal{I}(A) \cup \mathcal{I}(A)^*$, it is clear that $x \notin c^{\mathcal{I}(A)} \cap m$. On the other hand, it is easy to see that $\lim^A x = d^A(B)$, thus $x \in c^A \cap m$. \square

Below we provide two applications of Proposition 4.4 to some known matrix summability methods, namely to the Cesàro summability (Proposition 4.5) and to the Nörlund summability (Proposition 4.8).

Proposition 4.5 (Schoenberg [26, Lemma 4]). *The statistical convergence is strictly contained in the Cesàro summability in the realm of bounded sequences.*

Proof. If C is the Cesàro matrix, then $\mathcal{I}(C) = \mathcal{I}_d$, \mathcal{I}_d -convergence is the same as the statistical convergence, and C -summability is the Cesàro summability. Moreover the set of all even numbers has the asymptotic density $1/2$. Thus Proposition 4.4 finishes the proof. \square

Definition 4.6. Let $p = (p_n)$ be a sequence of reals with $s_i = \sum_{k=1}^i p_k \neq 0$ for every $i \in \mathbb{N}$. The matrix $N_p = (a_{i,k})$ given by $a_{i,k} = p_k/s_i$ for $k \leq i, i \in \mathbb{N}$ and $a_{i,k} = 0$ otherwise is called *Nörlund matrix with respect to the sequence p* (e.g. [4, Definition 3.3.1]). It is known that a Nörlund matrix $N_p = (a_{i,k})$ is regular if and only if $\lim_{n \rightarrow \infty} a_{n,n} = 0$ (e.g. [4, Corollary 3.3.4]).

Remark. If p is the constant sequence equal to 1, then the Nörlund matrix N_p is equal to the Cesàro matrix C .

Definition 4.7. Let $p = (p_n)$ be a sequence of nonnegative reals with $s_i = \sum_{k=1}^i p_k \neq 0$ for every $i \in \mathbb{N}$ such that $\lim_{i \rightarrow \infty} s_i = +\infty$ and $\lim_{i \rightarrow \infty} p_i/s_i = 0$. The family

$$\mathcal{EU}_p = \left\{ A \subseteq \mathbb{N} : \lim_{n \rightarrow \infty} \frac{\sum_{i \in A, i \leq n} p_i}{\sum_{i \leq n} p_i} = 0 \right\}$$

is called the *Erdős-Ulam ideal generated by p* . It is known that \mathcal{EU}_p is an analytic P-ideal ([17] or [9]).

Remark. If N_p is a nonnegative regular Nörlund matrix such that $\lim_{i \rightarrow \infty} s_i = +\infty$ and $\lim_{i \rightarrow \infty} p_i/s_i = 0$ then it is not difficult to see that $\mathcal{I}(N_p) = \mathcal{EU}_p$.

Remark. The ideal \mathcal{I}_d is the Erdős-Ulam ideal \mathcal{EU}_p with any constant sequence p .

Proposition 4.8. Let $p = (p_n)$ be a sequence of nonnegative reals with $s_i = \sum_{k=1}^i p_k \neq 0$ for every $i \in \mathbb{N}$ such that $\lim_{i \rightarrow \infty} s_i = +\infty$ and $\lim_{i \rightarrow \infty} p_i/s_i = 0$. The ideal convergence generated by an Erdős-Ulam ideal \mathcal{EU}_p is strictly contained in the matrix summability generated by a Nörlund matrix N_p in the realm of bounded sequences.

Proof. Since $a_{i,k} \geq 0$ for all i, k and $\lim_{i \rightarrow \infty} a_{i,i} = 0$, N_p is regular and nonnegative. Thus, by Proposition 4.4(1), $\lim^{\mathcal{EU}_p} \subseteq \lim^{N_p}$.

Now we show that the inclusion is strict. In [21, Theorem 1.3] the authors proved that the function d^{N_p} has Darboux property for regular N_p . In particular, there is a set $B \subseteq \mathbb{N}$ such that $d^{N_p}(B) = 1/2$. Now Proposition 4.4(2) finishes the proof. \square

4.2. Necessary conditions for ideal convergence to be equal to (or contained in) some matrix summability. Below (Proposition 4.9) we provide some necessary conditions for ideal convergence to be equal to (or contained in) some matrix summability in the realm of bounded sequences which seem easier to check than showing that an ideal is not equal to (or contained in) a matrix ideal. In Proposition 4.11 and Theorem 4.12 we show that these conditions are not sufficient.

Proposition 4.9. Let \mathcal{I} be an ideal.

- (1) If there is a nonnegative regular matrix A with $\lim^{\mathcal{I}} \upharpoonright m = \lim^A \upharpoonright m$, then \mathcal{I} is an $F_{\sigma\delta}$ P-ideal.
- (2) If there is a nonnegative regular matrix A with $\lim^{\mathcal{I}} \upharpoonright m \subseteq \lim^A \upharpoonright m$, then \mathcal{I} is contained in an $F_{\sigma\delta}$ P-ideal.

Proof. (1) Apply Corollary 4.2 and Propositions 2.29 and 2.30. (2) Apply Corollary 4.3 and Propositions 2.29 and 2.30. \square

Definition 4.10. For every $f : \mathbb{N} \rightarrow [0, \infty)$ such that $\sum_{n=1}^{\infty} f(n) = \infty$ we define a *summable ideal generated by a function f* by $\mathcal{I}_f = \{B \subseteq \mathbb{N} : \sum_{n \in B} f(n) < \infty\}$. In particular, if $f(n) = 1/n^\alpha$ with $0 < \alpha \leq 1$ we obtain the ideal $\mathcal{I}_{(1/n^\alpha)} = \{B \subseteq \mathbb{N} : \sum_{n \in B} \frac{1}{n^\alpha} < \infty\}$. It is known that summable ideals are F_σ P-ideals (see e.g. [9, Example 1.2.3]).

Proposition 4.11. If $f : \mathbb{N} \rightarrow [0, \infty)$ is such that $\sum_{n=1}^{\infty} f(n) = \infty$ and $\lim_{n \rightarrow \infty} f(n) = 0$, then $\mathcal{I}_f \neq \mathcal{I}(A)$ for any nonnegative regular matrix A .

Proof. First we notice that the ideal \mathcal{I}_f is dense and the coideal $\mathcal{I}_f^+ = \{B \subseteq \mathbb{N} : B \notin \mathcal{I}_f\}$ is a P-coideal i.e. for every decreasing sequence of sets $B_n \in \mathcal{I}_f^+$ ($n \in \mathbb{N}$) there is a set $B \notin \mathcal{I}_f$ such that $B \setminus B_n$ is finite for every n .

Density of \mathcal{I}_f easily follows from the fact that $\lim_{n \rightarrow \infty} f(n) = 0$ (see e.g. [9, Lemma 1.12.3]).

The fact that \mathcal{I}_f^+ is a P-coideal can easily be shown directly or we can just note that every summable ideal is an F_σ ideal (see e.g. [9, Example 1.2.3]) and for every F_σ ideal \mathcal{J} it is known that \mathcal{J}^+ is a P-coideal (see e.g. [10, Proposition 5.1]).

Now we show that any matrix ideal $\mathcal{I}(A)$ generated by a nonnegative regular matrix A is either non-dense or the coideal $\mathcal{I}(A)^+$ is not a P-coideal (and that will finish the proof of the proposition).

