PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 118, Number 3, July 1993

RESTRICTIONS TO CONTINUOUS FUNCTIONS AND BOOLEAN ALGEBRAS

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(Communicated by Andrew M. Bruckner)

ABSTRACT. We show that every Borel function $f : \mathbb{R} \to \mathbb{R}$ is continuous on a set $A \notin \mathscr{J}$ if $B(\mathbb{R})/\mathscr{J}$ is weakly distributive. We also show that CCC is not sufficient. We investigate some other conditions considering the problem of restrictions to continuous functions.

There are many theorems that say some kinds of functions have restrictions to continuous functions on large sets (see, e.g., [Bl, Br, BP, SZ]). We recall a few.

Theorem A. *For every Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ there is a set D of positive measure such that $f|D$ is continuous.*

Theorem B. *For every function with the Baire property $f : \mathbb{R} \to \mathbb{R}$ there is a set D of second category and with the Baire property such that $f|D$ is continuous.*

Theorem C (Sierpiński and Zygmund [SZ]). *There is a function $f : \mathbb{R} \to \mathbb{R}$ such that f is not continuous on any set of size continuum.*

Theorem D (Blumberg [Bl]). *For every function $f : \mathbb{R} \to \mathbb{R}$ there is a countable dense set D such that $f|D$ is continuous.*

Let us try to express all these theorems in a common language.

Definition 1. Let $\mathscr{A}$ be a σ-algebra on a topological space X and $\mathscr{J} \subseteq \mathscr{A}$ be a proper ideal. We say that $(\mathscr{A}, \mathscr{J})$ has CRP (continuous restrictions property) if for every $\mathscr{A}$-measurable function $f : X \to \mathbb{R}$ there exists $D \in \mathscr{A} \setminus \mathscr{J}$ such that $f|D$ is continuous.

Let L be the σ-algebra of Lebesgue measurable sets; let L_0 be the σ-ideal of Lebesgue negligible sets; let B_w be the σ-algebra of sets with Baire property; and let FC be the σ-ideal of sets of first category.

As simple corollaries we get

Received by the editors September 29, 1991.

1991 *Mathematics Subject Classification.* Primary 26A15, 06E10; Secondary 28A10, 04A15.

Key words and phrases. Continuity, weak distributivity, Boolean algebra, Lusin set.

Most of the results in this paper were obtained when the author was visiting Auburn University, supported by the Kosciuszko Foundation.

Corollary 1.1. *Pairs* (L, L_0), (B_w, FC), $(P(\mathbb{R}), [\mathbb{R}]^{<\omega})$, $((s), (s_0))$ (*see* [Ma]) *have CRP.*

Corollary 1.2. *The pair* $(P(\mathbb{R}), [\mathbb{R}]^{<2^\omega})$ *does not have CRP.*

Throughout this paper we say that a function $f : Z \to \mathbb{R}$ is B-measurable if it is measurable with respect to $B(Z)$, the σ-algebra generated by all closed G_δ-sets.

We say that a subset $L \subseteq \mathbb{R}$ is a Lusin (Sierpiński) set if L is uncountable and its intersection with every set of first category (of measure zero) is countable. A set is a λ-set if every countable set is relatively G_δ. A set $X \subseteq \mathbb{R}^n$ is universally null if $\mu^*(X) = 0$ for every continuous Borel measure μ on $\mathbb{R}^n$.

In this paper we will investigate the property CRP considering properties of the quotient Boolean algebra $\mathscr{A}/\mathscr{J}$. First we will consider $\mathscr{A} = B(\mathbb{R})$ and $\mathscr{J} \subseteq B(\mathbb{R})$ such that $B(\mathbb{R})/\mathscr{J}$ is CCC. We see that for $\mathscr{J} = L_0$ and $\mathscr{J} = FC$, $(B(\mathbb{R}), \mathscr{J})$ has CRP.

Theorem 1. *There is a σ-ideal $\mathscr{J} \subseteq B(\mathbb{R})$ such that $B(\mathbb{R})/\mathscr{J}$ has CCC and $(B(\mathbb{R}), \mathscr{J})$ does not have CRP. Moreover, $B(\mathbb{R})/\mathscr{J}$ is isomorphic to $B(\mathbb{R})/FC$.*

A similar result was shown independently in [CMPS].

Proof. Lusin [L] showed that there is a continuous one-to-one function $g : \omega^\omega \to \omega^\omega$ such that for every perfect set $D \subseteq \omega^\omega$ there is an open dense set $U \subseteq \omega^\omega$ such that $g[U] \cap D$ is first category in D.

