IDEAL VERSION OF RAMSEY'S THEOREM

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(Received July 30, 2010)

Abstract. We consider various forms of Ramsey's theorem, the monotone subsequence theorem and the Bolzano-Weierstrass theorem which are connected with ideals of subsets of natural numbers. We characterize ideals with considered properties. We show that, in a sense, Ramsey's theorem, the monotone subsequence theorem and the Bolzano-Weierstrass theorem characterize the same class of ideals. We use our results to show some versions of density Ramsey's theorem (these are similar to generalizations shown in [P. Frankl, R. L. Graham, and V. Rödl: Iterated combinatorial density theorems. J. Combin. Theory Ser. A 54 (1990), 95–111]).

Keywords: Ideal of subsets of natural numbers, Bolzano-Weierstrass theorem, Bolzano-Weierstrass property, ideal convergence, statistical density, statistical convergence, subsequence, monotone sequence, Ramsey's theorem.

MSC 2010: 05D10, 40A35, 05A17, 11B05, 54A20.

1. Introduction

In this paper we consider various forms of Ramsey's theorem, the monotone subsequence theorem and the Bolzano-Weierstrass theorem which are connected with ideals of subsets of natural numbers. Recall

- Ramsey's theorem says that for any finite coloring of two-element subsets of natural numbers there exists an *infinite* homogeneous set.
- The monotone subsequence theorem says that every sequence of reals contains an *infinite* monotone subsequence.
- The Bolzano-Weierstrass theorem says that every bounded sequence of reals contains *infinite* convergent subsequence.

[0]The work of all but the second author was supported by BW grant 5100-5-0204-6. The work of all authors was supported by BW grant 5100-5-0292-7.

One can ask a question how "big" a homogeneous set (a monotone subsequence, a convergent subsequence, respectively) can be? Here, we replace the word "infinite" with "not belonging to the ideal". It is not difficult to show that "ideal" versions of the above theorems do not hold for some ideals, so it is reasonable to characterize ideals for which the "ideal" forms of these theorems hold.

In our consideration we use the notion of ideal convergence extensively. In case of Ramsey's theorem it seems to be natural because well-known argument shows that Ramsey's theorem implies that every sequence of reals contains a monotone subsequence (see e.g. [4]), hence every bounded sequence contains a convergent subsequence. The same way one can easily show that the ideal Ramsey's theorem implies the ideal monotone subsequence theorem which implies the ideal Bolzano-Weierstrass theorem. We find some conditions on the ideals of naturals under which the above implications can be reversed.

The above considerations are discussed in Section 3.

In Section 4 we weaken definitions of "Ramsey" ("Mon", "BW") ideals by requiring that almost all pairs are of the same color (the subsequence is almost monotone, ideal convergent, respectively). We characterize ideals for which this generalizations hold and prove that these properties are equivalent for every ideal.

In Sections 3 and 4 we also describe some methods of constructing ideals with properties we are interested in. We investigate how properties of ideals are connected with properties of the direct sum and the Fubini product of ideals.

In [9] Frankl, Graham and Rödl consider some generalization of Ramsey's theorem for the ideal of sets of statistical density zero. In [8] the authors show how this result can be generalized on the class of all analytic P-ideals (see definitions below). This generalization uses the notion of a submeasure. In Section 5 we show how one can strengthen the submeasure version of Ramsey's theorem for ideals with the Bolzano-Weierstrass property.

2. Preliminaries

The set of natural numbers we denote by the symbol ω. The cardinality of a set X is denoted by $|X|$. We do not distinguish between natural number n and the set $\{0, 1, \ldots, n-1\}$.

An *ideal* on ω is a family $\mathcal{I} \subset \mathcal{P}(\omega)$ (where $\mathcal{P}(\omega)$ denotes the power set of ω) which is closed under taking subsets and finite unions. By Fin we denote the ideal of all finite subsets of ω. If not explicitly said we assume that all considered ideals are proper ($\neq \mathcal{P}(\omega)$) and contain all finite sets. We can talk about ideals on any countable set by identifying this set with ω via a fixed bijection.

If $\mathcal{I}$ is an ideal on ω, then $I^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ and $\mathcal{I}^* = \{\omega \setminus A : A \in \mathcal{I}\}$. $\mathcal{I}^+$ is called the *coideal* and $\mathcal{I}^*$ is called the *dual filter*.

If $\mathcal{I}$ is an ideal on ω and $A \in \mathcal{I}^+$, then the *restriction* of $\mathcal{I}$ to A, denoted by $\mathcal{I} \restriction A$, is the ideal on ω given by $\mathcal{I} \restriction A = \{B \subset \omega : B \cap A \in \mathcal{I}\}$.

An ideal $\mathcal{I}$ is a *P-ideal* if for every sequence $(A_n)_{n\in\omega}$ of sets from $\mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \setminus A \in \mathrm{Fin}$ for all n, i.e. A_n is almost contained in A for each n.

A coideal $\mathcal{I}^+$ is *selective* if for every sequence

$$F_1 \supset F_2 \supset F_3 \supset \cdots$$

of sets from $\mathcal{I}^+$ there is F_∞ in $\mathcal{I}^+$ such that $j \in F_i$ for all $i < j$ and $i, j \in F_\infty$. (The set F_∞ is called a *diagonalization* of $\{F_i\}_i$.)

An ideal $\mathcal{I}$ is called *dense* if every $A \notin \mathcal{I}$ contains an infinite subset that belongs to the ideal.

For two ideals $\mathcal{I}, \mathcal{J}$ define their *direct sum*, $\mathcal{I} \oplus \mathcal{J}$, to be the ideal on $\omega \times 2$ given by

$$A \in \mathcal{I} \oplus \mathcal{J} \text{ iff } \{n \in \omega \ : \ \langle n, 0\rangle \in A\} \in \mathcal{I} \text{ and } \{n \in \omega \ : \ \langle n, 1\rangle \in A\} \in \mathcal{J}.$$

For $A \subset \omega \times \omega$ and $n \in \omega$ by A_n we denote the vertical section of A at n, i.e.

$$A_n = \{m \in \omega \ : \ \langle n, m\rangle \in A\}.$$

We define the *Fubini product*, $\mathcal{I} \times \mathcal{J}$ of $\mathcal{I}$ and $\mathcal{J}$ to be the ideal on $\omega \times \omega$ given by

$$A \in \mathcal{I} \times \mathcal{J} \text{ iff } \{n \in \omega \ : \ A_n \notin \mathcal{J}\} \in \mathcal{I}.$$

In the context of Fubini products we use the "empty ideal" $\{\emptyset\}$ (sometimes we just write $\emptyset$ instead of $\{\emptyset\}$). Nevertheless this ideal does not contain all finite sets, its Fubini product with other ideals usually gives very interesting ideals which contain all finite sets (e.g. $\emptyset \times \mathrm{Fin}$ is an ideal on $\omega \times \omega$ which is made up of sets with finite vertical sections).

By order on $\omega \times 2$ we will mean the lexicographical order i.e. $(n, i) \leq (m, j)$ iff $n < m$ or $m = n \wedge i \leq j$. Whenever we say that a sequence $(x_{n,i})_{(n,i)\in\omega\times 2}$ is monotone we mean monotone with respect to this order.

