EXTENDING THE IDEAL OF NOWHERE DENSE SUBSETS OF RATIONALS TO A P-IDEAL

RAFAŁ FILIPÓW, NIKODEM MROŻEK, IRENEUSZ RECŁAW, AND PIOTR SZUCA

Abstract. We show that the ideal of nowhere dense subsets of rationals cannot be extended to an analytic P-ideal, F_σ ideal or maximal P-ideal. We also consider a problem of extendability to a non-meager P-ideals (in particular, to maximal P-ideals).

Our notation and terminology conforms to that used in the most recent set-theoretic literature. The cardinality of the set X is denoted by $|X|$ and $\overline{X}$ means the closure of X. The set of all natural numbers is denoted by ω. An *ideal on* ω is a family of subsets of ω closed under taking finite unions and subsets of its elements. If not explicitly said we assume that an ideal is proper ($\neq \mathcal{P}(\omega)$) and contains all finite sets. If $\mathcal{I}$ is an ideal then by $\mathcal{I}^+$ and $\mathcal{I}^\star$ we mean a coideal and a dual filter to $\mathcal{I}$, i.e. $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ and $\mathcal{I}^\star = \{\omega \setminus A : A \in \mathcal{I}\}$. We can talk about ideals on any other countable set by identifying this set with ω via a fixed bijection. We say that $A \subseteq \omega$ is *almost contained* in $B \subseteq \omega$ ($A \subseteq^\star B$ in symbols) if $A \setminus B$ is finite. An ideal $\mathcal{I}$ is a *P-ideal* if for every family $\{A_n : n \in \omega\}$ of sets from $\mathcal{I}$ there is an $A \in \mathcal{I}$ such that $A_n \subseteq^\star A$ for all n.

By nwd we denote the ideal of nowhere dense subsets of rationals,

$$\mathsf{nwd} = \{A \subseteq \mathbb{Q} : A \text{ is nowhere dense in } \mathbb{R}\}.$$

It is not difficult to observe that nwd is not a P-ideal. Indeed, for every rational $r \in \mathbb{Q}$ let $(q_n^r)_{n\in\omega}$ be a one-to-one sequence of rationals which is convergent to r. Let $A_r = \{q_n^r : n \in \omega\}$ and suppose that nwd is a P-ideal. Then there is a set $A \in \mathsf{nwd}$ which almost contains every set A_r. But that means that the set A has points which are arbitrarily close to any rational, so it is dense in $\mathbb{Q}$—a contradiction.

In [2], Dow proved that it is consistent with ZFC (in particular it holds under the Continuum Hypothesis) that the ideal nwd can be extended to a P-ideal (his construction of this extension is implicit in the proof of [2, Theorem 3.4].) In the same paper, Dow asked a question (see also Dow's Questions [9, Question 12]) whether the ideal nwd can be extended to a P-ideal in ZFC.

In Section 1 we show that nwd cannot be extended in ZFC to an analytic P-ideal. In Section 2 we note that nwd cannot be extended to an F_σ ideal or maximal P-ideal. In Section 3 we show that under the Continuum Hypothesis the ideal nwd can be extended to a non-meager P-ideal. In this section we also give some necessary conditions for ideals which can be extended to a non-meager P-ideal.

Date: May 3, 2013.

2010 *Mathematics Subject Classification.* Primary: 54D35; secondary: 54G10, 54D80, 40A05, 40A35.

Key words and phrases. P-ideal, nowhere dense set, extension, analytic ideal, maximal ideal, meager ideal, ideal convergence.

1. Analytic ideals

By identifying sets of naturals with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the Cantor-space topology and therefore we can assign the topological complexity to the ideals of sets of integers. For instance an ideal $\mathcal{I}$ is *analytic* if it is a continuous image of a G_δ subset of the Cantor space.