Let $A = (a_{i,k})$ be a nonnegative regular matrix and suppose that $\mathcal{I}(A)$ is dense. By Proposition 2.31, we have $\lim_{i,k \rightarrow \infty} a_{i,k} = 0$. By [8, Theorem 6.2] for every set B such that $\overline{d^A}(B) = \alpha > 0$ and each $0 < \beta < \alpha$ there is such $C \subseteq B$ that $\overline{d^A}(C) = \beta$. Therefore, there is a decreasing sequence (B_n) such that $\overline{d^A}(B_n) \leq 1/n$ while $B_n \in \mathcal{I}(A)^+$ for every n . Now, if $B \setminus B_n$ is finite, then $0 \leq d^A(B) \leq \overline{d^A}(B_n) \leq 1/n$. Thus, when $B \setminus B_n$ is finite for all n , $\overline{d^A}(B) = 0$, hence $B \in \mathcal{I}(A)$. \square

In [20, Lemma 11] Laczkovich and Reclaw proved that the ideal convergence generated by the ideal $\text{Exh}(\Phi)$ with a nonpathological submeasure Φ is always weaker than the matrix summability generated by some nonnegative regular matrix in the realm of bounded sequences (i.e. $\text{Exh}(\Phi) \subseteq \mathcal{I}(A)$ for some nonnegative regular matrix A). Below we show that there are analytic P-ideals generated by pathological submeasures that are not contained in any matrix ideal.

Theorem 4.12. *There is an F_σ P-ideal which is not contained in any matrix ideal.*

Proof. In [11, Example 3.6] the authors constructed an F_σ P-ideal which cannot be extended to any summable ideal. Below we show that the same ideal is not contained in any matrix ideal.

First, for the readers convenience, let us recall the definition of this ideal. By [25, Lemma 1.8] for every $n > 0$ there exists a finite set K_n and a family $\mathcal{S}_n \subseteq \mathcal{P}(K_n)$ such that:

- (1) $\forall \omega_1, \dots, \omega_n \in \mathcal{S}_n$ ($\omega_1 \cup \dots \cup \omega_n \neq K_n$);
- (2) if P is a probability distribution on K_n then there exists $\omega \in \mathcal{S}_n$ such that $P(\omega) \geq 1/2$.

Assume that $\{K_n : n \in \mathbb{N}\}$ is a partition of \mathbb{N} into intervals and define $\Phi_n : \mathcal{P}(K_n) \rightarrow [0, \infty)$ by

$$\Phi_n(A) = \min \left\{ |\mathcal{S}| : \mathcal{S} \subseteq \mathcal{S}_n \text{ and } A \subseteq \bigcup \mathcal{S} \right\}$$

for any $A \subseteq K_n$. Notice that $\Phi_n(K_n) > n$. For any $B \subseteq \mathbb{N}$ let

$$\Phi(B) = \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{\Phi_n(B \cap K_n)}{\Phi(K_n)}.$$

Let $\mathcal{I} = \text{Fin}(\Phi)$. Then \mathcal{I} is F_σ ideal (see Theorem 2.16). Since $\text{Fin}(\Phi) = \text{Exh}(\Phi)$, \mathcal{I} is a P-ideal (see Theorem 2.18). It is also clear that $\mathcal{J} \subseteq \mathcal{I}$ when $\mathcal{J} = \text{Fin}(\Psi)$ where $\Psi(B) = \sup\{\Phi_n(B \cap K_n) : n \in \mathbb{N}\}$ for any $B \subseteq \mathbb{N}$. It suffices to show that \mathcal{J} is not contained in any matrix ideal.

Take any nonnegative regular matrix $A = (a_{i,k})$. We show that $\mathcal{J} \not\subseteq \mathcal{I}(A)$. By Lemma 2.28 we can assume that A has only finitely many nonzero elements in each row and the sum of each row is one.

For a given $i \in \mathbb{N}$ let m_i be the smallest natural number such that $a_{i,k} = 0$ for all $k > m_i$. We find such n_1 that $m_1 \in K_{n_1}$.

Let $M_1 = \{n \leq n_1 : \sum_{k \in K_n} a_{1,k} \neq 0\}$. For $n \in M_1$ we define a probability measure P_n on K_n by

$$P_n(B) = \frac{\sum_{k \in B} a_{1,k}}{\sum_{k \in K_n} a_{1,k}}$$

for any $B \subseteq K_n$.

We can now find $C_n \in \mathcal{S}_n$ ($n \in M_1$) such that $P_n(C_n) \geq 1/2$. For $n \leq n_1, n \notin M_1$, we put $C_n = \emptyset$. It is easy to see that $d_1^A(\bigcup_{n \leq n_1} C_n) \geq 1/2$.

Suppose we have defined $i_1 = 1, i_2, \dots, i_N, n_1, \dots, n_N$ and appropriate sets C_n for $n \leq n_N$.

Now we find such an i_{N+1} that for all $i \geq i_{N+1}$, $d_i^A(\bigcup_{n \leq n_N} K_n) < 1/2$. We find such n_{N+1} that $m_{i_{N+1}} \in K_{n_{N+1}}$. Let $M_{N+1} = \{n \in (n_N, n_{N+1}] \cap \mathbb{N} : \sum_{k \in K_n} a_{i_{N+1},k} \neq 0\}$. For $n \in M_{N+1}$ we define a probability measure P_n on K_n by

$$P_n(B) = \frac{\sum_{k \in B} a_{i_{N+1},k}}{\sum_{k \in K_n} a_{i_{N+1},k}}$$

for any $B \subseteq K_n$. We can now find $C_n \in \mathcal{S}_n$ ($n \in M_{N+1}$) such that $P_n(C_n) \geq 1/2$. For $n \notin M_{N+1}, n_N < n \leq n_{N+1}$, we put $C_n = \emptyset$. Since $d_{i_{N+1}}^A(\bigcup_{n_N < n \leq n_{N+1}} K_n) > 1/2$, it follows that $d_{i_{N+1}}^A(\bigcup_{n_N < n \leq n_{N+1}} C_n) > 1/4$.

Let $C = \bigcup_{n \in \mathbb{N}} C_n$. Since every C_n belongs to \mathcal{S}_n or is empty, $\Psi(C) = 1$, thus $C \in \mathcal{J}$. On the other hand, for each i_j we have $d_{i_j}^A(C) \geq d_{i_j}^A(\bigcup_{n_j < n \leq n_{j+1}} C_n) > 1/4$, hence $C \notin \mathcal{I}_A$. Thus, $\mathcal{J} \not\subseteq \mathcal{I}(A)$. \square

5. IDEALS WITH MAZUR'S PROPERTY

Definition 5.1. We say that an ideal \mathcal{I} has the *property (M)* if for every regular nonnegative matrix A such that $\lim^A \upharpoonright m \subseteq \lim^{\mathcal{I}} \upharpoonright m$ there is $F \in \mathcal{I}^*$ such that for every $x \in m \cap c^A$ the subsequence $x \upharpoonright F$ is ordinarily convergent.

Remark. In the ‘‘Scottish Book’’ (e.g. [24, p. 56] or [23, p. 69]) Mazur claims that the ideal \mathcal{I}_d has the property (M). Below we show (Corollary 5.8(3)) that Mazur’s claim about the ideal \mathcal{I}_d was incorrect.

Proposition 5.2. *Let \mathcal{I} be an ideal with the property (M). If \mathcal{J} is isomorphic to \mathcal{I} , then \mathcal{J} has the property (M) as well.*

Proof. Suppose that \mathcal{J} does not have the property (M). Let $B = (b_{i,k})$ be a nonnegative regular matrix such that $\lim^B \upharpoonright m \subseteq \lim^{\mathcal{I}} \upharpoonright m$ and for every $G \in \mathcal{J}^*$ there is $y \in m \cap c^B$ such that $y \upharpoonright G$ is not convergent.

Let $\phi : \mathbb{N} \rightarrow \mathbb{N}$ be a bijection such that $C \in \mathcal{I} \iff \phi[C] \in \mathcal{J}$. Let $a_{i,k} = (b_{i,\phi(k)})$. Then $A = (a_{i,k})$ is nonnegative regular and we claim that A witnesses the lack of the property (M) of \mathcal{I} i.e. we have to show

- (1) $\lim^A \upharpoonright m \subseteq \lim^{\mathcal{I}} \upharpoonright m$;
- (2) for every $F \in \mathcal{I}^*$ there is $x \in m \cap c^A$ such that $x \upharpoonright F$ is not convergent.