By order on $\omega \times \omega$ we will mean the order given by $(n, m) \leq (k, l)$ iff $\phi(n, m) \leq \phi(k, l)$, where $\phi : \omega\times\omega \to \omega$ is a bijection given by $\phi(n, m) = (n+m)(n+m+1)/2+m$. Whenever we say that a sequence $(x_{n,m})_{(n,m)\in\omega\times\omega}$ is monotone we mean monotone with respect to this order.

Recall that $A \subset \omega$ is *monochromatic* (or homogeneous) for coloring $c\colon [\omega]^2 \to r$ $(r \in \omega)$ if $c \restriction [A]^2$ is constant.

2.1. **Analytic ideals.** By identifying sets of naturals with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the Cantor-space topology and therefore we can assign the topological complexity to the ideals of sets of integers. In particular, an ideal $\mathcal{I}$ is F_σ (*analytic*) if it is an F_σ subset of the Cantor space (if it is a continuous image of a G_δ subset of the Cantor space, respectively.)

A map $\phi\colon \mathcal{P}(\omega) \to [0,\infty]$ is a *submeasure on* ω if

$$\phi(\emptyset) = 0,$$

$$\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B),$$

for all $A, B \subset \omega$. It is *lower semicontinuous* if for all $A \subset \omega$ we have

$$\phi(A) = \lim_{n\to\infty} \phi(A \cap n).$$

For any lower semicontinuous submeasure on ω, let $\|\cdot\|_\phi\colon \mathcal{P}(\omega) \to [0,\infty]$ be the submeasure defined by

$$\|A\|_\phi = \limsup_{n\to\infty} \phi(A \setminus n) = \lim_{n\to\infty} \phi(A \setminus n),$$

where the second equality follows by the monotonicity of ϕ. Let

$$\mathrm{Exh}(\phi) = \left\{ A \subset \omega \ : \ \|A\|_\phi = 0 \right\},$$

$$\mathrm{Fin}(\phi) = \{ A \subset \omega \ : \ \phi(A) < \infty \}.$$

It is clear that $\mathrm{Exh}(\phi)$ and $\mathrm{Fin}(\phi)$ are ideals (not necessarily proper) for an arbitrary submeasure ϕ.

All analytic P-ideals are characterized by the following theorem of Solecki.

Theorem 2.1 ([14])**.** *The following conditions are equivalent for an ideal $\mathcal{I}$ on ω.*

(1) *$\mathcal{I}$ is an analytic P-ideal;*

(2) *$\mathcal{I} = \mathrm{Exh}(\phi)$ for some lower semicontinuous submeasure ϕ on ω.*

Moreover, for F_σ ideals the following characterization holds.

Theorem 2.2 ([11])**.** *The following conditions are equivalent for an ideal $\mathcal{I}$ on ω.*

(1) *$\mathcal{I}$ is an F_σ ideal;*

(2) *$\mathcal{I} = \mathrm{Fin}(\phi)$ for some lower semicontinuous submeasure ϕ on ω.*

Below we present a few examples of analytic ideals. A lot more examples can be found in Farah's book [6].

Example 2.3. The ideal of sets of density 0

$$\mathcal{I}_d = \left\{ A \subset \omega \;:\; \limsup_{n\to\infty} \frac{|A \cap n|}{n} = 0 \right\},$$

is an analytic P-ideal. If we denote

$$\phi_d(A) = \sup \left\{ \frac{|A \cap n|}{n} \;:\; n \in \omega \right\},$$

then $\overline{d}\,(A) = \|A\|_{\phi_d}$ and $\mathcal{I}_d = \mathrm{Exh}\,(\phi_d)$.

Example 2.4. The ideal

$$\mathcal{I}_{\frac{1}{n}} = \left\{ A \subset \omega : \sum_{n\in A} \frac{1}{n} < \infty \right\}$$

is an F_σ P-ideal. If ϕ is a submeasure defined by the formula

$$\phi(A) = \sum_{n\in A} \frac{1}{n}$$

then $\mathcal{I}_{\frac{1}{n}} = \mathrm{Fin}(\phi)$.

Example 2.5. The ideal of arithmetic progressions free sets

$$\mathcal{W} = \{W \subset \omega \;:\; W \text{ does not contain arithmetic progressions of all lengths}\}$$

is an F_σ ideal which is not a P-ideal. The fact that $\mathcal{W}$ is an ideal follows from the non-trivial theorem of van der Waerden. This ideal was first considered by Kojman in [10].

Example 2.6. The ideal of nowhere dense subsets of rational numbers $\mathbb{Q}$

$$\mathrm{NWD}(\mathbb{Q}) = \{A \subset \mathbb{Q} \;:\; A \text{ is nowhere dense in } \mathbb{R}\}$$

is an analytic ideal which is neither a P-ideal nor F_σ.

2.2. **Bolzano-Weierstrass property.** Let $\mathcal{I}$ be an ideal on ω, $A \subset \omega$ and $(x_n)_{n\in\omega}$ be a sequence of reals. By $(x_n) \upharpoonright A$ we mean a subsequence $(x_n)_{n\in A}$. We say that $(x_n) \upharpoonright A$ is *$\mathcal{I}$-convergent* to $x \in \mathbb{R}$ if $\{n \in A : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$.

An ideal $\mathcal{I}$ on ω is called:

(1) *Fin-BW* if for any bounded sequence $(x_n)_{n\in\omega}$ of reals there is $A \in \mathcal{I}^+$ such that $(x_n) \upharpoonright A$ is convergent;

(2) *h-Fin-BW* if $\mathcal{I} \restriction A$ is Fin-BW for every $A \in \mathcal{I}^+$;
(3) *BW* if for any bounded sequence $(x_n)_{n\in\omega}$ of reals there is $A \in \mathcal{I}^+$ such that $(x_n) \restriction A$ is $\mathcal{I}$-convergent;
(4) *h-BW* if $\mathcal{I} \restriction A$ is BW for every $A \in \mathcal{I}^+$;

By the well-known Bolzano-Weierstrass theorem, the ideal Fin is Fin-BW. For the discussion and applications of these properties see [7], where we examine all BW-like properties. In particular, it is known that the ideal $\mathcal{I}_d$ of sets of density 0 is not BW, and every F_σ ideal is h-Fin-BW.

In the sequel we will use the following characterizations of BW-like properties.

Theorem 2.7 ([7])**.** *Let ϕ be a lower semicontinuous submeasure. The following conditions are equivalent.*

(1) *The ideal $\mathrm{Exh}(\phi)$ is BW.*
(2) *The ideal $\mathrm{Exh}(\phi)$ is Fin-BW.*
(3) *There is $\delta > 0$ such that for any partition $A_1, A_2, \ldots, A_n$ of ω there exists $i \leq n$ with $\|A_i\|_\phi \geq \delta$.*

Next propositions are slight generalizations of Proposition 3.3 from [7].

Proposition 2.8. *Let $r \in \omega$. An ideal $\mathcal{I}$ is Fin-BW if and only if for every family of sets $\{A_s : s \in r^{<\omega}\}$ fulfilling the following conditions*

(S1) $A_\emptyset = \omega$,
(S2) $A_s = A_{s^\frown 0} \cup \cdots \cup A_{s^\frown (r-1)}$,
(S3) $A_{s^\frown i} \cap A_{s^\frown j} = \emptyset$ *for every* $i \neq j$,

there exist $x \in r^\omega$ and $B \subset \omega$, $B \notin \mathcal{I}$ such that $B \setminus A_{x\restriction n}$ is finite for all n.