A map $\phi\colon \mathcal{P}(\omega) \to [0,\infty]$ is a *submeasure on* ω if $\phi(\emptyset) = 0$ and $\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B)$ for all $A, B \subseteq \omega$. It is *lower semicontinuous* if $\phi(A) = \lim_{n\to\infty} \phi(A \cap \{0,1,\ldots,n\})$ for all $A \subseteq \omega$. For any lower semicontinuous submeasure on ω, let $\|\cdot\|_\phi\colon \mathcal{P}(\omega) \to [0,\infty]$ be the submeasure defined by $\|A\|_\phi = \lim_{n\to\infty} \phi(A \setminus \{0,1,\ldots,n\})$. It is easy to see that $\mathrm{Exh}(\phi) = \left\{A \subseteq \omega \ : \ \|A\|_\phi = 0\right\}$ is an ideal (not necessarily proper) for an arbitrary submeasure ϕ. All analytic P-ideals are characterized by the following theorem of Solecki.

Theorem 1.1 ([7]). *$\mathcal{I}$ is an analytic P-ideal $\iff$ $\mathcal{I} = \mathrm{Exh}(\phi)$ for some lower semicontinuous submeasure ϕ on ω.*

The following theorem will be essential to prove that nwd cannot be extended to an analytic P-ideal.

Theorem 1.2. *Let $\mathcal{I}$ be an analytic P-ideal on $\mathbb{Q}$. For every countable set $C \subseteq \mathbb{R}\setminus\mathbb{Q}$ there exists a set $X \subseteq \mathbb{Q}$ such that $X \notin \mathcal{I}$ and $\left|\overline{X} \cap C\right| \leq 1$.*

Proof. Let ϕ be a lower semicontinuous submeasure with $\mathcal{I} = \mathrm{Exh}(\phi)$. Let $\mathbb{Q} = \{q_i \ : \ i \in \omega\}$ and $C = \{c_i \ : \ i \in \omega\}$. Let

$$\alpha = \lim_n \phi(\mathbb{Q} \setminus \{q_i \ : \ i < n\}) > 0.$$

We have two cases:

(1) for each $n \in \omega$ there exists set V_n open in $\mathbb{R}$ such that $c_n \in V_n$ and $\phi(V_n \cap \mathbb{Q}) < \frac{\alpha}{2^{n+2}}$;
(2) there is $\beta > 0$ and $N \in \omega$ such that for any open set $V \ni c_N$, $\phi(V \cap \mathbb{Q}) > \beta$.

In the first case we can take $X = \mathbb{Q} \setminus \bigcup_{n\in\omega} V_n$. Clearly, $\overline{X} \cap C = \emptyset$. Since ϕ is lower semicontinuous we have

$$\phi\left(\bigcup_{n\in\omega} V_n \cap \mathbb{Q}\right) \leq \sum_{n\in\omega} \phi(V_n \cap \mathbb{Q}) \leq \sum_{n\in\omega} \frac{\alpha}{2^{n+2}} = \frac{\alpha}{2}$$

so

$$\left\|\bigcup_{n\in\omega} V_n \cap \mathbb{Q}\right\|_\phi \leq \frac{\alpha}{2}.$$

On the other hand

$$\alpha = \|\mathbb{Q}\|_\phi \leq \|X\|_\phi + \left\|\bigcup_{n\in\omega} V_n \cap \mathbb{Q}\right\|_\phi \leq \|X\|_\phi + \frac{\alpha}{2}$$

hence $\|X\|_\phi > 0$, so $X \notin \mathcal{I}$. Therefore, we will assume the second case in the sequel. Moreover, we will assume that $N = 0$. We define a sequence $(U_n)_n$ of open subsets of $\mathbb{R}$, and a sequence $(X_n)_n$ of subsets of $\mathbb{Q}$ such that for each natural number $n > 0$:

(1) $c_0 \notin \overline{U_n}$ and $\{c_1, c_2, \ldots, c_n\} \subseteq U_n$;
(2) $\bigcup_{1\leq i\leq n} X_i \cap \overline{U_n} = \emptyset$ and $X_n \cap \bigcup_{1\leq i<n} \overline{U_i} = \emptyset$;

(3) $X_n \subseteq \mathbb{Q} \setminus \{q_i \ : \ i \leq n\}$ and $\phi(X_n) > \beta$.