Let $B = g[\omega^\omega]$ and $f = g^{-1}|B$ and let $\mathscr{J} = \{g[F] : F \text{ is first category}\}$. Since g is a Borel isomorphism, we have that f is Borel and $\mathscr{J}$ is a σ-ideal such that $B(B)/\mathscr{J}$ is CCC. Let us assume that there is a set $X \in B(B) \setminus \mathscr{J}$ such that $f|X$ is continuous and hence, a homeomorphism. We can assume that X is dense in itself. Then $D = \overline{X}$ is perfect. Let $U \subseteq \omega^\omega$ be an open dense set such that $g[U] \cap D$ is first category in D. Thus $A = g[U] \cap X$ is first category in X. Since $f|X$ is a homeomorphism, $f[A]$ is first category in $f[X]$, and, so, in ω^ω. Thus $A \in \mathscr{J}$ because $A = g[f[A]]$. $X \setminus A$ also belongs to $\mathscr{J}$ because $X \setminus A = g[(\omega^\omega \setminus U)] \cap X$. So $X \in \mathscr{J}$—a contradiction.

Now we can extend a function f to a Borel function h defined on $\mathbb{R}$ and let $\mathscr{J}^* = \{C \in B(\mathbb{R}) : C \cap B \in \mathscr{J}\}$. Then the function h and the σ-ideal $\mathscr{J}^*$ have the required properties. □

From this theorem we have that the properties of $B(\mathbb{R})/FC$ are too weak to imply that $(B(\mathbb{R}), \mathscr{J})$ has CRP. So we will look more carefully at the measure Boolean algebra. Of course if $\mathscr{A}$ is the σ-algebra of measurable sets with respect to a finite measure μ and $\mathscr{J}$ is the σ-ideal of μ-measure zero sets then $(\mathscr{A}, \mathscr{J})$ has CRP. We consider a weaker condition: weak distributivity.

Definition 2.1. $\mathscr{C} \subseteq \mathscr{A}$ is predense if for every $D \in \mathscr{A} \setminus \mathscr{J}$ there is $C \in \mathscr{C}$ such that $C \cap D \notin \mathscr{J}$.

Definition 2.2. $\mathscr{A}/\mathscr{J}$ is weakly distributive if for every $D \in \mathscr{A} \setminus \mathscr{J}$ and for every sequence of predense families $\mathscr{C}_n \subseteq \mathscr{A}$ there is sequence of finite subfamilies $\mathscr{C}_n' \subseteq \mathscr{C}_n$ such that $D \cap \bigcap_n \bigcup \mathscr{C}_n' \notin \mathscr{J}$.

Definition 3. Let $\mathscr{F} \subseteq \mathbb{R}^X$. Then let $\mathscr{F}_c$ be the smallest family of functions containing $\mathscr{F}$ closed under pointwise limits.

We write $f_n \rightrightarrows f$ if f_n is converging uniformly to f.

Theorem 2. *Let $\mathscr{F}$ be a family of $\mathscr{A}$-measurable functions. If $\mathscr{A}/\mathscr{J}$ is weakly distributive then for every $D \in \mathscr{A} \setminus \mathscr{J}$ and for every $f \in \mathscr{F}_c$ there is a sequence of functions $\{f_n : n \in \omega\} \subseteq \mathscr{F}$ and there is $C \in \mathscr{A}$ such that $C \cap D \notin \mathscr{J}$ and $f_n \rightrightarrows f$ on $C \cap D$, or, equivalently, $\{C : \exists_{\{f_n : n \in \omega\} \subseteq \mathscr{F}} f_n \rightrightarrows f \text{ on } C\}$ is predense for every $f \in \mathscr{F}_c$.*