Proposition 2.9. *Let $r \in \omega$. An ideal $\mathcal{I}$ is BW if and only if for every family of sets $\{A_s : s \in r^{<\omega}\}$ fulfilling the following conditions*

(S1) $A_\emptyset = \omega$,
(S2) $A_s = A_{s^\frown 0} \cup \cdots \cup A_{s^\frown (r-1)}$,
(S3) $A_{s^\frown i} \cap A_{s^\frown j} = \emptyset$ *for every* $i \neq j$,

there exist $x \in r^\omega$ and $B \subset \omega$, $B \notin \mathcal{I}$ such that $B \setminus A_{x\restriction n} \in \mathcal{I}$ for all n.

2.3. **Some points in $\beta\omega$ and its ideal generalizations.** An ultrafilter $\mathcal{U}$ on ω is called:

(1) a *P-point* if for every partition $A_0, A_1, \ldots$ of ω into sets not belonging to $\mathcal{U}$ there exists $S \in \mathcal{U}$ such that $|A_n \cap S| < \omega$ for every $n \in \omega$;
(2) a *Q-point* if for every partition $A_0, A_1, \ldots$ of ω into finite sets there exists $S \in \mathcal{U}$ such that $|A_n \cap S| \leq 1$ for every $n \in \omega$;
(3) *selective* (or *Ramsey*) if for every partition $A_0, A_1, \ldots$ of ω into sets not belonging to $\mathcal{U}$ there exists $S \in \mathcal{U}$ such that $|A_n \cap S| \leq 1$ for every $n \in \omega$.

In [2], authors introduced the following generalizations of the above notions.

An ideal $\mathcal{I}$ on ω is called:

(1) a *local P-point* if for every partition $A_0, A_1, \dots$ of ω into sets from $\mathcal{I}$ there exists $S \in \mathcal{I}^+$ such that $|A_n \cap S| < \omega$ for every $n \in \omega$;
(2) a *local Q-point* if for every partition $A_0, A_1, \dots$ of ω into finite sets there exists $S \in \mathcal{I}^+$ such that $|A_n \cap S| \leq 1$ for every $n \in \omega$;
(3) *locally selective* if for every partition $A_0, A_1, \dots$ of ω into sets from $\mathcal{I}$ there exists $S \in \mathcal{I}^+$ such that $|A_n \cap S| \leq 1$ for every $n \in \omega$;
(4) a *weak P-point* if $\mathcal{I} \restriction A$ is local P-point for every $A \in \mathcal{I}^+$;
(5) a *weak Q-point* if $\mathcal{I} \restriction A$ is local Q-point for every $A \in \mathcal{I}^+$;
(6) *weakly selective* if $\mathcal{I} \restriction A$ is locally selective for every $A \in \mathcal{I}^+$.

3. Ramsey's theorem

3.1. **Local version.** An ideal $\mathcal{I}$ on ω will be called:

(1) *Ramsey* if for every finite coloring of $[\omega]^2$ there exists homogeneous $A \in \mathcal{I}^+$;
(2) *Mon* if for every sequence $(x_n)_{n\in\omega}$ there exists $A \in \mathcal{I}^+$ such that $(x_n) \restriction A$ is monotone.

Using the same argument as in the classical case we can prove the following fact.

Fact 3.1. *Let $\mathcal{I}$ be an ideal on ω. In the following list of conditions on $\mathcal{I}$ each implies the next.*

(1) *$\mathcal{I}$ is Ramsey,*
(2) *$\mathcal{I}$ is Mon,*
(3) *$\mathcal{I}$ is Fin-BW.*

Remark. The ideal $\mathcal{I}_{\frac{1}{n}}$ is Fin-BW but is not Mon. However, we do not know if the first implication of the above fact is reversible.

Proposition 3.2. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. Then $\mathcal{I} \oplus \mathcal{J}$ is Ramsey if and only if $\mathcal{I}$ is Ramsey or $\mathcal{J}$ is Ramsey. The same holds for Mon and Fin-BW properties.*

Proof. Straightforward. □

Proposition 3.3. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. If $\mathcal{I} \neq \emptyset$ and $\mathcal{J} \neq \emptyset$ then $\mathcal{I} \times \mathcal{J}$ is not Fin-BW (hence is not Mon and Ramsey).*

Proof. If $\mathcal{I} \neq \emptyset$ and $\mathcal{J} \neq \emptyset$ then $\text{Fin} \times \text{Fin} \subset \mathcal{I} \times \mathcal{J}$. In [7] we showed that $\text{Fin} \times \text{Fin}$ is not Fin-BW so $\mathcal{I} \times \mathcal{J}$ is also not Fin-BW. □

Proposition 3.4. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω.*

(1) *$\mathcal{I} \times \mathcal{J}$ is Ramsey $\iff$ either $\mathcal{I}$ is Ramsey and $\mathcal{J} = \emptyset$, or $\mathcal{I} = \emptyset$ and $\mathcal{J}$ is Ramsey.*

(2) *If either* $\mathcal{I}$ *is Mon and* $\mathcal{J} = \emptyset$*, or* $\mathcal{I} = \emptyset$ *and* $\mathcal{J}$ *is Mon then* $\mathcal{I} \times \mathcal{J}$ *is Mon.*

(3) *If* $\emptyset \times \mathcal{J}$ *is Mon then* $\mathcal{J}$ *is Mon.*

(4) $\mathcal{I} \times \mathcal{J}$ *is Fin-BW* $\iff$ *either* $\mathcal{I}$ *is Fin-BW and* $\mathcal{J} = \emptyset$*, or* $\mathcal{I} = \emptyset$ *and* $\mathcal{J}$ *is Fin-BW.*

Proof. (1). ($\Rightarrow$). By Proposition 3.3 we know that either $\mathcal{I} = \emptyset$ or $\mathcal{J} = \emptyset$.

Suppose that $\mathcal{I} = \emptyset$, $\mathcal{J} \neq \emptyset$ and $\emptyset \times \mathcal{J}$ is Ramsey. Let $c : [\omega]^2 \to r$. We define $\chi : [\omega \times \omega]^2 \to r$ in the following way. If $k \neq l$ then $\chi(\{(n,k),(m,l)\}) = c(\{k,l\})$. If $k = l$ then $\chi(\{(n,k),(m,l)\}) = 0$. There is $A \notin \emptyset \times \mathcal{J}$ which is homogeneous for χ. So there is $n \in \omega$ with $A_n \notin \mathcal{J}$. It is easy to see that A_n is homogeneous for c. Thus $\mathcal{J}$ is Ramsey ideal.

Similar argument shows that if $\mathcal{I} \neq \emptyset$, $\mathcal{J} = \emptyset$ and $\mathcal{I} \times \emptyset$ is Ramsey then $\mathcal{I}$ is Ramsey.

($\Leftarrow$). Suppose that $\mathcal{I} \neq \emptyset$ is a Ramsey ideal and $\mathcal{J} = \emptyset$. For any coloring $\chi : [\omega \times \omega]^2 \to r$ we define a coloring $c : [\omega]^2 \to r$ by $c(\{n,k\}) = \chi(\{(n,0),(k,0)\})$. Let $A \in \mathcal{I}^+$ be a homogeneous set for c. Then it is easy to see that $A \times \{0\} \notin \mathcal{I} \times \emptyset$ is homogeneous for χ.

