ON SOME QUESTIONS OF DREWNOWSKI AND ŁUCZAK CONCERNING SUBMEASURES ON $\mathbb{N}$

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Abstract. For a given submeasure ϕ on $\mathbb{N}$ a sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$ is called a *ϕ-sequence* if $\phi\left(\bigcup_{n\in\mathbb{N}} F_n\right) = 0$ for every choice of finite sets $F_n \subset A_n$ $(n \in \mathbb{N})$. We show an example of a submeasure ϕ which is not the lim sup of lower semicontinuous submeasures, but $\lim \phi(A_n) = 0$ for any ϕ-sequence $(A_n)_n$. Moreover, we show that it is enough to consider only decreasing sequences $(A_n)_n$ in the above.

We also construct a submeasure on $\mathbb{N}$ which is not the core of a σ-submeasure, but has the property that for every sequence $(A_n)_n$ of subsets of $\mathbb{N}$ if $\lim \phi(A_n) = 0$ then there is a subsequence $(n_k)_k$ and finite sets $E_{n_k} \subset A_{n_k}$ such that $(A_{n_k} \setminus E_{n_k})_k$ is a ϕ-sequence.

These answer questions of Drewnowski and Łuczak from [2].

1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of all natural numbers. A function $\phi : \mathcal{P}(\mathbb{N}) \to [0, +\infty]$ is called a *submeasure* if $\phi(\emptyset) = 0$, ϕ is monotone (i.e. $A \subset B \Rightarrow \phi(A) \leq \phi(B)$) and ϕ is subadditive (i.e. $\phi(A \cup B) \leq \phi(A) + \phi(B)$). We always assume that $\phi(\mathbb{N}) > 0$ for a submeasure ϕ.

A submeasure ϕ is a *σ-submeasure* if it is countably subadditive.

We say that a submeasure ϕ is *dominated* by a submeasure ψ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\phi(A) < \varepsilon$ whenever $\psi(A) < \delta$. We say that submeasures ϕ and ψ are *equivalent* if ϕ is dominated by ψ and ψ is dominated by ϕ.

A submeasure ϕ is *lower semi-continuous* (or *lsc*, for short) if $\phi(A) = \lim_{n\to\infty} \phi(A \cap \{0, 1, \dots, n\})$ for every $A \subset \mathbb{N}$. We say that ϕ is the lim sup *of lsc submeasures* if there is a sequence of lsc submeasures $(\phi_n)_{n\in\mathbb{N}}$, such that $\phi(A) = \limsup_{n\to\infty} \phi_n(A)$ for every $A \subset \mathbb{N}$.

For a submeasure ϕ, by the *core* of ϕ we mean the submeasure $\phi^\bullet$ defined by $\phi^\bullet(A) = \lim_{n\to\infty} \phi(A \setminus \{0, 1, \dots, n\})$.

An *ideal* on $\mathbb{N}$ is a family $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ which is closed under taking subsets and finite unions.

For a submeasure ϕ, we define the ideal $\mathcal{Z}(\phi) = \{A \subset \mathbb{N} : \phi(A) = 0\}$ of ϕ-zero sets. It is not difficult to see that if submeasures ϕ and ψ are equivalent then $\mathcal{Z}(\phi) = \mathcal{Z}(\psi)$.

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For a given submeasure ϕ on $\mathbb{N}$, a sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$ is called a *ϕ-sequence* if $\phi\left(\bigcup_{n\in\mathbb{N}} F_n\right) = 0$ for every choice of finite sets $F_n \subset A_n$ $(n \in \mathbb{N})$.

We say that a submeasure ϕ satisfies *condition*

- (A) if $\lim_{n\to\infty} \phi(A_n) = 0$ for every ϕ-sequence $(A_n)_{n\in\mathbb{N}}$;
- (B) if $\lim_{n\to\infty} \phi(A_n) = 0$ for every decreasing sequence $(A_n)_{n\in\mathbb{N}}$ such that there is no $Z \subset \mathbb{N}$ with $\phi(Z) > 0$ and $Z \setminus A_n$ finite for every $n \in \mathbb{N}$;
- (C) if for every sequence $(A_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \phi(A_n) = 0$ there is a subsequence $(n_k)_{k\in\mathbb{N}}$ and finite sets $E_{n_k} \subset A_{n_k}$ such that $(A_{n_k} \setminus E_{n_k})_{k\in\mathbb{N}}$ is a ϕ-sequence.