Let U_1 be an open set such that $c_0 \notin \overline{U_1}$ and $c_1 \in U_1$. Then by lower semicontinuity of ϕ there is a finite set $X_1 \subseteq \mathbb{Q} \setminus (\{q_0\} \cup \overline{U_1})$ such that $\phi(X_1) > \beta$. Inductively, let U_n be an open set with

$$\{c_1, c_2 \ldots, c_n\} \subseteq U_n \text{ and } \left(\{c_0\} \cup \bigcup_{1 \leq i < n} X_i\right) \cap \overline{U_n} = \emptyset.$$

Then there is a finite set

$$X_n \subseteq \mathbb{Q} \setminus \left(\{q_i \ : \ i \leq n\} \cup \bigcup_{i \leq n} \overline{U_i}\right)$$

with $\phi(X_n) > \beta$. Let $X = \bigcup_{n \in \omega} X_n$. Since $\phi(X_n) > \beta$ and $X_n \cap \{q_0, q_1, \ldots, q_n\} = \emptyset$ for each n, $\|X\| \geq \beta$ hence $X \notin \mathcal{I}$. By (2) $X \cap \bigcup_{n \in \omega} U_n = \emptyset$, and so by (1) $\overline{X} \cap C \subseteq \{c_0\}$. □

Remark. Let $c \in \mathbb{R} \setminus \mathbb{Q}$ and $(t_i)_{i \in \omega}$ be a sequence of rationals convergent to c. Let $T = \{t_i \colon i \in \omega\}$ and

$$\mathcal{I} = \{A \subseteq \mathbb{Q} \ : \ A \cap T \text{ is finite}\}.$$

$\mathcal{I}$ is an analytic P-ideal and for every C with $c \in C$, if $X \notin \mathcal{I}$ then $c \in \overline{X} \cap C$. So, in Theorem 1.2 we cannot assert that $\overline{X} \cap C = \emptyset$.

Remark. In Theorem 1.2 we can replace $\mathbb{R}$ with any Hausdorff topological space and $\mathbb{Q}$ with any countable set (not necessarily dense).

Note that if $\mathcal{I}$ is an ideal, $\mathsf{nwd} \subseteq \mathcal{I}$ and $X \notin \mathcal{I}$ then $\overline{X}$ contains an interval, and so $\overline{X} \cap C$ is infinite for every dense set C. So, we have the following corollary.

Corollary 1.3. *There is no analytic P-ideal $\mathcal{I}$ such that* $\mathsf{nwd} \subseteq \mathcal{I}$.

In [2], Dow considered the assertion Mel: "there are disjoint countable dense subsets A, B of $\mathbb{R} \setminus \mathbb{Q}$ and a P-ideal $\mathcal{I}$ on $\mathbb{Q}$ such that for each $X \in \mathcal{I}^+$ both $\overline{X} \cap A$ and $\overline{X} \cap B$ are non-empty". He proved that Mel is consistent with ZFC. The following corollary shows that an ideal $\mathcal{I}$ in Mel cannot be analytic.

Corollary 1.4. *Let $\mathcal{I}$ be an analytic P-ideal on $\mathbb{Q}$. For every disjoint countable dense sets $A, B \subseteq \mathbb{R} \setminus \mathbb{Q}$ there exists a set $X \subseteq \mathbb{Q}$ such that $X \notin \mathcal{I}$ and either $\overline{X} \cap A = \emptyset$ or $\overline{X} \cap B = \emptyset$.*

Proof. Use Theorem 1.2 with $C = A \cup B$. □

2. Maximal ideals and ideals with Bolzano-Weierstrass property

An ideal $\mathcal{I}$ satisfies FinBW (*finite Bolzano-Weierstrass property*) if for any bounded sequence $(x_n)_{n \in \omega}$ of reals there is an $A \in \mathcal{I}^+$ such that $(x_n)_{n \in A}$ is convergent ([3]). By the well-known Bolzano-Weierstrass theorem the ideal of finite subsets of ω satisfies FinBW. By a folklore argument the same is true for every maximal P-ideal. In [3] we have shown that every F_σ ideal satisfies FinBW. In the same paper we have also shown that the ideal nwd does not possess Bolzano-Weierstrass property.