Similar argument shows that if $\mathcal{I} = \emptyset$ and $\mathcal{J} \neq \emptyset$ is a Ramsey ideal then $\emptyset \times \mathcal{J}$ is a Ramsey ideal as well.

(2). This can be done by the argument similar to one from the proof of the implication ($\Leftarrow$) from point (1).

(3). Let $(x_n)_{n\in\omega}$ be a sequence of reals. We define a sequence $(y_{n,k})_{(n,k)\in\omega\times\omega}$ by $y_{n,k} = x_k$ for any $n, k \in \omega$. Now there is $B \notin \emptyset \times \mathcal{J}$ such that $(y_{n,k})_{(n,k)\in B}$ is monotone. Since $B \notin \emptyset \times \mathcal{J}$ so there is $n \in \omega$ with $A = \{k \in \omega : (n,k) \in B\} \notin \mathcal{J}$. It is easy to see that $(x_k)_{k\in B}$ is monotone. Thus $\mathcal{J}$ is Mon.

(4). ($\Rightarrow$). By Proposition 3.3 we know that either $\mathcal{I} = \emptyset$ or $\mathcal{J} = \emptyset$.

Let $\mathcal{I} = \emptyset$ and $(x_n)_{n\in\omega}$ be a bounded sequence. Let $y_{n,m} = x_m$ for $n, m \in \omega$. Let $A \notin \emptyset \times \mathcal{J}$ be such that $(y_{n,m})_{(n,m)\in A}$ is convergent. Let $n \in \omega$ be such that $A_n = \{m \in \omega : (n,m) \in A\} \notin \mathcal{J}$. Then the subsequence $(x_m)_{m\in A_n}$ is also convergent.

Now assume that $\mathcal{I} = \emptyset$ and $(x_n)_{n\in\omega}$ is a bounded sequence. We define a sequence $y_{n,m} = x_n$ for $n, m \in \omega$. Let $A \notin \mathcal{I} \times \emptyset$ be such that $(y_{n,m})_{(n,m)\in A}$ is convergent. Let $B \notin \mathcal{I}$ and $f : B \to \omega$ be such that $f \subset A$. Then $y_{(n,f(n))} = x_n$ so the subsequence $(x_n)_{n\in B}$ is convergent.

($\Leftarrow$). It is easy to show that $\mathcal{I} \times \emptyset$ and $\emptyset \times \mathcal{J}$ are Fin-BW whenever $\mathcal{I}$ and $\mathcal{J}$ are Fin-BW. □

Remark. We do not know if the implication (3) of the above proposition can be reversed.

Lemma 3.5. *Let $\mathcal{I}$ be an ideal on ω which is not dense. Then $\mathcal{I}$ is Ramsey (so Mon and Fin-BW as well).*

Proof. Apply ordinary Ramsey's theorem to a set which witnesses that the ideal $\mathcal{I}$ is not dense. □

The following example shows that there are dense Ramsey ideals.

Example 3.6. Let $\mathcal{A}$ be an infinite maximal almost disjoint family of infinite subsets of ω. Let $\mathcal{I}_{\mathcal{A}}$ be the ideal which consists of all subsets of ω which can be covered by finitely many members of $\mathcal{A}$. It is easy to see that $\mathcal{I}_{\mathcal{A}}$ is dense. On the other hand, coideal $\mathcal{I}_{\mathcal{A}}^{+}$ is selective (see [15]), so $\mathcal{I}_{\mathcal{A}}$ is h-Ramsey (by Proposition 3.17) thus the ideal $\mathcal{I}_{\mathcal{A}}$ is Ramsey (see the first remark in Subsection 3.2).

Lemma 3.7. *Let $\mathcal{I}$ be an analytic P-ideal on ω. If $\mathcal{I}$ is a local Q-point then $\mathcal{I}$ is not dense.*

Proof. Let $\mathcal{I} = \mathrm{Exh}\,(\phi)$ be an analytic P-ideal which is a local Q-point.

We claim that there is $\delta > 0$ such that $\{n \in \omega : \phi(\{n\}) \geq \delta\}$ is infinite. Indeed, suppose that for every $\delta > 0$ the set $\{n \in \omega : \phi(\{n\}) \geq \delta\}$ is finite. Let $A_0 = \{i \in \omega : \phi(\{i\}) \geq 1\}$ and $A_{n+1} = \{i \in \omega : \frac{1}{2^{n+1}} \leq \phi(\{i\}) < \frac{1}{2^n}\}$ and $B = \{n \in \omega : \phi(\{n\}) = 0\}$. Then $\{A_n : n \in \omega\} \cup \{\{n\} : n \in B\}$ is a partition of ω into finite sets, hence there is a selector $S \notin \mathcal{I}$ of this family. Since $B \in \mathcal{I}$ so $S \setminus B \notin \mathcal{I}$. On the other hand $\|S \setminus B\|_{\phi} \leq \sum_{i>n} \frac{1}{2^i} \to 0$, so $S \setminus B \in \mathcal{I}$, a contradiction.

Let $A = \{n \in \omega : \phi(\{n\}) \geq \delta\} \notin \mathcal{I}$. Let $B \subset A$. Then $B \notin \mathcal{I} \iff |B| < \omega$. □

Lemma 3.8. *Let $\mathcal{I}$ be an analytic P-ideal on ω which is a local Q-point. Then $\mathcal{I}$ is Ramsey (hence Mon and Fin-BW).*

Proof. It is Ramsey by Lemmas 3.7 and 3.5. □

Lemma 3.9. *Let $\mathcal{I}$ be an ideal on ω. If $\mathcal{I}$ is Mon then it is locally selective.*

Proof. Let $(y_n)_{n\in\omega}$ be a decreasing sequence in the interval $(0,1)$. For each $i \in \omega$ let $\left(x_n^i\right)_{n\in\omega}$ be an increasing sequence in the interval (y_{i+1}, y_i).

Let $\{A_i\}_{i\in\omega}$, $A_i \in \mathcal{I}$ be a partition of ω. Let $i, j \colon \omega \to \omega$ be such that for each $n \in \omega$ $n \in A_{i(n)}$ and n is the $j(n)$'th element of $A_{i(n)}$ (i.e. $j(n) = \left|A_{i(n)} \cap n\right|$.) Define a sequence $(a_n)_{n\in\omega}$ by the formula

$$a_n = x_{j(n)}^{i(n)}.$$

By the Mon property there exists an $A \notin \mathcal{I}$ such that $(a_n)_n \restriction A$ is monotone. Since $(a_n)_n$ is one-to-one, we have only two possibilities:

(1) $(a_n)_n \restriction A$ is strictly increasing, or

(2) $(a_n)_n \restriction A$ is strictly decreasing.

Suppose the first case and take any $k \in A$. Since

$$\{n \in A \ : \ a_n > a_k\} = \left\{n \in A \ : \ a_n > x_{j(k)}^{i(k)}\right\} = \big(A_{i(k)} \cap (n, \infty)\big) \cup \bigcup_{m=0}^{i(k)-1} A_m,$$

we have

$$A \subset \{n \in A \ : \ a_n \leq a_k\} \cup \{n \in A \ : \ a_n > a_k\} \subset (k+1) \cup \bigcup_{m=0}^{i(k)} A_m \in \mathcal{I},$$

which is a contradiction.