In [2], the authors used conditions (A), (B) and (C) to show the equivalence of lsc submeasures ϕ and ψ for which $\mathcal{Z}(\phi^\bullet) = \mathcal{Z}(\psi^\bullet)$. They also showed that (A) implies (B), that the lim sup of a sequence of lsc submeasures has (A), and if a submeasure ϕ is the core of a σ-submeasure then ϕ has (C). They formulated main theorems of their paper using properties (A), (B) and (C). They asked the following questions.

1. Is condition (A) stronger than (B)?
2. Does there exist a submeasure with property (A) which is not equivalent to the $\limsup$ of a sequence of lsc submeasures?
3. Does there exist a submeasure with property (C) which is not equivalent to the core of a σ-submeasure?

We answer these questions in Section 2, 3 and 4, respectively. The answer to question (3) is only partial (it needs some additional set theoretic assumption).

The authors of [2] focus their considerations on nonatomic submeasures (a submeasure ϕ is said to be *nonatomic* if for every $\varepsilon > 0$ there exists a finite partition $A_0, A_1, \ldots, A_{n-1}$ of $\mathbb{N}$ with $\phi(A_i) \le \varepsilon$ for each i). We answer questions (2) and (3) affirmatively, however our examples are not nonatomic. We do not know the answer to those questions if we additionally require that a submeasure is nonatomic.

2. The equivalence of properties (A) and (B)

We say that a submeasure ϕ satisfies *property* (A') if $\lim_{n\to\infty} \phi(A_n) = 0$ for every decreasing ϕ-sequence $(A_n)_{n\in\mathbb{N}}$.

Proposition 1. *Let ϕ be a submeasure. Then the following conditions are equivalent.*

1. *ϕ satisfies (A).*
2. *ϕ satisfies (A').*

Proof. The implication "(1) $\Rightarrow$ (2)" is obvious. The implication (2) $\Rightarrow$ (1) is an immediate consequence of

Fact 2. *For any ϕ-sequence $(A_n)_n$, also the sequence $(B_n)_n$, where $B_n = \bigcup_{i\ge n} A_n$, is a ϕ-sequence.*

To see this fact, let $F_k \subset B_k$, $k \in \mathbb{N}$, be finite sets. For any $k \le n$ let $F'_{k,n} = F_k \cap A_n$. Note that $F_k = \bigcup_{n\ge k} F'_{k,n}$ and, if we denote $E_n = \bigcup_{k\le n} F'_{k,n}$, then E_n is finite and $E_n \subset A_n$. Since $(A_n)_n$ is a ϕ-sequence and $\bigcup_{n\in\mathbb{N}} F_n = \bigcup_{n\in\mathbb{N}} E_n$, $\phi\left(\bigcup_{n\in\mathbb{N}} F_n\right) = \phi\left(\bigcup_{n\in\mathbb{N}} E_n\right) = 0$. $\square$

Theorem 3. *Let ϕ be a submeasure such that $\phi(\{n\}) = 0$ for every $n \in \mathbb{N}$. Then the following conditions are equivalent.*

(1) ϕ *satisfies* (A).
(2) ϕ *satisfies* (A').
(3) ϕ *satisfies* (B).

Proof. (1) $\iff$ (2). By Proposition 1.

(2) $\Rightarrow$ (3). Let $(A_n)_n$ be a decreasing sequence with $\lim_{n\to\infty}\phi(A_n)\neq 0$. Then there are finite sets $F_n\subset A_n$ with $\phi(\bigcup_{n\in\mathbb{N}}F_n)>0$ (by property (A')). Let $Z=\bigcup_{n\in\mathbb{N}}F_n$. Then $Z\setminus A_n\subset F_0\cup\cdots\cup F_{n-1}$ is finite. And $\phi(Z)>0$. This shows that ϕ satisfies (B).

(3) $\Rightarrow$ (2). Suppose, that ϕ does not satisfy (A'). Then there is a decreasing ϕ-sequence $(A_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}\phi(A_n)\neq 0$.

Let $A=\bigcap_{n\in\mathbb{N}}A_n$. We have two cases.