Proposition 2.1 ([3, Prop. 4.1])**.** *If $\mathcal{I}, \mathcal{J}$ are ideals, $\mathcal{I} \subseteq \mathcal{J}$ and $\mathcal{J}$ satisfies FinBW then $\mathcal{I}$ satisfies FinBW.*

Hence, nwd cannot be extended to an ideal with FinBW property. In particular we get the following corollary.

Corollary 2.2. *The ideal* nwd *cannot be extended to any* F_σ *ideal or to a maximal P-ideal.*

Remark. In [10] Zapletal proved that if an analytic ideal $\mathcal{J}$ can be extended to a maximal P-ideal $\mathcal{I}$ then there is an F_σ ideal $\mathcal{K}$ such that $\mathcal{J} \subseteq \mathcal{K} \subseteq \mathcal{I}$.

3. Non-meager ideals

All ideals with Baire property are characterized by the following theorem proved independently by Jalali-Naini [4] and Talagrand [8].

Theorem 3.1 ([4], [8]). *The following conditions are equivalent for an ideal* $\mathcal{I}$ *on* ω.

(1) $\mathcal{I}$ *has the Baire property;*
(2) $\mathcal{I}$ *is meager;*
(3) *there exists* $n_0 < n_1 < \dots$ *such that for every* $A \in \mathcal{I}$

$$\exists_{N\in\omega}\forall_{k>N}\, \{n_k+1,\dots,n_{k+1}\} \not\subseteq A.$$

Theorem 3.2. *Assume the Continuum Hypothesis. There exists non-meager P-ideal* $\mathcal{I}$ *such that* $\mathsf{nwd} \subseteq \mathcal{I}$.

Proof. Fix a bijection $\sigma\colon \omega \to \mathbb{Q}$ identifying $\mathbb{Q}$ with ω. Let $\mathsf{nwd} = \{A_\alpha \ : \ \alpha \in \omega_1\}$, and $\{(n_k^\alpha)_{k\in\omega} \ : \ \alpha \in \omega_1\}$ be a family of all increasing sequences of naturals. Firstly, we will construct a sequence $\{G_\alpha : \alpha \in \omega_1\}$ such that

(1) G_α is dense in $\mathbb{Q}$ for each $\alpha < \omega_1$,
(2) $G_\beta \subseteq^\star G_\alpha$ for $\alpha < \beta < \omega_1$,
(3) $G_\alpha \cap A_\alpha = \emptyset$ for each $\alpha < \omega_1$,
(4) $G_\alpha \cap \sigma[\{n_k^\alpha+1,\dots,n_{k+1}^\alpha\}] = \emptyset$ for infinitely many k.

Let $\{B_n \ : \ n \in \omega\}$ be a basis of the topology on $\mathbb{Q}$. Suppose that we have already constructed sets G_β for $\beta < \alpha$. Let

$$\begin{cases} \{H_n \ : \ n\in\omega\} = \{G_\beta \ : \ \beta<\alpha\} & \text{if } \alpha > 0,\\ H_n = \mathbb{Q} \text{ for each } n\in\omega & \text{if } \alpha = 0.\end{cases}$$