Suppose the second case. Since for each $i \in \omega$ $(a_n)_n \restriction A_i$ is increasing, $|A \cap A_i| \leq 1$. By extending A to a selector we get a thesis. □

By Lemma 3.9 every Mon ideal is locally selective. Since every locally selective ideal is a local Q-point, and if an analytic P-ideal is a local Q-point then it is Ramsey (Lemma 3.8), we get the following corollary.

Corollary 3.10. *Let $\mathcal{I}$ be an analytic P-ideal on ω. The following are equivalent:*

(1) *$\mathcal{I}$ is Ramsey,*
(2) *$\mathcal{I}$ is Mon.*

Remark. Every Ramsey ideal is locally selective, but the inverse implication does not hold, for the ideal NWD($\mathbb{Q}$) is locally selective but is not Mon. Moreover, there is no connection between Fin-BW and locally selective ideals. Indeed, NWD($\mathbb{Q}$) is not Fin-BW (see [7]) but it is locally selective. On the other hand, $\mathcal{I}_{\frac{1}{n}}$ is Fin-BW and is not locally selective.

By Corollary 3.10 and the above remark, $\mathcal{I}_{\frac{1}{n}}$ is an example of Fin-BW ideal which is not Mon. The authors do not know any example of Mon ideal which is not Ramsey. We are only able to prove the equivalence of both properties with additional assumptions on the ideal (note that in the proof of Theorem 3.11 we use the assumption that $\mathcal{I}$ is a weak Q-point only to show the implication "(3) $\Rightarrow$ (4)").

Theorem 3.11. *Let $\mathcal{I}$ be an ideal on ω which is a weak Q-point. The following are equivalent.*

(1) *$\mathcal{I}$ is Ramsey;*
(2) *$\mathcal{I}$ is Mon;*
(3) *$\mathcal{I}$ is Fin-BW;*
(4) *For every family $\{A_s : s \in r^{<\omega}\}$ of sets fulfilling conditions (S1), (S2), (S3) from assumption of Proposition 2.8, there is $x \in r^\omega$ and $C \notin \mathcal{I}$ such that C is a diagonalization of $\{A_{x\restriction n}\}_n$, i.e. $m \in A_{x\restriction n}$ for all $n < m$ and $m, n \in C$.*

Proof. (3) $\Rightarrow$ (4). There are possible two cases:

(i) there is an $x \in r^\omega$ such that $C = \bigcap_{n\in\omega} A_{x\restriction n} \notin \mathcal{I}$, or
(ii) for each $x \in r^\omega$, $\bigcap_{n\in\omega} A_{x\restriction n} \in \mathcal{I}$.

In the first case C is a desired diagonalization, so assume the second case.

Since $\mathcal{I}$ is Fin-BW we can apply Proposition 2.8 to get $x \in r^\omega$ and $B \subset \omega$, $B \notin \mathcal{I}$ such that $B \setminus A_{x\restriction n}$ is finite for all n. We can assume that $\bigcap_{n\in\omega} A_{x\restriction n} = \emptyset$.

We can choose a strictly increasing sequence $n_0 < n_1 < n_2 < \ldots$ such that for every k, $B \setminus A_{x\restriction n_k} \subset n_{k+1}$. Using the fact that $\mathcal{I}$ is a weak Q-point, there is $C \subset B$, $C \in \mathcal{I}^+$ which has at most one point in each of the intervals $[n_k, n_{k+1})$. Let $C = \{k_n : n \in \omega\}$ be an increasing enumeration of C. Let $C_0 = \{k_{2n} : n \in \omega\}$ and $C_1 = \{k_{2n+1} : n \in \omega\}$. Then C_0 and C_1 are diagonalizations of the sequence $(A_{x\restriction n})_{n\in\omega}$ and one of them has to be in $\mathcal{I}^+$.

(4) $\Rightarrow$ (1). We define a family $\{A_s : s \in r^{<\omega}\}$ of subsets of ω.

- $A_\emptyset = \omega$.
- $A_{s^\frown i} = \{n \in A_s : c(|s^\frown i|, n) = i\}$, $i = 0, 1, \ldots, r-1$.

Let $x \in r^\omega$ and $C \notin \mathcal{I}$ be such that $m \in A_{x\restriction n}$ for all $n < m$ and $m, n \in C$.

For every $i \in r$ let

$$C_i = \{n \in C \;:\; x(n-1) = i\}.$$

Take $j < r$ with $C_j \notin \mathcal{I}$. Since C is a diagonalization of $\{A_{x\restriction n}\}_n$, for each $m, n \in C_j$ $(n < m)$, $m \in A_{x\restriction n}$, so $c(n, m) = j$. Thus C_j is homogeneous. $\square$

Remark. There are weak Q-points which are Ramsey (e.g. all ideals which are not dense, the ideal from Example 3.6). But there also are weak Q-points which are not Ramsey (e.g. NWD($\mathbb{Q}$)).

Remark.
(1) If we consider the class of maximal ideals then by [3, Thm 4.9] Mon and Ramsey property are equivalent to the fact that $\mathcal{I}^*$ is a Ramsey ultrafilter.
(2) By [3, Thm 4.7] in the class of maximal ideals, $\mathcal{I}$ is Fin-BW if and only if $\mathcal{I}^*$ is a P-point.
(3) It is consistent with ZFC that there are maximal ideals which are Ramsey (hence Mon and Fin-BW) — for, under CH, there are selective ultrafilters.
(4) It is consistent with ZFC that there is a maximal Fin-BW ideal which is not Ramsey — for, under CH, there is a P-point ultrafilter which is not a Ramsey ultrafilter.
(5) It is consistent with ZFC that there is no maximal ideals which are Fin-BW (hence Ramsey and Mon) — for it is consistent that there are no P-point ultrafilters (see [13]).

Remark. In the working version of this paper we have asked some questions about multicolor versions of Ramsey's theorem. We have asked, among others, if for every ideal $\mathcal{I}$ which is Ramsey and for every coloring $\chi : [\omega]^2 \to \{0,1,2\}$ there is a homogeneous set $A \notin \mathcal{I}$ for χ? This question was solved in the negative by Alcántara in [1, Th. 2.7.6]. He also gives some criterion for Mon (and h-Mon, see definitions below) ideals.

3.2. **Global version.** An ideal $\mathcal{I}$ on ω will be called:

(1) *h-Ramsey* if $\mathcal{I} \restriction A$ is Ramsey for every $A \in \mathcal{I}^+$;
(2) *h-Mon* if $\mathcal{I} \restriction A$ is Mon for every $A \in \mathcal{I}^+$.

Remark. It is easy to see that if an ideal $\mathcal{I}$ is h-Ramsey (h-Mon, h-Fin-BW) then $\mathcal{I}$ is Ramsey (Mon, Fin-BW resp.). However the reverse implications do not hold. Indeed, take an ideal $\mathcal{I}$ which is Ramsey (e.g. Fin) and an ideal $\mathcal{J}$ which is not Fin-BW (e.g. $\mathcal{I}_d$), then $\mathcal{I} \oplus \mathcal{J}$ is Ramsey (by Proposition 3.2) but is not h-Fin-BW.