(1) Let $x \in m \cap c^A$ with $\lim^A x = L$. Let $y_k = x_{\phi^{-1}(k)}$. Since x is bounded, the series $\sum_k a_{i,k} x_k$ is unconditionally convergent. Then

$$\lim_{i \rightarrow \infty} \sum_k b_{i,k} y_k = \lim_{i \rightarrow \infty} \sum_k a_{i,\phi^{-1}(k)} x_{\phi^{-1}(k)} = \lim_{i \rightarrow \infty} \sum_k a_{i,k} x_k = L.$$

Thus $y \in m \cap c^B$, $\lim^B y = L$. Hence $\lim^{\mathcal{J}} y = L$. Now it is not difficult to see that $\lim^{\mathcal{I}} x = L$.

(2) Let $F \in \mathcal{I}^*$. Since $G = \phi[F] \in \mathcal{J}^*$, there is $y \in m \cap c^B$ such that $y \upharpoonright G$ is not convergent. Let $x_k = y_{\phi(k)}$. A similar argument as in (1) shows that $x \in c^A$. Since $x \upharpoonright F = x \upharpoonright \phi^{-1}[G] = y \upharpoonright G$, $x \upharpoonright F$ is not convergent. \square

Proposition 5.3. *Fin and $\text{Fin} \oplus \mathcal{P}(\mathbb{N})$ have the property (M).*

Proof. Obviously $\mathcal{I} = \text{Fin}$ has the property (M). Let $\mathcal{I} = \text{Fin} \oplus \mathcal{P}(\mathbb{N})$. Let A be a nonnegative regular matrix such that $\lim^A \upharpoonright m \subseteq \lim^{\mathcal{I}} \upharpoonright m$. We claim that $F = 2\mathbb{N}$ is the required set. Let $x \in m \cap c^A$. Then $x \in c^{\mathcal{I}}$. If $L = \lim^{\mathcal{I}} x$ and $\varepsilon > 0$, then $B_\varepsilon = \{n \in \mathbb{N} : |x_n - L| > \varepsilon\} \in \mathcal{I}$. Hence $B_\varepsilon \cap F = B_\varepsilon \cap 2\mathbb{N} \in \text{Fin}$. Thus $\{n \in F : |x_n - L| > \varepsilon\} \in \text{Fin}$, so $x \upharpoonright F$ is ordinarily convergent. \square

Proposition 5.4. *An ideal \mathcal{I} has the property (M) if and only if for every regular nonnegative matrix A such that $\lim^A \upharpoonright m \subseteq \lim^{\mathcal{I}} \upharpoonright m$ there is $F \in \mathcal{I}^*$ such that $\lim^A \upharpoonright m \subseteq \lim^B \upharpoonright m$, where the matrix $B = (b_{i,k})$ is given by $b_{i,e_F(i)} = 1$ for $i \in \mathbb{N}$ and $b_{i,k} = 0$ otherwise.*

Proof. (\Rightarrow) Take any regular nonnegative matrix A such that $\lim^A \upharpoonright m \subseteq \lim^{\mathcal{I}} \upharpoonright m$. Since \mathcal{I} has the property (M), there is $F \in \mathcal{I}^*$ such that for all $x \in m \cap c^A$, $x \upharpoonright F$ is ordinarily convergent to $\lim^{\mathcal{I}} x$. If we take the appropriate matrix B then $B_i(x) = \sum_{k \in \mathbb{N}} b_{i,k} x_k = x_{e_F(i)}$, hence $\lim^B x = \lim(x \upharpoonright F)$. Thus, $\lim^A \upharpoonright m \subseteq \lim^B \upharpoonright m$.

(\Leftarrow) Take any regular nonnegative matrix A such that $\lim^A \upharpoonright m \subseteq \lim^{\mathcal{I}} \upharpoonright m$. We now have a set $F \in \mathcal{I}^*$ such that for every $x \in m \cap c^A$, $\lim^A x = \lim_{i \rightarrow \infty} \sum_{k \in \mathbb{N}} b_{i,k} x_k = \lim_{i \rightarrow \infty} x_{e_F(i)}$. Therefore, $x \upharpoonright F$ is ordinarily convergent. \square

Theorem 5.5. *An ideal \mathcal{I} has the property (M) if and only if the ideal $\mathcal{I} \upharpoonright C$ has the property (M) for every $C \notin \mathcal{I}$.*

Proof. We only need to prove the “only if” part of the proposition. Let $C \notin \mathcal{I}$ and $B = (b_{i,k})$ be a nonnegative regular matrix such that $\lim^B \upharpoonright m \subseteq \lim^{\mathcal{I} \upharpoonright C} \upharpoonright m$. Let $e = e_C$.

We define a matrix $A = (a_{i,k})$ by $a_{e(i),e(k)} = b_{i,k}$ for all $i, k \in \mathbb{N}$, $a_{i,i} = 1$ for $i \in \mathbb{N} \setminus C$ and $a_{i,k} = 0$ otherwise. It is not difficult to see that A is nonnegative and regular. Moreover $\lim^A \upharpoonright m \subseteq \lim^{\mathcal{I}} \upharpoonright m$. Indeed, let $\lim^A x = L$. Since

$$A_{e(i)}(x) = \sum_k a_{e(i),k} x_k = \sum_k a_{e(i),e(k)} x_{e(k)} = \sum_k b_{i,k} x_{e(k)},$$

the sequence $(x_{e(k)})_k$ is B -summable to L . Hence it is $\mathcal{I} \upharpoonright C$ -convergent to L . On the other hand, since $A_i(x) = x_i$ for $i \notin C$, the sequence $(x_i)_{i \in \mathbb{N} \setminus C}$ is ordinarily convergent to L . Since for any $\varepsilon > 0$

$$\{k \in \mathbb{N} : |x_k - L| > \varepsilon\} = e[\{k : |x_{e(k)} - L| > \varepsilon\}] \cup \{k \in \mathbb{N} \setminus C : |x_k - L| > \varepsilon\},$$

x is \mathcal{I} -convergent to L .

Since \mathcal{I} has the property (M), there is $F \in \mathcal{I}^*$ such that for every $x \in m \cap c^A$ the subsequence $x \upharpoonright F$ is convergent. Let $G = e^{-1}[F]$. Then $G \in (\mathcal{I} \upharpoonright C)^*$. Let $y \in m \cap c^B$ with $\lim^B y = L$. We define the sequence x by $x_k = y_{e^{-1}(k)}$ for $k \in C$ and $x_k = L$ for $k \notin C$.

Since $A_{e(i)}(x) = \sum_k b_{i,k} y_k$ and $A_i(x) = x_i = L$ for $i \notin C$, $\lim_{i \rightarrow \infty} A_i(x) = L$. Thus $x \in c^A$. So $x \upharpoonright F$ is convergent. On the other hand, $y \upharpoonright G = y \upharpoonright e^{-1}[F] = x \upharpoonright F$, so $y \upharpoonright G$ is convergent. \square

Corollary 5.6. *If an ideal \mathcal{I} does not have the property (M), and \mathcal{J} is an arbitrary ideal or $\mathcal{J} = \mathcal{P}(\mathbb{N})$, then the ideal $\mathcal{I} \oplus \mathcal{J}$ does not have the property (M).*

Proposition 5.7. *Let \mathcal{I} be an ideal. If there are $C \subseteq \mathbb{N}$ and a regular nonnegative matrix A such that the ideal $\mathcal{I}(A) \upharpoonright C$ is dense and $\mathcal{I}(A) \upharpoonright C \subseteq \mathcal{I} \upharpoonright C$, then \mathcal{I} does not have the property (M).*

Proof. By Theorem 5.5 we only need to show that $\mathcal{I} \upharpoonright C$ does not have the property (M). By Theorem 4.1 there is a regular nonnegative matrix B such that $\mathcal{I}(A) = \mathcal{I}(B)$ and $\lim^{\mathcal{I}(A)} \upharpoonright m = \lim^B \upharpoonright m$. Let $F \in (\mathcal{I} \upharpoonright C)^*$. Since $\mathcal{I}(A) \upharpoonright C$ is dense and F is infinite, there is an infinite $D \subseteq F$ with $D \in \mathcal{I}(A) \upharpoonright C$. Let $D_0, D_1 \subseteq D$ be disjoint infinite sets such that $D = D_0 \cup D_1$. Let $x = \chi_{e_C[D_1]}$. Since $D_1 \subseteq D$, $e_C[D_1] \in \mathcal{I}(A)$. Hence $x \in c^{\mathcal{I}(A)} \cap m = c^B \cap m$. On the other hand $x \upharpoonright F$ is not ordinarily convergent. \square