(1) $\phi(A)>0$, or
(2) $\phi(A)=0$.

In the first case, let $A=\{a_n : n\in\mathbb{N}\}$. Let $F_n=\{a_n\}$ for every $n\in\mathbb{N}$. Then $F_n\subset A_n$ and F_n are finite. On the other hand, $\phi(\bigcup_{n\in\mathbb{N}}F_n)=\phi(A)>0$, a contradiction.

Now, consider the second case. Then, by property (B), there is $Z\subset\mathbb{N}$ such that $\phi(Z)>0$ and $Z\setminus A_n$ is finite (so $\phi(Z\setminus A_n)=0$) for every $n\in\mathbb{N}$. Let $X=Z\setminus A$. Let $G_n=X\cap(A_n\setminus A_{n+1})$. Then $G_n\subset A_n$ and G_n are finite. On the other hand, $Z\subset(Z\setminus A_0)\cup\bigcup_{n\in\mathbb{N}}G_n\cup A$, so $\phi(\bigcup_{n\in\mathbb{N}}G_n)>0$, a contradiction. □

Remark. The assumption that ϕ vanishes on singletons is only used in the proof of "(3) $\Rightarrow$ (2)".

Below we will consider properties of submeasures for which the assumption of Theorem 3 does not hold.

Lemma 4. *For a submeasure ϕ, let $S(\phi)=\{n\in\mathbb{N}:\phi(\{n\})=0\}$.*

(1) ϕ *satisfies* (A) $\iff$ $\phi\restriction\mathcal{P}(S(\phi))$ *satisfies* (A).
(2) *If* $\phi(S(\phi))=0$ *then* ϕ *satisfies* (A).
(3) *If* $\mathbb{N}\setminus S(\phi)\neq\emptyset$ *then* ϕ *satisfies* (B).

Proof. (1). ($\Rightarrow$). Obvious.

($\Leftarrow$). Let $(A_n)_n$ be a ϕ-sequence. Clearly, also $(A_n\cap S(\phi))_n$ and $(A_n\setminus S(\phi))_n$ are ϕ-sequences. Since $\phi\restriction\mathcal{P}(S(\phi))$ satisfies (A), $\lim_{n\to\infty}\phi(A_n\cap S(\phi))=0$. Since $\phi(\{k\})>0$ for all $k\notin S(\phi)$, $A_n\setminus S(\phi)=\emptyset$ for every n. In consequence, $\lim_{n\to\infty}\phi(A_n)=0$.

(2). Follows from (1).

(3). Let $(A_n)_n$ be a sequence of subsets of $\mathbb{N}$ such that $\lim_{n\to\infty}\phi(A_n)\neq 0$. Let $x\in\mathbb{N}\setminus S(\phi)$ and $Z=\{x\}$. Then $Z\setminus A_n$ is finite for every $n\in\mathbb{N}$ and $\phi(Z)>0$. Thus ϕ satisfies (B). □

Example 5. There is a submeasure ϕ which satisfies (B) but does not satisfy (A).

Proof. Let $\mathbb{N}\setminus\{0\}=A_0\cup A_1\cup\ldots$ be a partition of $\mathbb{N}\setminus\{0\}$ into infinite pairwise disjoint sets. We define a submeasure ϕ by $\phi(A)=0$ if $0\notin A$ and $\{n\in\mathbb{N}: A\cap A_n$ is infinite$\}$ is finite, and $\phi(A)=1$ otherwise.

Since $\phi(\{0\})=1$, so ϕ satisfies (B) (by Proposition 4).

Now, we show that ϕ does not satisfy (A). Suppose, to the contrary, that ϕ satisfies (A). Let $B_n=\bigcup_{i\geq n}A_i$. Then $\phi(B_n)=1$ for every $n\in\mathbb{N}$. So there are finite $F_n \subset B_n$ with $\phi(\bigcup_{n\in\mathbb{N}} F_n) > 0$. Let $Z = \bigcup_{n\in\mathbb{N}} F_n$. Then $Z \cap A_n \subset F_0 \cup \cdots \cup F_n$ is finite for every $n \in \mathbb{N}$. Thus $\phi(Z) = 0$, a contradiction. □

3. A submeasure with property (A)

If $\mathcal{I}$ is an ideal on $\mathbb{N}$, then $I^+ = \mathcal{P}(\mathbb{N}) \setminus \mathcal{I}$ is called the *coideal associated with* $\mathcal{I}$.