Remark. h-Ramsey ideals were already considered by Farah in [5]. He considered in his paper coideals instead of ideals and called them Ramsey coideals. Recently, h-Ramsey ideals have been also considered in [12]. Authors call them Ramsey superfilters and apply them to topological selection principles.

One can easily see that the following fact holds.

Fact 3.12. *Let $\mathcal{I}$ be an ideal on ω. In the following list of conditions on $\mathcal{I}$ each implies the next.*

(1) *$\mathcal{I}$ is h-Ramsey,*
(2) *$\mathcal{I}$ is h-Mon,*
(3) *$\mathcal{I}$ is h-Fin-BW.*

Remark. Here, as in the case of local version, the second implication of the above proposition cannot be reversed. The ideal $\mathcal{I}_{\frac{1}{n}}$ is h-Fin-BW (see [7]) but is not h-Mon. Whereas, the second implication is in fact equivalence as is shown in Theorem 3.16.

Proposition 3.13. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. Then $\mathcal{I} \oplus \mathcal{J}$ is h-Ramsey if and only if $\mathcal{I}$ and $\mathcal{J}$ are h-Ramsey. The same holds for h-Mon and h-Fin-BW properties.*

Proof. Straightforward. □

Proposition 3.14. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω.*

(1) *$\mathcal{I} \times \mathcal{J}$ is h-Ramsey $\iff$ either $\mathcal{I}$ is h-Ramsey and $\mathcal{J} = \emptyset$, or $\mathcal{I} = \emptyset$ and $\mathcal{J}$ is h-Ramsey.*
(2) *$\mathcal{I} \times \mathcal{J}$ is h-Fin-BW $\iff$ either $\mathcal{I}$ is h-Fin-BW and $\mathcal{J} = \emptyset$, or $\mathcal{I} = \emptyset$ and $\mathcal{J}$ is h-Fin-BW.*

Proof. The proof is very similar to the proof of Proposition 3.4. □

Remark. The above proposition also holds for h-Mon property because it is equivalent to h-Ramsey property (Theorem 3.16).

The same argument as in the proof of Lemma 3.9 gives us the following.

Lemma 3.15. *Let $\mathcal{I}$ be an ideal on ω. If $\mathcal{I}$ is h-Mon then it is weakly selective.*

Remark. The ideal $\mathrm{NWD}(\mathbb{Q})$ is weakly selective but is not h-Mon. Moreover, there is no connection between h-Fin-BW and weakly selective ideals. Indeed, $\mathrm{NWD}(\mathbb{Q})$ is not h-Fin-BW (see [7]) but it is weakly selective. On the other hand, $\mathcal{I}_{\frac{1}{n}}$ is h-Fin-BW and is not weakly selective.

And below is a global version of Theorem 3.11.

Theorem 3.16. *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent.*

(1) *$\mathcal{I}$ is h-Ramsey;*
(2) *$\mathcal{I}$ is h-Mon;*
(3) *$\mathcal{I}$ is h-Fin-BW and a weak Q-point;*
(4) *For every $A \in \mathcal{I}^+$ and every family $\{A_s : s \in r^{<\omega}\}$ of sets fulfilling the condition $A_\emptyset = A$ and conditions (S2), (S3) from assumption of Proposition 2.8, there is $x \in r^\omega$ and $C \notin \mathcal{I}$ such that C is a diagonalization of $\{A_{x\restriction n}\}_n$, i.e. $m \in A_{x\restriction n}$ for all $n < m$ and $m, n \in C$.*

Proof. (2) ⇒ (3). By Lemma 3.15 every h-Mon ideal is weakly selective hence it is a weak Q-point.

(3) ⇒ (4) and (4) ⇒ (1) — like in the local case. □

Remark. There are weak Q-points which are h-Ramsey (e.g. Fin and the ideal from Example 3.6). But there also are weak Q-points which are not h-Ramsey (e.g. $\mathrm{NWD}(\mathbb{Q})$).

Next proposition and remark were already noted by Farah in [5].

Proposition 3.17 ([5])**.** *If $\mathcal{I}^+$ is a selective coideal then $\mathcal{I}$ is h-Ramsey.*

Proof. Using selectivity of $\mathcal{I}^+$ it is easy to show that condition (4) of Theorem 3.16 holds. □

Remark. The above proposition cannot be reversed. The ideal $\mathcal{I} = \emptyset \times \mathrm{Fin}$ is h-Ramsey but $\mathcal{I}^+$ is not selective.

3.3. **Filter version.** One could consider some stronger property than Ramsey, requiring that homogeneous set is from dual filter $\mathcal{I}^*$. Analogously, one could consider "filter" version of Mon and Fin-BW properties. However, as we show below, these properties are, in a sense, too strong.

Proposition 3.18. *Let $\mathcal{I}$ be an ideal on ω. If for every bounded sequence $(x_n)_{n\in\omega}$ of reals there is $A \in \mathcal{I}^*$ such that $(x_n) \restriction A$ is convergent then $\mathcal{I}$ is a maximal ideal.*

Proof. Suppose that $\mathcal{I}$ is an ideal with the required property which is not maximal. Let $A \subset \omega$ be such that $A \notin \mathcal{I}$ and $\omega \setminus A \notin \mathcal{I}$. Let $(x_n)_{n\in\omega}$ be a sequence such that $x_n = 0$ if $n \in A$ and $x_n = 1$ if $n \in \omega \setminus A$. Let $C \in \mathcal{I}^*$ such that $(x_n) \restriction C$ is convergent. Then either $C \cap A$ is finite or $C \setminus A$ is finite, a contradiction. □

Corollary 3.19. *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent.*

1. *For every bounded sequence $(x_n)_{n\in\omega}$ of reals there is $A \in \mathcal{I}^*$ such that $(x_n) \restriction A$ is convergent;*
2. *$\mathcal{I}^*$ is a P-point.*

Corollary 3.20. *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent.*

1. *For every finite coloring $[\omega]^2$ there exists homogeneous $A \in \mathcal{I}^*$;*
2. *for every sequence $(x_n)_{n\in\omega}$ there exists $A \in \mathcal{I}^*$ such that $(x_n) \restriction A$ is monotone;*
3. *$\mathcal{I}^*$ is selective.*

4. Ideal version of Ramsey's theorem

Let $\mathcal{I}$ be an ideal on ω and $r \in \omega$. We say that $A \subset \omega$ is *$\mathcal{I}$-monochromatic* (or $\mathcal{I}$-homogeneous) for coloring $c\colon [\omega]^2 \to r$ if there is $k \in r$ such that for every $a \in A$

$$\{b \in A \ : \ c(\{a,b\}) \neq k\} \in \mathcal{I}.$$

The following easy propositions characterize $\mathcal{I}$-homogeneous and homogeneous sets in terms of Fubini products of ideals.