Corollary 5.8. (1) *If there is a regular nonnegative matrix A such that the ideal $\mathcal{I}(A)$ is dense and $\mathcal{I}(A) \subseteq \mathcal{I}$, then \mathcal{I} does not have the property (M).*
(2) *If A is a nonnegative regular matrix such that the ideal $\mathcal{I}(A)$ is dense then it does not have the property (M).*
(3) *No Erdős-Ulam ideal has the property (M). In particular the ideal \mathcal{I}_d does not have the property (M).*

Proof. (1) Follows from Proposition 5.7. (2) Follows from (1). (3) Follows from (2) and Proposition 2.31. \square

Corollary 5.9. *If a dense ideal \mathcal{I} has the property (M), then $\lim^{\mathcal{I}} \upharpoonright m \neq \lim^A \upharpoonright m$ for any nonnegative regular matrix A .*

Proof. Suppose to the contrary that $\lim^{\mathcal{I}} \upharpoonright m = \lim^A \upharpoonright m$. Then $\mathcal{I}(A) = \mathcal{I}$. By Corollary 5.8, \mathcal{I} does not have the property (M), a contradiction. \square

Question 1. Is the converse of Proposition 5.7 true?

An *ultrafilter* is a filter dual to a maximal ideal. An ultrafilter \mathcal{U} is *selective* if for every partition $\{A_n : n \in \mathbb{N}\}$ of \mathbb{N} into sets not in \mathcal{U} there is $U \in \mathcal{U}$ such that $|U \cap A_n| = 1$ for every $n \in \mathbb{N}$. It is known that consistently (for instance under the Continuum Hypothesis) there are selective ultrafilters (see e.g. [2, Theorem 4.4.5])

Remark. If the converse of Proposition 5.7 is true at least for P-coideals, then it is consistent that there is an ideal with the property (M) which is not isomorphic to Fin nor $\text{Fin} \oplus \mathcal{P}(\mathbb{N})$.

Proof. Let \mathcal{U} be a selective ultrafilter and $\mathcal{I} = \mathcal{U}^*$. Then \mathcal{I}^+ is a P-coideal (see e.g. [2, Theorem 4.5.2]) and of course \mathcal{I} is not isomorphic to Fin nor $\text{Fin} \oplus \mathcal{P}(\mathbb{N})$. Assuming that the converse of Proposition 5.7 is true for P-coideals, we only need to show that \mathcal{I} does not extend any dense ideal $\mathcal{I}(A)$.

In [22, Theorem 9.31], Mathias proved that \mathcal{U} is selective if and only if $\mathcal{U} \cap \mathcal{J} \neq \emptyset$ for every dense analytic ideal \mathcal{J} . Since every ideal $\mathcal{I}(A)$ is analytic (Proposition 2.30), \mathcal{I} cannot extend any dense ideal $\mathcal{I}(A)$. \square

Question 2. Does there exist an ideal with the property (M) which is not isomorphic to Fin nor $\text{Fin} \oplus \mathcal{P}(\mathbb{N})$?

6. INTERSECTION OF MATRIX SUMMABILITY METHODS

Definition 6.1. For an ideal \mathcal{I} on \mathbb{N} we define

$$\mathcal{M}(\mathcal{I}) = \{A : A \text{ is a nonnegative regular matrix such that } \mathcal{I} \subseteq \mathcal{I}(A)\}$$

where $\mathcal{I}(A)$ is a matrix ideal generated by A (see Definition 2.25).

Definition 6.2. An ideal \mathcal{I} has the *property GMV* if

$$\lim^{\mathcal{I}} x = L \iff (\forall A \in \mathcal{M}(\mathcal{I}))(\lim^A x = L)$$

for every bounded sequence $x \in \mathbb{R}^{\mathbb{N}}$.

Proposition 6.3. Let \mathcal{I} be an ideal on \mathbb{N} .

- (1) If $\mathcal{M}(\mathcal{I}) = \emptyset$, then \mathcal{I} does not have the property GMV.
- (2) If $\mathcal{M}(\mathcal{I}) \neq \emptyset$, then in Definition 6.2 the implication “ \Rightarrow ” always holds.

Proof. (1) Let \mathcal{I} be an ideal with $\mathcal{M}(\mathcal{I}) = \emptyset$. Let x be a constant sequence with value 0. Then $(\forall A \in \mathcal{M}(\mathcal{I}))(\lim^A x = 1)$ holds. But $\lim^{\mathcal{I}} x \neq 1$. Thus \mathcal{I} does not have the property GMV.

(2) Let \mathcal{I} be an ideal such that $\mathcal{M}(\mathcal{I}) \neq \emptyset$. Let x be a bounded sequence with $\lim^{\mathcal{I}} x = L$. Let $A \in \mathcal{M}(\mathcal{I})$. Then $\mathcal{I} \subseteq \mathcal{I}(A)$, so $\lim^{\mathcal{I}(A)} x = L$. By Proposition 4.4 we obtain that $\lim^A x = L$. \square

In [14, Theorem 1] Fridy and Miller proved that the ideal \mathcal{I}_d has the property GMV, and they wrote ([14, Theorem 4]) that in a similar manner one can show that every matrix ideal has the property GMV. Below we provide, using the result of Khan and Orhan, a short proof of this fact.

Proposition 6.4 (Fridy-Miller [14, Theorem 4]). *If A is a nonnegative regular matrix, then the matrix ideal $\mathcal{I}(A)$ has the property GMV.*

Proof. Since $A \in \mathcal{M}(\mathcal{I}(A))$, by Proposition 6.3(2) we only need to show the implication “ \Leftarrow ” of Definition 6.2. Let $x \in m$. Since $A \in \mathcal{M}(\mathcal{I}(A))$, $\lim^A x = L$. By Theorem 4.1 there is a nonnegative regular matrix B such that $\mathcal{I}(A) = \mathcal{I}(B)$ and $\lim^{\mathcal{I}(A)} x = \lim^B x$. Then $B \in \mathcal{M}(\mathcal{I}(A))$, so $\lim^B x = L$. Hence $\lim^{\mathcal{I}(A)} x = L$. \square

In [15, Theorem 4.4] Gogola, Maćaj and Visnyai proved that the ideals $\mathcal{I}_{(1/n^\alpha)}$ with $0 < \alpha \leq 1$ (see Definition 4.10) have the property GMV. Proposition 4.11 shows that their theorem does not follow from Proposition 6.4. Moreover they posed a problem ([15, Problem 4.6]) if every ideal has the property GMV. Below (Propositions 6.5 and 6.8) we show the the answer to this problem is negative. Moreover, we prove (Theorem 6.9) a characterization of ideals with the property GMV, and as a corollary (Corollary 6.15) we show that the answer to the problem is also negative for ideals with $\mathcal{M}(\mathcal{I}) \neq \emptyset$.

Proposition 6.5. *For any maximal ideal \mathcal{I} , $\mathcal{M}(\mathcal{I}) = \emptyset$. Hence it does not have the property GMV.*

Proof. Let \mathcal{I} be a maximal ideal. By Proposition 6.3(1) we are done once we show that $\mathcal{M}(\mathcal{I}) = \emptyset$.

Suppose to the contrary that $\mathcal{M}(\mathcal{I}) \neq \emptyset$. Let $A \in \mathcal{M}(\mathcal{I})$. Then $\mathcal{I} \subseteq \mathcal{I}(A)$, and using maximality of \mathcal{I} , we obtain that $\mathcal{I} = \mathcal{I}(A)$. By Proposition 2.30 the ideal $\mathcal{I}(A)$ is Borel, but it is known (see e.g. [2, p. 205]) that any maximal ideal is not Borel, a contradiction. \square

Since any maximal ideal does not have the Baire property (see e.g. [2, p. 205]), Theorem 6.6 together with Corollary 6.7 is a strengthening of Proposition 6.5.