A coideal $\mathcal{I}^+$ is a *P-coideal* if for every decreasing sequence $(A_n)_n$, $A_n \in \mathcal{I}^+$, there is a set $A \in \mathcal{I}^+$ such that $A \setminus A_n$ is finite for every $n \in \mathbb{N}$.

Lemma 6. *Let ϕ be a submeasure which takes only two values* 0 *and* 1, *and* $\phi(\{n\}) = 0$ *for every* $n \in \mathbb{N}$. *Then the following conditions are equivalent.*

1. *ϕ satisfies (A).*
2. *ϕ satisfies (B).*
3. *$\mathcal{Z}(\phi)^+$ is a P-coideal.*

Proof. The equivalence of (1) and (2) follows from Theorem 3.

(1) ⇒ (3). Let $(A_n)_n$ be a decreasing sequence of sets from $\mathcal{Z}(\phi)^+$. Denote $A = \bigcap_{n\in\mathbb{N}} A_n$. If $A \notin \mathcal{Z}(\phi)$, then we are done. So suppose that $A \in \mathcal{Z}(\phi)$. Let $B_n = A_n \setminus A \notin \mathcal{Z}(\phi)$. Since $\lim_{n\to\infty} \phi(B_n) = 1$ so $(B_n)_n$ is not a ϕ-sequence. Thus, there are finite sets $F_n \subset B_n$ with $\phi(\bigcup_{n\in\mathbb{N}} F_n) > 0$. Then $B = \bigcup_{n\in\mathbb{N}} F_n \notin \mathcal{Z}(\phi)$ and $B \setminus A_n \subset F_0 \cup \cdots \cup F_{n-1}$ is finite for every $n \in \mathbb{N}$. Thus, $\mathcal{Z}(\phi)^+$ is a P-coideal.

(3) ⇒ (2). Let $(A_n)_n$ be a decreasing sequence such that $\lim_{n\to\infty} \phi(A_n) \neq 0$. Then $A_n \notin \mathcal{Z}(\phi)$ for every $n \in \mathbb{N}$. Since $\mathcal{Z}(\phi)^+$ is a P-coideal, so there is $Z \notin \mathcal{Z}(\phi)$ with $Z \setminus A_n$ finite for every $n \in \mathbb{N}$. This shows that ϕ satisfies (B). □

Remark. The assumption that ϕ takes only two values is only used in the proof of "(1) ⇒ (3)". And the assumption that ϕ vanishes on singletons is only used in the proof of "(2) ⇒ (1)".

By identifying subsets of $\mathbb{N}$ with their characteristic functions, we equip $\mathcal{P}(\mathbb{N})$ with the Cantor-space topology and therefore we can assign the topological complexity to the ideals of sets of integers. In particular, an ideal $\mathcal{I}$ is an $F_{\sigma\delta}$ (*analytic*) if it is an $F_{\sigma\delta}$ subset of the Cantor space (if it is a continuous image of a G_δ subset of the Cantor space, respectively). Moreover, an lsc submeasure is also lsc (in the topological sense) when viewed as a function on the Cantor cube.

Fact 7 (Folklore)**.** *Let ϕ be the* lim sup *of lsc submeasures. The ideal $\mathcal{Z}(\phi)$ is an $F_{\sigma\delta}$ subset of $\mathcal{P}(\mathbb{N})$.*

Proof. We provide an argument for the completeness.

$$\mathcal{Z}(\phi) = \left\{A \subset \mathbb{N} : \lim_{n\to\infty} \phi_n(A) = 0\right\} = \bigcap_{k\geq 1} \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} \left\{A \subset \mathbb{N} : \phi_n(A) \leq \frac{1}{k}\right\}$$

is $F_{\sigma\delta}$ because the families $\{A \subset \mathbb{N} : \phi_n(A) \leq 1/k\}$ are closed in $\mathcal{P}(\mathbb{N})$. □

For an ideal $\mathcal{I}$, we denote by $\phi_{\mathcal{I}}$ the submeasure defined by $\phi_{\mathcal{I}}(A) = 0$ if $A \in \mathcal{I}$ and $\phi(A) = 1$ otherwise.