Proposition 4.1. *Let $\mathcal{I}$ be an ideal on ω and $c\colon [\omega]^2 \to r$. The following are equivalent.*

1. *There is $A \in \mathcal{I}^+$ which is $\mathcal{I}$-homogeneous for c.*
2. *There is $A \in \mathcal{I}^+$ and $k \in r$ such that $\left\{(a,b) \in A^2 : c(\{a,b\}) \neq k\right\} \in \emptyset \times \mathcal{I}$.*
3. *There is $A \in \mathcal{I}^+$ and $k \in r$ such that $\left\{(a,b) \in A^2 : c(\{a,b\}) \neq k\right\} \in \mathcal{I} \times \mathcal{I}$.*

Proposition 4.2. *Let $\mathcal{I}$ be an ideal on ω and $c\colon [\omega]^2 \to r$. The following are equivalent.*

(1) *There is $A \in \mathcal{I}^+$ which is homogeneous for c.*
(2) *There is $A \in \mathcal{I}^+$ and $k \in r$ such that $\{(a,b) \in A^2 : c(\{a,b\}) \neq k\} \in \mathcal{I} \times \emptyset$.*

We say that the sequence $(x_n)_{n\in A}$ is *$\mathcal{I}$-increasing* if for every $N \in A$

$$\{n \in A : \; x_N \geq x_n\} \in \mathcal{I}.$$

Analogously we define $\mathcal{I}$-decreasing, $\mathcal{I}$-nonincreasing and $\mathcal{I}$-nondecreasing sequence. We say that $(x_n)_n$ is *$\mathcal{I}$-monotone* if $(x_n)_n$ is $\mathcal{I}$-nonincreasing or $\mathcal{I}$-nondecreasing.

An ideal $\mathcal{I}$ on ω will be called:

(1) *Ramsey** if for every finite coloring of $[\omega]^2$ there exists $\mathcal{I}$-homogeneous $A \in \mathcal{I}^+$;
(2) *Mon** if for every sequence $(x_n)_{n\in\omega}$ there exists $A \in \mathcal{I}^+$ such that $(x_n) \restriction A$ is $\mathcal{I}$-monotone;
(3) *h-Ramsey** if $\mathcal{I} \restriction A$ is locally Ramsey* for every $A \in \mathcal{I}^+$;
(4) *h-Mon** if $\mathcal{I} \restriction A$ is locally Mon* for every $A \in \mathcal{I}^+$.

Remark. It is easy to see that if the ideal $\mathcal{I}$ is h-Ramsey* (h-Mon*, h-BW) then $\mathcal{I}$ is Ramsey* (Mon*, BW resp.). However the reverse implications do not hold. Indeed, the ideal $\text{Fin} \oplus \mathcal{I}_d$ is BW but is not h-BW. By Theorem 4.3 and Corollary 4.4 this example also works for other properties.

Remark. Clearly if $\mathcal{I}$ is Ramsey (h-Ramsey) then $\mathcal{I}$ is Ramsey* (h-Ramsey*). Analogous implications hold for Mon and BW properties, respectively. These implications do not reverse. Indeed, any maximal ideal which is not a P-point is h-BW but is not Fin-BW (by Corollary 3.19). Theorem 4.3 and Corollary 4.4 shows that the same example works for other properties.

Theorem 4.3. *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent:*

(1) *$\mathcal{I}$ is Ramsey*,*
(2) *$\mathcal{I}$ is Mon*,*
(3) *$\mathcal{I}$ is BW.*

Proof. (1) $\Rightarrow$ (2). For every $a, b \in \omega$, $a < b$ let

$$\chi(\{a,b\}) = \begin{cases} 0 & \text{if } x_a < x_b; \\ 1 & \text{otherwise.} \end{cases}$$

Since $\mathcal{I}$ is Ramsey*, there is an $A \notin \mathcal{I}$ which is $\mathcal{I}$-monochromatic. Observe that if $\{b \in A \; : \; \chi(\{a,b\}) \neq 0\} \in \mathcal{I}$ for all $a \in A$ then $(x_n)_n \restriction A$ is $\mathcal{I}$-increasing, and if $\{b \in A \; : \; \chi(\{a,b\}) \neq 1\} \in \mathcal{I}$ for each $a \in A$ then $(x_n)_n \restriction A$ is $\mathcal{I}$-nonincreasing.

(2) $\Rightarrow$ (3). Suppose that $\mathcal{I}$ is Mon* and $(x_n)_{n\in\omega}$ is a bounded sequence. Let $A \notin \mathcal{I}$ be such that $(x_n) \restriction A$ is $\mathcal{I}$-monotone. Without lost of generality we can assume that $(x_n) \restriction A$ is $\mathcal{I}$-nondecreasing.

Let $x = \sup_{n\in A} x_n$, and fix $\varepsilon > 0$. Let $N \in A$ be such that $x_N > x - \varepsilon$. Then

$$\{n \in A \ : \ |x_n - x| \geq \varepsilon\} \subset \{n \in A \ : \ x_n < x_N\},$$

and the last set is an element of $\mathcal{I}$ by Mon* property of $\mathcal{I}$. Thus $(x_n) \restriction A$ is $\mathcal{I}$-convergent to x.

(3) $\Rightarrow$ (1). Let $c : [\omega]^2 \to r$. We define a family $\{A_s : s \in r^{<\omega}\}$ of subsets of ω.

- $A_\emptyset = \omega$.
- $A_{s^\frown i} = \{n \in A_s : c(|s^\frown i|, n) = i\}$, $i = 0, 1, \ldots, r-1$.

The family $\{A_s : s \in r^{<\omega}\}$ satisfies (S1), (S2) and (S3) from Proposition 2.9. But $\mathcal{I}$ is BW hence there is $x \in r^\omega$ and a set $B \notin \mathcal{I}$ such that $B \setminus A_{x\restriction n} \in \mathcal{I}$ for every $n \in \omega$. Moreover, there is $C \subset B$ and $j \in r$ such that $C \notin \mathcal{I}$ and $x(k-1) = j$ for every $k \in C$.

We claim that C is $\mathcal{I}$-homogeneous for coloring c. Let $n \in C$. Then

$$\{k \in C \ : \ c(\{n,k\}) \neq j\} \subset C \setminus A_{x\restriction n} \subset B \setminus A_{x\restriction n} \in \mathcal{I}.$$

□

The global versions of these properties are also equivalent.

Corollary 4.4. *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent:*

(1) *$\mathcal{I}$ is h-Ramsey*,*
(2) *$\mathcal{I}$ is h-Mon*,*
(3) *$\mathcal{I}$ is h-BW.*

And filter versions of these properties are very strong.

Corollary 4.5. *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent.*

(1) *For every finite coloring of $[\omega]^2$ there exists $\mathcal{I}$-homogeneous $A \in \mathcal{I}^*$,*
(2) *For every sequence $(x_n)_{n\in\omega}$ there exists $A \in \mathcal{I}^*$ such that $(x_n) \restriction A$ is $\mathcal{I}$-monotone,*
(3) *for any bounded sequence $(x_n)_{n\in\omega}$ of reals there is $A \in \mathcal{I}^*$ such that $(x_n) \restriction A$ is $\mathcal{I}$-convergent,*
(4) *$\mathcal{I}$ is a maximal ideal.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are proved like above.

(3) $\Rightarrow$ (4). Like in the proof of Proposition 3.18.