Theorem 6.6. *If $\mathcal{M}(\mathcal{I}) \neq \emptyset$, then \mathcal{I} has the Baire property.*

Proof. Let \mathcal{I} be such that $\mathcal{M}(\mathcal{I}) \neq \emptyset$. If we construct an increasing sequence $(k_n)_n$ such that for every $A \in \mathcal{I}$ there is only finitely many n with $[k_n, k_{n+1}) \cap \mathbb{N} \subseteq A$, then the ideal \mathcal{I} has the Baire property (by Talagrand's characterization of ideals with the Baire property [28], see also [2]).

Let A be a nonnegative regular matrix with $\mathcal{I} \subseteq \mathcal{I}(A)$. By Lemma 2.28 we may assume that rows of A have only finitely many nonzero elements and the sum of every row is 1. It is not difficult to show that then there exist increasing sequences $(k_n)_n$ and $(i_n)_n$ such that $k_1 = 1, i_1 = 1$ and

$$\sum_{k_n \leq k < k_{n+1}} a_{i_n, k} \geq \frac{1}{2} \quad \text{and} \quad \sum_{1 \leq k < k_{n+1}} a_{i_{n+1}, k} < \frac{1}{4}.$$

Let $B \subseteq \mathbb{N}$ be such that there is infinitely many n with $[k_n, k_{n+1}) \cap \mathbb{N} \subseteq B$. Then for these n ,

$$\sum_{k \in B} a_{i_n, k} \geq \sum_{k_n \leq k < k_{n+1}} a_{i_n, k} \geq \frac{1}{2},$$

so $B \notin \mathcal{I}(A)$. Thus $B \notin \mathcal{I}$. \square

Corollary 6.7. *If \mathcal{I} has the property GMV, then \mathcal{I} has the Baire property.*

Proof. Apply Theorem 6.6 and Proposition 6.3(1). \square

The following proposition shows that Theorem 6.6 and Corollary 6.7 do not reverse.

Proposition 6.8. *There exists an F_σ ideal \mathcal{I} for which $\mathcal{M}(\mathcal{I}) = \emptyset$. Hence it does not have the property GMV.*

Proof. For F_σ ideals \mathcal{I} and \mathcal{J} described in Theorem 4.12, $\mathcal{M}(\mathcal{I}) = \mathcal{M}(\mathcal{J}) = \emptyset$. \square

Theorem 6.9. *Let \mathcal{I} be an ideal on \mathbb{N} such that $\mathcal{M}(\mathcal{I}) \neq \emptyset$. \mathcal{I} has the property GMV $\iff \mathcal{I} = \bigcap \{\mathcal{I}(A) : A \in \mathcal{M}(\mathcal{I})\}$.*

Proof. (\implies) Let \mathcal{I} be an ideal with the property GMV. We show that $\mathcal{I} = \bigcap \{\mathcal{I}(A) : A \in \mathcal{M}(\mathcal{I})\}$.

(\subseteq) If $A \in \mathcal{M}(\mathcal{I})$ then $\mathcal{I} \subseteq \mathcal{I}(A)$. Thus $\mathcal{I} \subseteq \bigcap \{\mathcal{I}(A) : A \in \mathcal{M}(\mathcal{I})\}$.

(\supseteq) Let $B \in \bigcap \{\mathcal{I}(A) : A \in \mathcal{M}(\mathcal{I})\}$ and $x = \chi_B$. Then x is A -summable to 0 for every $A \in \mathcal{M}(\mathcal{I})$. Since \mathcal{I} has the property GMV, x is \mathcal{I} -convergent to 0. Thus $B = \{n \in \mathbb{N} : |x_n - 0| > 1/2\} \in \mathcal{I}$.

(\Leftarrow) Since $\mathcal{M}(\mathcal{I}) \neq \emptyset$, by Proposition 6.3(2), we only need to show the implication " \Leftarrow " of Definition 6.2. Let $x \in m$ such that $(\forall A \in \mathcal{M}(\mathcal{I}))(\lim^A x = L)$. Suppose to the contrary that x is not \mathcal{I} -convergent. Let $\varepsilon > 0$ be such that $C = \{n \in \mathbb{N} :$

$|x_n - L| > \varepsilon\} \notin \mathcal{I}$. Then there is $A \in \mathcal{M}(\mathcal{I})$ with $C \notin \mathcal{I}(A)$. By Theorem 4.1 there is a nonnegative regular matrix B such that $\mathcal{I}(A) = \mathcal{I}(B)$ and $\lim^{\mathcal{I}(A)} = \lim^B$. Then $B \in \mathcal{M}(\mathcal{I})$, so $\lim^B x = L$. Thus $\lim^{\mathcal{I}(A)} x = L$ as well. But $C \notin \mathcal{I}(A)$, a contradiction. \square

Lemma 6.10. *Let A be a regular nonnegative matrix. If $2\mathbb{N} \notin \mathcal{I}(A)$, then $\mathcal{I}(A) \upharpoonright 2\mathbb{N}$ is contained in some matrix ideal (i.e. $\mathcal{M}(\mathcal{I}(A) \upharpoonright 2\mathbb{N}) \neq \emptyset$).*

Proof. Since $2\mathbb{N} \notin \mathcal{I}(A)$, the sequence $(d_i^A(2\mathbb{N}))_i$ is not convergent to zero. Let w_i be an increasing sequence of natural numbers such that $\lim_{i \rightarrow \infty} d_{w_i}^A(2\mathbb{N}) = \alpha > 0$.

We define the matrix $B = (b_{i,k})$ by $b_{i,k} = a_{w_i,2k}/\alpha$ for every $i, k \in \mathbb{N}$. It is easy to see that the matrix B is nonnegative. To see that B is regular first observe that for a fixed k we have $b_{i,k} = a_{w_i,2k}/\alpha$, so $(b_{i,k})_i$ tends to zero as a subsequence of the sequence $(a_{i,2k}/\alpha)_i$. Next we show that sums of rows tends to one. Indeed, for a fixed $i \in \mathbb{N}$ we have

$$\sum_{k \in \mathbb{N}} b_{i,k} = \sum_{k \in \mathbb{N}} a_{w_i,2k}/\alpha = \frac{1}{\alpha} \sum_{l \in 2\mathbb{N}} a_{w_i,l} = \frac{1}{\alpha} \cdot d_{w_i}^A(2\mathbb{N}) \xrightarrow{i \rightarrow \infty} \frac{1}{\alpha} \cdot \alpha = 1.$$

Now we show that $\mathcal{I}(A) \upharpoonright 2\mathbb{N} \subseteq \mathcal{I}(B)$. Let $D \in \mathcal{I}(A) \upharpoonright 2\mathbb{N}$. Since $C = \{2d : d \in D\} \in \mathcal{I}(A)$, $\lim_{i \rightarrow \infty} d_i^A(C) = 0$. On the other hand,

$$d_i^B(D) = \sum_{k \in D} b_{i,k} = \sum_{l \in C} b_{i,l/2} = \frac{1}{\alpha} \sum_{l \in C} a_{w_i,l} = \frac{1}{\alpha} d_{w_i}^A(C) \xrightarrow{i \rightarrow \infty} \frac{1}{\alpha} \cdot 0 = 0.$$

\square

Proposition 6.11. *Let \mathcal{I}, \mathcal{J} be ideals on \mathbb{N} . If \mathcal{I} does not have the property GMV, then the ideal $\mathcal{I} \oplus \mathcal{J}$ does not have the property GMV for any ideal \mathcal{J} .*

Proof. If $\mathcal{M}(\mathcal{I} \oplus \mathcal{J}) = \emptyset$, then we are done by Proposition 6.3. Assume that $\mathcal{M}(\mathcal{I} \oplus \mathcal{J}) \neq \emptyset$. Suppose to the contrary that $\mathcal{I} \oplus \mathcal{J}$ has the property GMV. By Theorem 6.9, $\mathcal{I} \oplus \mathcal{J} = \bigcap \{\mathcal{I}(A) : A \in \mathcal{M}(\mathcal{I} \oplus \mathcal{J})\}$, hence $\mathcal{I} \oplus \mathcal{J} \upharpoonright 2\mathbb{N} = \bigcap \{\mathcal{I}(A) \upharpoonright 2\mathbb{N} : A \in \mathcal{M}(\mathcal{I} \oplus \mathcal{J})\}$. Since $\mathcal{I} \oplus \mathcal{J} \upharpoonright 2\mathbb{N} = \mathcal{I} \neq \mathcal{P}(\mathbb{N})$, there is $A \in \mathcal{M}(\mathcal{I} \oplus \mathcal{J})$ with $\mathcal{I}(A) \upharpoonright 2\mathbb{N} \neq \mathcal{P}(\mathbb{N})$.