If we assume the Continuum Hypothesis, then it is not difficult to show an example of a submeasure with (A) which is not equivalent to the lim sup of a sequence of lsc submeasures. Namely, under the Continuum Hypothesis there exists a maximal ideal $\mathcal{I}$ containing all finite sets such that $\mathcal{I}^+$ is a P-coideal (see e.g. [3]

where the dual notion of p-point ultrafilters is considered). Then, by Lemma 6, $\phi_{\mathcal{I}}$ satisfies (A). Since any maximal ideal containing all finite sets does not have the Baire property (see, e.g. [1, Ch. 4, Sec. 4.1, Thm. 4.1.1]), $\mathcal{I}$ is not $F_{\sigma\delta}$, so $\phi_{\mathcal{I}}$ is not the lim sup of lsc submeasures (by Fact 7).

In Example 8, we construct a submeasure with the above properties without any additional set theoretic assumptions.

A coideal $\mathcal{I}^+$ is a *Q-coideal* if for every $A \in \mathcal{I}^+$ and every partition $A = \bigcup_{n\in\mathbb{N}} F_n$ of A into finite sets there is $S \subset A$ such that $S \in \mathcal{I}^+$ and S intersects each F_n in one point. A coideal $\mathcal{I}^+$ is *selective* if it is a P-coideal and Q-coideal.

We say that a family $\mathcal{A}$ of subsets of $\mathbb{N}$ is *almost disjoint* if $A \cap B$ is finite for every distinct $A, B \in \mathcal{A}$.

Example 8. There is a submeasure with property (A) which is not equivalent to the lim sup of lsc submeasures.

Proof. Let $\mathcal{A}$ be an infinite maximal almost disjoint family of infinite subsets of $\mathbb{N}$ such that $\bigcup \mathcal{A} = \mathbb{N}$. Let $\mathcal{I}_{\mathcal{A}}$ be the ideal generated by $\mathcal{A}$, i.e. the family of all subsets of $\mathbb{N}$ which can be covered by finitely many sets from $\mathcal{A}$ (this ideal was first considered by Mathias in [4].) Let $\phi = \phi_{\mathcal{I}_{\mathcal{A}}}$.

It is known that ${\mathcal{I}_{\mathcal{A}}}^+$ is a P-coideal (see e.g. [5, Sec. 9, Ex. 2, Le. 1]), hence (by Lemma 6) ϕ satisfies (A).

Now, we show that ϕ is not equivalent to the lim sup of lsc submeasures. By Fact 7, it is enough to show that $\mathcal{Z}(\phi) = \mathcal{I}_{\mathcal{A}}$ is not an $F_{\sigma\delta}$ subset of $\mathcal{P}(\mathbb{N})$. But it is known that $\mathcal{I}_{\mathcal{A}}$ is not even an analytic subset of $\mathcal{P}(\mathbb{N})$.

Indeed, since ${\mathcal{I}_{\mathcal{A}}}^+$ is a selective coideal (see e.g. [5, Sec. 9, Ex. 2, Le. 1]), so by [5, Sec. 12, Exercise 4] if $\mathcal{I}_{\mathcal{A}}$ was an analytic ideal on $\mathbb{N}$ then for every $B \notin \mathcal{I}_{\mathcal{A}}$ there would be an infinite $C \subset B$ such that

$$C \cap A \text{ is finite for all } A \in \mathcal{I}_{\mathcal{A}}. \tag{$\star$}$$

But such C is almost disjoint from any $A \in \mathcal{A}$, so by the maximality of the family $\mathcal{A}$ it is an element of $\mathcal{A} \subset \mathcal{I}_{\mathcal{A}}$. Then $C \cap C$ is infinite, a contradiction with $(\star)$. □

Remark. It can be shown that the ideal $\mathcal{I}_{\mathcal{A}}$ (from the above proof) is of the first category (hence has the Baire property).

4. A submeasure with property (C)

An ideal $\mathcal{I}$ is *dense* if for every infinite set $A \subset \mathbb{N}$ there is an infinite set $B \subset A$ with $B \in \mathcal{I}$.