(4) $\Rightarrow$ (3). Folklore (see e.g. [7]). □

Proposition 4.6. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. Then*

(1) $\mathcal{I} \oplus \mathcal{J}$ *is Ramsey** *if and only if* $\mathcal{I}$ *is Ramsey** *or* $\mathcal{J}$ *is Ramsey***. Of course, the same holds for Mon** *and BW properties.*

(2) $\mathcal{I} \oplus \mathcal{J}$ *is h-Ramsey** *if and only if* $\mathcal{I}$ *and* $\mathcal{J}$ *are h-Ramsey***. Of course, the same holds for h-Mon** *and h-BW properties.*

Proof. Straightforward. □

Proposition 4.7 ([7])**.** *Let* $\mathcal{I}$ *and* $\mathcal{J}$ *be ideals on* ω*.*

(1) *If Fin* $\subset \mathcal{I}$ *then* $\mathcal{I} \times \mathcal{J}$ *is Ramsey** *(h-Ramsey***) if and only if* $\mathcal{I}$ *is Ramsey** *(h-Ramsey***).*

(2) $\emptyset \times \mathcal{J}$ *is Ramsey** *(h-Ramsey***) if and only if* $\mathcal{J}$ *is Ramsey** *(h-Ramsey***).*

Remark. Our paper [7] is devoted to BW and h-BW ideals. For instance, we show examples of ideals which are and which are not BW and h-BW, we give characterizations of BW property in terms of submeasures and extendability to a maximal P-ideal for analytic P-ideals. Moreover, we show applications to Rudin-Keisler and Rudin-Blass orderings of ideals and quotient Boolean algebras. In particular we show that an ideal $\mathcal{I}$ is not BW if and only if its quotient Boolean algebra $\mathcal{P}(\omega)/\mathcal{I}$ has a countably splitting family.

Remark. It seems interesting to examine ideal versions of Ramsey's theorem for coloring of triples, quadruples, and more. We leave these problems for further study. Let us just note here that it is not difficult to show that if a coideal I^+ is selective then ideal versions of Ramsey's theorem hold for coloring n-tuples for every $n \in \omega$.

5. Applications

For some classes of ideals we get the following strengthening of the fact that they are Ramsey*.

Proposition 5.1. *Let* $\mathcal{I}$ *be a Fin-BW ideal on* ω*. Then for every finite coloring of* $[\omega]^2$ *there exists Fin-homogeneous* $A \in \mathcal{I}^+$*.*

Proof. We proceed as in the proof of implication (3) $\Rightarrow$ (1) of Theorem 4.3, but we use Proposition 2.8 instead of Proposition 2.9. □

Corollary 5.2. *Let* $\mathcal{I}$ *be a P-ideal on* ω *which is Ramsey***. Then for every finite coloring of* $[\omega]^2$ *there exists Fin-homogeneous* $A \in \mathcal{I}^+$*.*

Proof. By Theorem 4.3 $\mathcal{I}$ is BW, so it is Fin-BW (see [7]). □

Theorem 5.3. *Let* $\mathcal{I} = Exh(\phi)$ *be an analytic P-ideal which is Ramsey***. Then there exists* $\delta = \delta(\phi)$ *such that for every finite coloring of* $[\omega]^2$ *there exists Fin-homogeneous* $A \subseteq \omega$ *with* $\|A\| > \delta$*.*

Proof. We say that $\{F_1, \ldots, F_N\}$ is an *(N,δ)-partition of the set $A \subset \omega$* if $F_1 \cup \cdots \cup F_N = A$ and $\phi(F_i) \leq \delta$ for every $i \leq N$. In [7, proof of Theorem 4.2] it is shown that there is $\delta > 0$ such that

$$\mathcal{I}_\delta = \{A \subset \omega \ : \ (\exists N, k \in \omega)\,(\forall n \in \omega)\,(\exists \mathcal{F}) \ \mathcal{F} \text{ is } (N,\delta)\text{-partition of } A \cap [k,n]\}\,.$$

is a proper F_σ ideal which extends $\mathcal{I}$. Moreover, it is easy to see that $\|A\|_\phi \geq \delta$ for every $A \notin \mathcal{I}_\delta$.

Ideal $\mathcal{I}_\delta$ is F_σ so it is Fin-BW (see [7, Proposition 3.4]). Hence for every coloring $c : [\omega]^2 \to r$ there is $i \in r$ and $A \in \mathcal{I}_\delta{}^+$ such that $\{b \in A : c(\{a,b\}) \neq i\}$ is finite for every $a \in A$ (by Proposition 5.1). Thus A is Fin-homogeneous for c and $\|A\| > \delta$. $\square$

Remark. Note that in the above theorem the constant δ does not depend on r.

In [9] authors prove, among others, the following theorem.

Theorem 5.4 ([9]). *For any finite coloring $c : [\omega]^2 \to r$ there exist $\delta = \delta(r) > 0$ and $i \leq r$ such that*

$$\underline{d}\left(\left\{x \in \omega : \overline{d}\left(\left\{y \in \omega : \overline{d}\,(Z(x,y)) \geq \delta\right\}\right) \geq \delta\right\}\right) \geq \delta,$$

where

$$Z(x,y) = \{z \in \omega : c(\{x,y\}) = c(\{x,z\}) = c(\{y,z\}) = i\}\,,$$

and $\overline{d}\,(\cdot)$ and $\underline{d}\,(\cdot)$ denote the upper density and lower density, respectively.

In [8] the following variant of Ramsey's theorem was proved (note that in Theorem 5.5 the constant δ depends on r).

Theorem 5.5 ([8]). *Let $\mathcal{I} = Exh(\phi)$ be an analytic P-ideal. For any finite coloring $c : [\omega]^2 \to r$ there exist $\delta = \delta(\phi, r) > 0$ and $i \leq r$ such that*

$$\|\{x \in \omega : \|\{y \in \omega : \|Z(x,y)\| \geq \delta\}\| \geq \delta\}\| \geq \delta.$$

For Ramsey* ideals we can prove similar result.

Proposition 5.6. *Let $\mathcal{I}$ be Ramsey* ideal. Then for any finite coloring $c : [\omega]^2 \to r$ there exist $i \leq r$ and $A \in \mathcal{I}^+$ such that for every $x \in A$*

$$\{y \in A : \{z \in A : c(\{x,y\}) \neq i \text{ or } c(\{x,z\}) \neq i \text{ or } c(\{y,z\}) \neq i\} \in \mathcal{I}\} \in \mathcal{I}$$

Proof. Let $A \in \mathcal{I}^+$ be a $\mathcal{I}$-homogeneous for $c : [\omega]^2 \to r$ (with color i). For every $x \in A$ we put $B_x = \{y \in A : c(\{x,y\}) \neq i\} \in \mathcal{I}$. Then for every $x \in A$ and for every $y \in A \setminus B_x$ and for every $z \in A \setminus (B_x \cup B_y)$ we have $c(\{x,y\}) = c(\{y,z\}) = c(\{x,z\}) = i$. But $B_x, B_x \cup B_y \in \mathcal{I}$ and that finishes the proof. $\square$

Using Theorem 5.3 one can easily show the following. Note that the constant δ does not depend on the number of colors.

Corollary 5.7. *Let $\mathcal{I} = Exh(\phi)$ be an analytic P-ideal which is Ramsey*. Then there exists $\delta = \delta(\phi)$ such that for every finite coloring $c : [\omega]^2 \to r$ there exists $i \in r$ and $A \subseteq \omega$ with $\|A\| > \delta$ such that for every $x \in A$*

$$\|\{y \in A : \|\{z \in A : c(\{x, y\}) = c(\{x, z\}) = c(\{y, z\}) = i\}\| \geq \delta\}\| \geq \delta.$$

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