Take any $A \in \mathcal{M}(\mathcal{I} \oplus \mathcal{J})$ such that $2\mathbb{N} \notin \mathcal{I}(A)$. Now, for every $m \in \mathbb{N}$ such that $\overline{d^A(2\mathbb{N})} > 1/m$ we will construct a matrix $B_m = (b_{i,k})$ such that $\mathcal{I}(A) \subseteq \mathcal{I}(B_m)$, hence $B_m \in \mathcal{M}(\mathcal{I} \oplus \mathcal{J})$. Let $(n_i)_{i \in \mathbb{N}}$ be an increasing sequence of numbers such that $d_{n_i}^A(2\mathbb{N}) > 1/m$ for all natural i and denote $\sum_{k \in \mathbb{N}} a_{n_i,2k}$ by α_i . We define the matrix B_m by $b_{i,2k} = a_{n_i,2k}/\alpha_i$ for all $i, k \in \mathbb{N}$ and $b_{i,k} = 0$ otherwise. It is easy to see that the matrix B_m is regular and $2\mathbb{N} + 1 \in \mathcal{I}(B_m)$. We may also notice that $\mathcal{I}(A) \subseteq \mathcal{I}(B_m)$ since α_i is always greater than $1/m$. Therefore, $B_m \in \mathcal{M}(\mathcal{I} \oplus \mathcal{J})$, thus $\mathcal{I} \subseteq \mathcal{I}(B_m) \upharpoonright 2\mathbb{N}$.

For every matrix B_m we now define the matrix $C_m = (c_{i,k})$ by $c_{i,k} = b_{i,2k}$ for every $i, k \in \mathbb{N}$. Clearly, $C_m \in \mathcal{M}(\mathcal{I})$. We will show that \mathcal{I} is equal to the intersection of all such $\mathcal{I}(C_m)$, which contradicts the assumption that \mathcal{I} does not have the property GMV.

Take any $D \notin \mathcal{I}$. Then $2D \notin \bigcap \{\mathcal{I}(A) \upharpoonright 2\mathbb{N} : A \in \mathcal{M}(\mathcal{I} \oplus \mathcal{J})\}$, thus there is a matrix $A \in \mathcal{M}(\mathcal{I} \oplus \mathcal{J})$ and a natural m such that $\overline{d^A(2D)} > 1/m$. Therefore, for the appropriate matrix B_m we have $\overline{d^{B_m}(2D)} > 1/m$, thus $\overline{d^{C_m}(D)} > 1/m$, hence $D \notin \mathcal{I}(C_m)$. It follows that D does not belong to the intersection of all $\mathcal{I}(C_m)$. \square

Corollary 6.12. *Let \mathcal{I}, \mathcal{J} be ideals on \mathbb{N} . If $\mathcal{M}(\mathcal{I}) = \emptyset$, then the ideal $\mathcal{I} \oplus \mathcal{J}$ does not have the property GMV for any ideal \mathcal{J} .*

Proof. Apply Propositions 6.3 and 6.11. \square

Proposition 6.13. *Let \mathcal{I}, \mathcal{J} be ideals on \mathbb{N} . If $\mathcal{M}(\mathcal{I}) = \emptyset$ and $\mathcal{M}(\mathcal{J}) \neq \emptyset$, then $\mathcal{M}(\mathcal{I} \oplus \mathcal{J}) \neq \emptyset$.*

Proof. Let $A = (a_{i,k}) \in \mathcal{M}(\mathcal{J})$. We define a matrix $B = (b_{i,k})$ by $b_{i,2k-1} = a_{i,k}$ and $b_{i,2k} = 0$ for every $i, k \in \mathbb{N}$. We show that $B \in \mathcal{M}(\mathcal{I} \oplus \mathcal{J})$. Of course B is nonnegative and regular. To finish the proof we show that $\mathcal{I} \oplus \mathcal{J} \subseteq \mathcal{I}(B)$. Let $C \in \mathcal{I} \oplus \mathcal{J}$. Then $D = e_{2\mathbb{N}+1}^{-1}[C \cap (2\mathbb{N}+1)] \in \mathcal{J} \subseteq \mathcal{I}(A)$, so $\lim_{i \rightarrow \infty} d_i^A(D) = 0$. Furthermore,

$$d_i^B(C) = \sum_{k \in C} b_{i,k} = \sum_{k \in C \cap (2\mathbb{N}+1)} b_{i,k} = \sum_{l \in D} b_{i,2l-1} = \sum_{l \in D} a_{i,l} = d_i^A(D) \xrightarrow{i \rightarrow \infty} 0,$$

so $C \in \mathcal{I}(B)$. \square

Corollary 6.14. *Let \mathcal{I}, \mathcal{J} be ideals on \mathbb{N} . If $\mathcal{M}(\mathcal{I}) = \emptyset$ and \mathcal{J} has the property GMV, then the ideal $\mathcal{I} \oplus \mathcal{J}$ does not have the property GMV and $\mathcal{M}(\mathcal{I} \oplus \mathcal{J}) \neq \emptyset$.*

Proof. Apply Corollary 6.12 and Proposition 6.13 and 6.3(1). \square

Corollary 6.15. *There exists an F_σ ideal \mathcal{I} such that $\mathcal{M}(\mathcal{I}) \neq \emptyset$ and \mathcal{I} does not have the property GMV.*

Proof. Apply Corollary 6.14 and Proposition 6.8. \square

All examples of ideals with GMV property we know are Borel.

Question 3. Does there exist a non-Borel ideal with the property GMV? In particular, does the ideal generated by a maximal almost disjoint family have the property GMV?

7. IDEAL STATISTICAL CONVERGENCE

Definition 7.1. Let \mathcal{I} be an ideal on \mathbb{N} . For a set $B \subseteq \mathbb{N}$ we define \mathcal{I} -density of B by

$$d^{\mathcal{I}}(B) = \lim^{\mathcal{I}} d_i(B)$$

provided that the considered \mathcal{I} -limit exists.

Definition 7.2 (Das-Ghosal-Savas [7]). Let \mathcal{I} be an ideal on \mathbb{N} . A sequence $x \in \mathbb{R}^{\mathbb{N}}$ is said to be \mathcal{I} -statistically convergent to L if

$$d^{\mathcal{I}}(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$$

for any $\varepsilon > 0$.

Example 7.3. For the ideal $\mathcal{I} = \text{Fin}$, Fin-density is equal to the asymptotic density (see Definition 2.12), and Fin-statistical convergence is equal to statistical convergence (see Example 2.23).

In [6, Problem 6.1], Das posed a problem to characterize those ideals for which \mathcal{I} -statistical convergence is different from the statistical convergence. Below (Theorem 7.16) we provide a partial solution to the problem. In our solution we utilize the notion of the gap density introduced by Grekos and Volkman [16].

Definition 7.4. The *gap density* of a set $A \subseteq \mathbb{N}$ is given by

$$\lambda(A) = \limsup_{n \rightarrow \infty} \frac{e_A(n+1)}{e_A(n)}.$$

Using the notion of gap density we define two classes of ideals which are connected with the problem of Das.