Proposition 9. *Let $\mathcal{I}$ be a dense ideal such that $\mathcal{I}^+$ is a Q-coideal. There is no σ-submeasure ϕ with $\mathcal{I} = \mathcal{Z}(\phi^\bullet)$.*

Proof. Suppose that there is a σ-submeasure ϕ with $\mathcal{I} = \mathcal{Z}(\phi^\bullet)$.

Let $A_\omega = \{n \in \mathbb{N} : \phi(\{n\}) > 1\}$ and $A_k = \left\{n \in \mathbb{N} : \frac{1}{2^{k+1}} < \phi(\{n\}) \leq \frac{1}{2^k}\right\}$ for every $k \in \mathbb{N}$.

We have two cases.

1. There is $k \in \mathbb{N} \cup \{\omega\}$ such that A_k is infinite.
2. The sets A_k are finite for every $k \in \mathbb{N} \cup \{\omega\}$.

In the first case, it is not difficult to check that there is no infinite $B \subset A_k$ with $B \in \mathcal{I}$. But, $\mathcal{I}$ is a dense ideal, a contradiction.

Now, consider the second case. Since $\phi(\mathbb{N}) > 0$ and ϕ is a σ-submeasure, so $\phi\left(A_\omega \cup \bigcup_{k\in\mathbb{N}} A_k\right) > 0$. Let $K = \{k \in \mathbb{N} \cup \{\omega\} : A_k \neq \emptyset\}$. Since $\mathcal{I}^+$ is a Q-coideal, there is $S \in \mathcal{I}^+$ such that $S \cap A_k = \{a_k\}$ for every $k \in K$. On the other hand,

$$0 < \phi^\bullet(S) = \lim_{n\to\infty} \phi\left(S \setminus \{0,1,\ldots,n\}\right) \leq \lim_{n\to\infty} \sum_{k\in K, a_k > n} \phi\left(\{a_k\}\right) =$$

$$\lim_{n\to\infty} \sum_{k\in K\setminus\{\omega\}, a_k>n} \phi\left(\{a_k\}\right) \leq \lim_{n\to\infty} \sum_{k\in K\setminus\{\omega\}, a_k>n} \frac{1}{2^k} = 0,$$

a contradiction. □

An ideal $\mathcal{I}$ is called a *P-ideal* if for every family $\{A_n : n \in \mathbb{N}\} \subset \mathcal{I}$ there is an $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for every $n \in \mathbb{N}$. It is not difficult to check that we can assume that $A_n \subset A_{n+1}$ for every $n \in \mathbb{N}$ in the definition of a P-ideal.

Proposition 10. *If $\mathcal{I}$ is a P-ideal containing all finite sets, then the submeasure $\phi_\mathcal{I}$ satisfies (C).*

Proof. Let $\phi = \phi_\mathcal{I}$. Let $(A_n)_n$ be such that $\lim_{n\to\infty} \phi(A_n) = 0$. Then there is $n_0 \in \mathbb{N}$ such that $A_n \in \mathcal{I}$ for every $n > n_0$. Since $\mathcal{I}$ is a P-ideal, so there is $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for every $n > n_0$.

Let $E_n = A_n \setminus A$ for $n > n_0$. We claim that $(A_n \setminus E_n)_{n>n_0}$ is a ϕ-sequence. Indeed, let $F_n \subset A_n \setminus E_n$ be finite sets. Then

$$\bigcup_{n>n_0} F_n \subset \bigcup_{n>n_0} (A_n \setminus E_n) \subset A \in \mathcal{I},$$

so $\phi\left(\bigcup_{n>n_0} F_n\right) = 0$. □

Theorem 11. *Assume the Continuum Hypothesis. There is a submeasure which satisfies (C) but is not equivalent to the core of a σ-submeasure.*

Proof. Let $\mathcal{I}$ be a maximal ideal containing all finite sets such that $\mathcal{I}^+$ is a selective coideal (there is one under CH, see e.g. [3] where the dual notion of Ramsey ultrafilter is considered). Let $\phi = \phi_\mathcal{I}$.