For every $n \in \omega$ we take

$$b_n \in B_n \cap H_0 \cap H_1 \cap \dots \cap H_n \cap \sigma\left[\left\{n^\alpha_{k(n)+1}+1, n^\alpha_{k(n)+1}+2,\dots\right\}\right],$$

where $k(n) = \min\{k\colon \{b_0,b_1,\dots,b_{n-1}\} \subseteq \sigma[\{0,1,\dots,n_k^\alpha\}]\}$ (Recall that since $\{H_n\}_n$ is almost-decreasing, $H_0 \cap H_1 \cap \dots \cap H_n$ is dense in $\mathbb{Q}$.) We put

$$G_\alpha = \{b_n \ : \ n\in\omega\} \setminus A_\alpha.$$

Note that for all $n \in \omega$

$$G_\alpha \cap \sigma\left[\left\{n^\alpha_{k(n)}+1,\dots,n^\alpha_{k(n)+1}\right\}\right] = \emptyset.$$

Define

$$\mathcal{I} = \{A \subseteq \mathbb{Q} \ : \ |A \cap G_\alpha| < \omega \text{ for some } \alpha\}.$$

Since $\mathcal{I} \supset \mathsf{nwd}$ it is enough to show that $\mathcal{I}$ is a non-meager P-ideal. First we show that $\mathcal{I}$ is a P-ideal. Indeed, let $\{C_n : n \in \omega\}$ be a countable family of sets from $\mathcal{I}$. For every $n \in \omega$ there is $\alpha_n < \omega_1$ with $|C_n \cap G_{\alpha_n}| < \omega$. Let $\alpha = \sup_n \alpha_n$. Then $|C_n \cap G_\alpha| < \omega$ for each n, and so $C_n \subseteq^\star \omega \setminus G_\alpha \in \mathcal{I}$ for every $n \in \omega$.

Next, observe that for each increasing sequence $(n_k)_{k\in\omega} = (n_k^\alpha)_{k\in\omega}$ there exists $A = \mathbb{Q} \setminus G_\alpha \in \mathcal{I}$ such that $\sigma[\{n_k^\alpha + 1, \dots, n_{k+1}^\alpha\}] \subseteq A$ for infinitely many k. Thus, by Theorem 3.1, $\mathcal{I}$ cannot be meager. □

Problem 1. The authors do not know if it is possible to prove that nwd can be extended to a meager P-ideal (under CH, for example).

Using notation of Laflamme ([6]), the game $\mathcal{G}(\mathcal{X}, [\omega]^{<\omega}, \mathcal{Y})$ is played by two players I and II as follows: at stage $k < \omega$, I chooses $X_k \in \mathcal{X}$, then II responds with finite $s_k \subseteq X_k$. At the end of the game, II is declared the winner if $\bigcup_k s_k \in \mathcal{Y}$.

Lemma 3.3 ([6], Th. 2.15). *Player I has no winning strategy in $\mathcal{G}(\mathcal{I}^\star, [\omega]^{<\omega}, \mathcal{I}^\star)$ if and only if $\mathcal{I}$ is a non-meager P-ideal.*

Let

$$\mathsf{Fin} \times \mathsf{Fin} = \{A \subseteq \omega \times \omega : (\exists N \in \omega)(\forall n > N)\ \{k : (n,k) \in A\} \text{ is finite}\}.$$

We say that an ideal $\mathcal{I}$ *contains an ideal isomorphic to the ideal* $\mathsf{Fin} \times \mathsf{Fin}$ if there exists a bijection $\sigma\colon \omega \to \omega \times \omega$ such that $\sigma^{-1}[A] \in \mathcal{I}$ whenever $A \in \mathsf{Fin} \times \mathsf{Fin}$.

Lemma 3.4 ([5], Le. 2). *Player I has a winning strategy in $\mathcal{G}(\mathcal{I}^\star, [\omega]^{<\omega}, \mathcal{I}^+)$ if and only if $\mathcal{I}$ contains an ideal isomorphic to* $\mathsf{Fin} \times \mathsf{Fin}$.

For a given ideal $\mathcal{I}$, Debs and Saint Raymond in [1] defined the rank of $\mathcal{I}$. In particular, the rank$(\mathcal{I}) \leq 1$ if and only if $\mathcal{I}$ can be separated from its dual filter by an F_σ set, i.e. if there exists an F_σ set $\mathcal{K}$ such that $\mathcal{I} \subseteq \mathcal{K}$ and $\mathcal{I}^\star \cap \mathcal{K} = \emptyset$.