Definition 7.5. An ideal \mathcal{I} has the *property (D)* if for every M there is $A \in \mathcal{I}^*$ such that $\lambda(A) > M$.

Definition 7.6. An ideal \mathcal{I} has the *property (D $^\infty$)* if there is $A \in \mathcal{I}^*$ such that $\lambda(A) = \infty$. (The sets having infinite gap density are called *thin sets* in [3].)

Obviously (D $^\infty$) implies (D). For P-ideals the reverse implications also holds (Proposition 7.8) but in general it is not true (Example 7.12). The property (D) ((D $^\infty$), resp.) is a necessary (sufficient, resp.) condition for an ideal \mathcal{I} to distinguish \mathcal{I} -statistical convergence and statistical convergence (Propositions 7.14 and 7.15). For P-ideals the property (D) is a characterization of such ideals (Theorem 7.16) – this gives a partial solution to a problem of Das [6, Problem 6.1] (we still lack a characterization for non P-ideals).

Proposition 7.7. *If \mathcal{I} does not have the property (D) ((D $^\infty$) resp.) and $\mathcal{J} \subseteq \mathcal{I}$, then \mathcal{J} does not have the property (D) ((D $^\infty$) resp.) .*

Proof. It is enough to note that in this case $\mathcal{J}^* \subseteq \mathcal{I}^*$. □

Proposition 7.8. *If \mathcal{I} is a P-ideal with the property (D), then it has the property (D $^\infty$).*

Proof. For $k \in \mathbb{N}$, let $A_k \in \mathcal{I}^*$ be such that $\limsup_{n \rightarrow \infty} e_{A_k}(n+1)/e_{A_k}(n) > k$. Without loss of generality we can assume that $A_k \supseteq A_{k+1}$ for all k . Let $A \in \mathcal{I}^*$ be such that $A \setminus A_k$ is finite for every $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Since $A \setminus A_k$ is finite, there is N such that $e_A(n) \in A_k$ for all $n > N$. Then $\lambda(A) \geq \lambda(A_k) > k$. Thus $\lambda(A) = \infty$. □

Example 7.9. Fin , $\mathcal{I}_{(1/n)}$ and \mathcal{I}_d are P-ideals without the property (D).

Proof. The case of the ideal Fin is obvious. For \mathcal{I}_d , it is enough to notice that if $\lambda(A) > 2$, then $\underline{d}(A) \leq 1/2$ (so $A \notin \mathcal{I}_d^*$). The case of $\mathcal{I}_{(1/n)}$ is done by Proposition 7.7 because it is known that $\mathcal{I}_{(1/n)} \subseteq \mathcal{I}_d$. □

Example 7.10. If $A = \{n! : n \in \mathbb{N}\}$, then $\mathcal{I} = \{B \subseteq \mathbb{N} : B \cap A \in \text{Fin}\}$ is a P-ideal with the property (D $^\infty$).

Example 7.11. Let $A = \{n! : n \in \mathbb{N}\}$ and $h : A \rightarrow \mathbb{N}$ be a bijection. For any ideal \mathcal{I} , the ideal $\mathcal{I} \oplus \mathcal{P}(\mathbb{N}) = \{B \subseteq \mathbb{N} : h[B \cap A] \in \mathcal{I}\}$ has the property (D $^\infty$). Moreover, $\mathcal{I} \oplus \mathcal{P}(\mathbb{N})$ is a P-ideal $\iff \mathcal{I}$ is a P-ideal.

Proof. It is enough to note that $A \in (\mathcal{I} \oplus \mathcal{P}(\mathbb{N}))^*$. □

Example 7.12. Let $A_k = \{(2^{2^k})^n : n \in \mathbb{N}\}$ for $k \in \mathbb{N}$. The ideal $\mathcal{I} = \{B \subseteq \mathbb{N} : B \cap A_k \in \text{Fin} \text{ for some } k\}$ (i.e. \mathcal{I} is the ideal generated by the sets $\mathbb{N} \setminus A_k$) has the property (D) and does not have the property (D $^\infty$).

Proof. Since $A_k \in \mathcal{I}^*$ for every $k \in \mathbb{N}$, \mathcal{I} has the property (D). To see that \mathcal{I} does not have the property (D $^\infty$) note that if $A \in \mathcal{I}^*$, then there is k with $(\mathbb{N} \setminus A) \cap A_k \in \text{Fin}$. Thus $\lambda(A) \leq \limsup_{n \rightarrow \infty} (2^{2^k})^{n+1} / (2^{2^k})^n = 2^{2^k} < \infty$. □

Example 7.13. The ideal $\text{Fin} \oplus \mathcal{I}$ does not have the property (D). Moreover $\text{Fin} \oplus \mathcal{I}$ is a P-ideal $\iff \mathcal{I}$ is a P-ideal.

Proof. If $A \in \mathcal{I}^*$, then $A \cap 2\mathbb{N}$ is co-finite. Thus $\lambda(A) \leq \limsup_{n \rightarrow \infty} 2(n+1)/2n = 1 < \infty$. \square

Proposition 7.14. *If there exists an \mathcal{I} -statistically convergent sequence which is not statistically convergent, then \mathcal{I} has the property (D).*

Proof. Suppose that \mathcal{I} does not have the property (D), and let M be such that $\lambda(A) < M$ for all $A \in \mathcal{I}^*$. We show that every \mathcal{I} -statistically convergent sequence is statically convergent. Let $x \in \mathbb{R}^{\mathbb{N}}$ be an \mathcal{I} -statistically convergent sequence with the limit L . For $\varepsilon > 0$, we define $K_\varepsilon = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$. Once we show that $d(K_\varepsilon) = \lim_{n \rightarrow \infty} d_n(K_\varepsilon) = 0$, the proof is completed.

Let $\delta > 0$. Since the sequence $(d_n(K_\varepsilon))_n$ is \mathcal{I} -convergent to 0, the set $A = \{n \in \mathbb{N} : d_n(K_\varepsilon) < \delta/M\} \in \mathcal{I}^*$. Since $\lambda(A) < M$, there is N such that $e_A(n+1)/e_A(n) \leq M$ for all $n > N$. Thus

$$0 \leq d_i(K_\varepsilon) \leq \frac{d_{a_{n+1}}(K_\varepsilon) \cdot e_A(n+1)}{i} \leq \frac{d_{a_{n+1}}(K_\varepsilon) \cdot e_A(n+1)}{e_A(n)} < \frac{\delta}{M} \cdot \frac{e_A(n+1)}{e_A(n)} \leq \delta$$

for $i \in [e_A(n), e_A(n+1))$ and $n > N$. Finally, $d_i(K_\varepsilon) < \delta$ for all but finitely many i , so $d(K_\varepsilon) = 0$. \square

Proposition 7.15. *If \mathcal{I} has the property (D^∞) , then there exists an \mathcal{I} -statistically convergent sequence which is not statistically convergent.*

Proof. Let $A \in \mathcal{I}^*$ such that $\lambda(A) = \infty$. Let k_n be an increasing sequence such that

$$\frac{e_A(k_n + 1)}{e_A(k_n)} > n + 1$$

for all k . We define a sequence $x \in \mathbb{R}^{\mathbb{N}}$ by $x_i = 1$ for $i \in (e_A(k_n), 2e_A(k_n)]$, $n \in \mathbb{N}$ and $x_i = 0$ otherwise. We claim that x is \mathcal{I} -statistically convergent to 0 and is not statistically convergent.

First we show that x is \mathcal{I} -statistically convergent. Let $\varepsilon > 0$. If $k_n + 1 \leq k \leq k_{n+1}$,

$$d_{e_A(k)}(\{i : |x_i - 0| \geq \varepsilon\}) \leq d_{e_A(k_{n+1})}(\{i : |x_i - 0| \geq \varepsilon\}) \leq \frac{2e_A(k_n)}{e_A(k_n + 1)} < \frac{2}{n+1}.$$

Thus the subsequence $(d_a(\{i : |x_i - 0| \geq \varepsilon\}))_{a \in A}$ is ordinarily convergent to 0. Since $A \in \mathcal{I}^*$, so $(d_n(\{i : |x_i - 0| \geq \varepsilon\}))_{n \in \mathbb{N}}$ is \mathcal{I} -convergent to 0 (and this means that x is \mathcal{I} -statistically convergent to 0).