Since $\mathcal{I}$ is dense and $\mathcal{I}^+$ is a Q-coideal, so there is no σ-submeasure ψ with $\mathcal{I} = \mathcal{Z}(\psi^\bullet)$ (by Proposition 9). Thus, ϕ is not equivalent to the core of a σ-submeasure. On the other hand, $\mathcal{Z}(\phi) = \mathcal{I}$ is a P-ideal, so ϕ satisfies (C) by Proposition 10. □

Remark. The ideal $\mathcal{I}_\mathcal{A}$ (from Example 8) is dense and ${\mathcal{I}_\mathcal{A}}^+$ is a selective coideal. Thus, the submeasure $\phi_{\mathcal{I}_\mathcal{A}}$, is not equivalent to the core of a σ-submeasure (by Proposition 9). It is not difficult to show that $\mathcal{I}_\mathcal{A}$ is not a P-ideal. Thus, by Proposition 12 (below), the submeasure ϕ does not satisfy (C).

Proposition 12. *Let $\mathcal{I}$ be an ideal containing all finite sets such that $\mathcal{I}^+$ is a P-coideal. The submeasure $\phi_\mathcal{I}$ satisfies (C) $\iff$ the ideal $\mathcal{I}$ is a P-ideal.*

Proof. The part "⇐" follows from Proposition 10, so it is enough to show the part "⇒".

Let $\phi = \phi_\mathcal{I}$, and let $(A_n)_{n\in\mathbb{N}}$ be an increasing sequence of sets from $\mathcal{I}$. Since ϕ satisfies (C) and $\lim_{n\to\infty} \phi(A_n) = 0$, so there is a subsequence (n_k) and finite

$E_{n_k} \subset A_{n_k}$ such that $(A_{n_k} \setminus E_{n_k})_k$ is a ϕ-sequence. Let $F_{n_k} = E_{n_0} \cup \cdots \cup E_{n_k}$. Then F_{n_k} are finite for every $k \in \mathbb{N}$ and $(A_{n_k} \setminus F_{n_k})_k$ is also a ϕ-sequence.

Let $A = \bigcup_{k\in\mathbb{N}}(A_{n_k} \setminus F_{n_k})$. If $A \in \mathcal{I}$ then $A_n \setminus A$ is finite for every $n \in \mathbb{N}$, so we are done. Thus, suppose that $A \notin \mathcal{I}$.

Let $B_{n_k} = A \setminus \bigcup_{i<k}(A_{n_i} \setminus F_{n_i})$. Then $B_{n_k} \notin \mathcal{I}$ and $B_{n_k} \supset B_{n_{k+1}}$. Since $\mathcal{I}^+$ is a P-coideal, so there is $B \in \mathcal{I}^+$ such that $B \setminus B_{n_k}$ if finite for every $k \in \mathbb{N}$.

Let $C = B \cap A$. Since $B = (B \cap A) \cup (B \setminus A)$ and $B \setminus A = B \setminus B_{n_0}$ is finite (hence in $\mathcal{I}$), so $C \in \mathcal{I}^+$.

Let $G_{n_k} = C \cap (A_{n_k} \setminus F_{n_k})$. Then $G_{n_k} \subset B \setminus B_{n_k}$ are finite and $G_{n_k} \subset A_{n_k} \setminus F_{n_k}$. Moreover, $\bigcup_{k\in\mathbb{N}} G_{n_k} = C \notin \mathcal{I}$, so $\phi(\bigcup_{k\in\mathbb{N}} G_{n_k}) > 0$. But $(A_{n_k} \setminus F_{n_k})_k$ is a ϕ-sequence, a contradiction. □

The authors do not know if there exists any ZFC example of a submeasure with property (C) which is not equivalent to the core of a σ-submeasure. However, for the submeasure of the form $\phi_{\mathcal{I}}$ it is not very hard to check that one cannot find such an example for $\mathcal{I}$ being a maximal ideal ($\phi_{\mathcal{I}}$ satisfies (C) iff $\mathcal{I}$ is a P-ideal), $\mathcal{I}$ being an F_σ ideal ($\phi_{\mathcal{I}}$ satisfies (C) iff $\mathcal{I}$ is a P-ideal iff $\phi_{\mathcal{I}}$ is equivalent to the core of a σ-measure), or $\mathcal{I}$ being an analytic P-ideal ($\phi_{\mathcal{I}}$ is equivalent to the core of a σ-submeasure for each $\mathcal{I}$.)

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