Lemma 3.5 ([1], Th. 7.5). *If $\mathcal{I}$ is an analytic ideal then* rank$(\mathcal{I}) \leq 1$ *if and only if $\mathcal{I}$ does not contain an ideal isomorphic to* $\mathsf{Fin} \times \mathsf{Fin}$.

Proposition 3.6. *If $\mathcal{I}$ is a P-ideal which is non-meager then every analytic ideal $\mathcal{J} \subseteq \mathcal{I}$ can be separated from its dual filter by an F_σ set, i.e.* rank$(\mathcal{J}) \leq 1$.

Proof. Consider two games $\mathcal{G}_1 = \mathcal{G}(\mathcal{I}^\star, [\omega]^{<\omega}, \mathcal{I}^\star)$ and $\mathcal{G}_2 = \mathcal{G}(\mathcal{I}^\star, [\omega]^{<\omega}, \mathcal{I}^+)$. By Lemma 3.3, I has no winning strategy for $\mathcal{G}_1$. Since the game $\mathcal{G}_2$ is easier for II, I has also no winning strategy for $\mathcal{G}_2$. Thus, by Lemma 3.4, $\mathcal{I}$ does not contain an ideal isomorphic to $\mathsf{Fin} \times \mathsf{Fin}$. Hence $\mathcal{J}$ does not contain an ideal isomorphic to $\mathsf{Fin} \times \mathsf{Fin}$, and thus, by Lemma 3.5, $\mathcal{J}$ can be separated from its dual filter by an F_σ set. □

Recall that nwd is an analytic ideal and rank$(\mathsf{nwd}) = 1$ (see [5]).

References

1. Gabriel Debs and Jean Saint Raymond, *Filter descriptive classes of Borel functions*, Fund. Math. **204** (2009), no. 3, 189–213.
2. Alan Dow, *The space of minimal prime ideals of $C(\beta\mathbf{N} - \mathbf{N})$ is probably not basically disconnected*, General topology and applications (Middletown, CT, 1988), Lecture Notes in Pure and Appl. Math., vol. 123, Dekker, New York, 1990, pp. 81–86.
3. Rafał Filipów, Nikodem Mrożek, Ireneusz Recław, and Piotr Szuca, *Ideal convergence of bounded sequences*, J. Symbolic Logic **72** (2007), no. 2, 501–512.
4. Sayyid Muhhamded Jalali-Naini, *The monotone subsets of Cantor space, filters and descriptive set theory*, PhD Thesis, Oxford, 1976.
5. Miklós Laczkovich and Ireneusz Recław, *Ideal limits of sequences of continuous functions*, Fund. Math. **203** (2009), no. 1, 39–46.

6. Claude Laflamme, *Filter games and combinatorial properties of strategies*, Set theory (Boise, ID, 1992–1994), Contemp. Math., vol. 192, Amer. Math. Soc., Providence, RI, 1996, pp. 51–67.
7. Sławomir Solecki, *Analytic ideals and their applications*, Ann. Pure Appl. Logic **99** (1999), no. 1-3, 51–72.
8. Michel Talagrand, *Compacts de fonctions mesurables et filtres non mesurables*, Studia Math. **67** (1980), no. 1, 13–43.
9. Jan van Mill and George M. Reed (eds.), *Open problems in topology*, North-Holland Publishing Co., Amsterdam, 1990.
10. Jindřich Zapletal, *Preserving P-points in definable forcing*, Fund. Math. **204** (2009), no. 2, 145–154.

Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland
E-mail address: Rafal.Filipow@mat.ug.edu.pl
URL: http://www.mat.ug.edu.pl/~rfilipow

Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland
E-mail address: nmrozek@mat.ug.edu.pl

Institute of Informatics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland

Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland
E-mail address: pszuca@radix.com.pl