Now we show that x is not statistically convergent. For all $n \in \mathbb{N}$,

$$d_{2e_A(k_n)}(\{i : |x_i - 0| \geq 1/2\}) \geq \frac{e_A(k_n)}{2e_A(k_n)} = \frac{1}{2}.$$

Thus, the sequence $(d_n(\{i : |x_i - 0| \geq \varepsilon\}))_{n \in \mathbb{N}}$ is not convergent to 0 (and this means that x is not statistically convergent to 0). \square

Theorem 7.16. *Let \mathcal{I} be a P-ideal. There exists an \mathcal{I} -statistically convergent sequence which is not statistically convergent if and only if \mathcal{I} has the property (D).*

Proof. Follows from Propositions 7.14, 7.15 and 7.8. \square

Now we present two examples that neither the (D) property nor the (D^∞) property work in the above characterization when \mathcal{I} is not a P-ideal.

Example 7.17. Let \mathcal{I} be the same as in Example 7.12. Then \mathcal{I} does not have the property (D^∞) and there exists an \mathcal{I} -statistically convergent sequence which is not statistically convergent.

Proof. Let $A = \bigcup_{n \in \mathbb{N}} (2^{2^n}, 2 \cdot 2^{2^n}]$ and define the sequence $x \in \mathbb{R}^{\mathbb{N}}$ by $x_i = 1$ for $i \in A$ and $x_i = 0$ otherwise. Obviously, for any $\varepsilon \in (0, 1)$ we have $\{n \in \mathbb{N} : |x_n| > \varepsilon\} = A$ and $\{n \in \mathbb{N} : |x_n - 1| > \varepsilon\} = \mathbb{N} \setminus A$. Clearly, x is not statistically convergent as $d_i(A) \geq 1/2$ for $i = 2 \cdot 2^{2^n}$, $n \in \mathbb{N}$, and $d_i(\mathbb{N} \setminus A) \geq 1/2$ for $i = 2^{2^n}$, $n \geq 2$.

To finish the proof, we will show that x is \mathcal{I} -statistically convergent to 0. Take $\delta > 0$ and let m be the smallest natural number such that $\delta > 1/2^{2^m-1}$. Notice that when $2^{2^n} \in A_k$ for some $k \in \mathbb{N}$ then $k \leq n$ and the smallest element greater than 2^{2^n} belonging to A_k is $2^{2^n} \cdot 2^{2^k}$. Furthermore, for $i = 2^{2^n} \cdot 2^{2^k}$ we get

$$d_i(A) \leq \frac{2 \cdot 2^{2^n}}{2^{2^n} \cdot 2^{2^k}} = \frac{1}{2^{2^k-1}}.$$

Therefore, for every $i \in A_m$ we have $d_i(A) \leq 1/2^{2^m-1} < \delta$. It follows that $\{i \in \mathbb{N} : d_i(A) > \delta\} \subseteq \mathbb{N} \setminus A_m \in \mathcal{I}$, thus x is \mathcal{I} -statistically convergent to 0. \square

Example 7.18. There is an ideal \mathcal{I} with the property (D) such that all \mathcal{I} -statistically convergent sequences are statistically convergent.

Proof. It is not difficult to see that one can construct inductively a sequence of pairwise disjoint intervals I_n^k , $k, n \in \mathbb{N}$, such that $\max I_n^k + 1 = 2^k \min I_n^k$ for every $k, n \in \mathbb{N}$ and for every two different intervals I_n^k, I_m^l , when $\min I_n^k \leq \min I_m^l$ then $\max I_n^k + 1 < \min I_m^l$ (the latter property guarantees that between every two intervals in the sequence there is at least one element not belonging to any interval in the sequence). Let $A_k = \mathbb{N} \setminus \bigcup_{n \in \mathbb{N}, m \leq k} I_n^m$. Notice that $A_{k+1} \subseteq A_k$ for every $k \in \mathbb{N}$ and $\bigcap_{k \in \mathbb{N}} A_k$ is infinite, because $\max I_n^k + 1$ belongs to this intersection for every $n, k \in \mathbb{N}$.

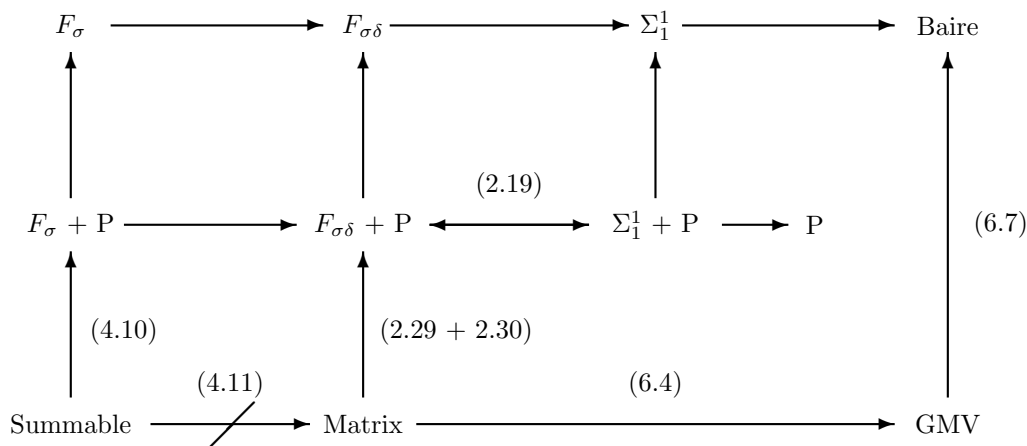
Define $\mathcal{I} = \{B \subseteq \mathbb{N} : B \cap A_k \in \text{Fin for some } k\}$. Then \mathcal{I} has the (D) property, because $A_k \in \mathcal{I}^*$ for every $k \in \mathbb{N}$ and $\lambda(A_k) \geq 2^k$ as $(\max I_n^k + 1)/(\min I_n^k - 1) \geq 2^k$. We will show that every \mathcal{I} -statistically convergent sequence is statistically convergent.

Suppose that there is a sequence $x \in \mathbb{R}^{\mathbb{N}}$ that is \mathcal{I} -statistically convergent to some $l \in \mathbb{R}$, but not statistically convergent. It means that there exist $\varepsilon, \delta > 0$ such that for the set $A = \{n \in \mathbb{N} : |x_n - l| > \varepsilon\}$, the set $B = \{i \in \mathbb{N} : d_i(A) > \delta\}$ is infinite. By the assumption that x is \mathcal{I} -statistically convergent, $B \in \mathcal{I}$, hence there is $k \in \mathbb{N}$ such that $B \cap A_k \in \text{Fin}$. Since B is infinite, there exists $m \leq k$ and infinitely many $n \in \mathbb{N}$ such that $B \cap I_n^m \neq \emptyset$. However, when $i \in B \cap I_n^m$ then $(\max I_n^m + 1)/i \leq 2^m$, thus for $j = \max I_n^m + 1$ we have $d_j(A) \geq d_i(A)/2^m > \delta/2^m$. Therefore, there are infinitely many elements $j \in \bigcap_{k \in \mathbb{N}} A_k$ such that $d_j(A) > \delta/2^m$. It means that $\{j \in \mathbb{N} : d_j(A) > \delta/2^m\} \notin \mathcal{I}$, which contradicts the assumption that x is \mathcal{I} -statistically convergent. \square

8. DIAGRAM

In the following diagram we summarize some relations between classes of ideals we consider in the paper (where “ $A \rightarrow B$ ” means that if an ideal \mathcal{I} has property A

then it also has property B , and a number over the arrow points to an appropriate theorem where the implication is proved).



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