Proof. We can define inductively $\mathscr{F}_\alpha$ as the family of all pointwise limits of functions from $\bigcup_{\beta<\alpha} \mathscr{F}_\beta$. It is easy to see that $\mathscr{F}_c = \bigcup_{\beta<\omega_1} \mathscr{F}_\beta$. We inductively show that for every function $f \in \mathscr{F}_c$ there is a family $\mathscr{C}_f$ such that $\mathscr{C}_f$ is closed under finite unions, predense, and for every $C \in \mathscr{C}_f$ there is a sequence $\{f_n : n \in \omega\} \subseteq \mathscr{F}$ converging uniformly to f on C. Let a function $f \in \mathscr{F}_\alpha$. Then there is a sequence of functions $\{f_n : n \in \omega\} \subseteq \bigcup_{\beta<\alpha} \mathscr{F}_\beta$ such that $f_n \to f$. Let $\mathscr{E}_k = \{A_{nk} : n, k \in \omega\}$ where $A_{nk} = \{x : \forall_{j \geq n} |f(x) - f_j(x)| < 1/k\}$. $\mathscr{E}_k$ is predense. So since A_{nk} is increasing with respect to n and $\bigcup_n A_{nk} = X$, for $D \notin \mathscr{J}$ there is a subsequence n_k such that $D \cap \bigcap_k A_{n_k k} \notin J$. Observe that f_n converges uniformly to f on $D \cap \bigcap_k A_{n_k k}$. So $\mathscr{E} = \{C : f_n \text{ is converging uniformly to } f \text{ on } C\}$ is predense and closed under finite unions. Let $\mathscr{C}_{f_n}$ be predense families given by the induction hypothesis. For every $D \in \mathscr{A} \setminus \mathscr{J}$ there is a sequence $\mathscr{C}'_{f_n} \subseteq \mathscr{C}_{f_n}$ of finite families such that $D \cap \bigcap_n \bigcup \mathscr{C}'_{f_n} \notin \mathscr{J}$. The family of sets of the form $\bigcap_n \bigcup \mathscr{C}'_{f_n}$ is predense and for every n there is a sequence $\{f_{nk} : k \in \omega\} \subseteq \mathscr{F}$ such that f_{nk} is uniformly converging to f_n on $\bigcap_n \bigcup \mathscr{C}'_{f_n}$. Let $\mathscr{C}$ be the closure under finite unions of that family; it also has that property. Let $\mathscr{C}_f$ be the closure under finite unions of $\{A \cap C : A \in \mathscr{E}$ and $C \in \mathscr{C}\}$. We can see that for every element C of $\mathscr{C}_f$, for every n there is a sequence $\{f_{nk} : k \in \omega\} \subseteq \mathscr{F}$ such that f_{nk} is uniformly converging to f_n on C and f_n is uniformly converging to f on C. So there is a sequence $\{f_{nk_n} : n \in \omega\} \subseteq \mathscr{F}$ such that f_{nk_n} is uniformly converging to f on C.

Corollary 2. *Let X be a normal topological space and $\mathscr{J} \subseteq B(X)$ such that $B(X)/\mathscr{J}$ is weakly distributive. Then the pair $(B(X), \mathscr{J})$ has CRP.*

Proof. Observe that every B-measurable function $f \in C(X)_{\mathrm{c}}$ where $C(X)$ is the family of all continuous functions on X. So the family of all sets on which the B-measurable function f is a uniform limit of continuous functions is predense.

The next theorem shows that we can consider more general σ-algebras.

Lemma 1. *Let $\mathscr{A}$ be a σ-algebra and $\mathscr{J}$ a σ-ideal such that $\mathscr{A}/\mathscr{J}$ is weakly distributive and for every $C \in \mathscr{A} \setminus \mathscr{J}$ there is $D \in B(X) \setminus \mathscr{J}$ such that $D \subseteq C$. Then for every $\mathscr{A}$-measurable function $f : X \to \mathbb{R}$ the set $\{D \in B(X) : f|D$ is B-measurable$\}$ is predense.*

Proof. Let $\{O_n : n \in \omega\}$ be a basis of $\mathbb{R}$. For every $\mathscr{A}$-measurable function we can define $\mathscr{C}_n = \{C \in B(X) : C \subseteq f^{-1}[O_n] \text{ or } C \subseteq X \setminus f^{-1}[O_n]\}$. For every n, $\mathscr{C}_n$ is predense, so for every $D \in \mathscr{A} \setminus \mathscr{J}$ there is a sequence of finite subfamilies $\mathscr{C}'_n \subseteq \mathscr{C}_n$ such that $D \cap \bigcap_n \bigcup \mathscr{C}'_n \notin \mathscr{J}$. Observe that $f^{-1}[O_n] \cap \bigcap_n \bigcup \mathscr{C}'_n = \bigcap_n \bigcup \mathscr{C}'_n \cap \bigcup \mathscr{C}''_n$ where $\mathscr{C}''_n \subseteq \mathscr{C}'_n$. So $f | \bigcap_n \bigcup \mathscr{C}'_n$ is B-measurable.

From the theorems above we get:

Theorem 3. *Let X be a normal topological space. Let $\mathscr{A}$ be a σ-algebra, $\mathscr{J}$ a σ-ideal such that $\mathscr{A}/\mathscr{J}$ is weakly distributive, and for every $C \in \mathscr{A} \setminus \mathscr{J}$ there is a $D \in B(X) \setminus \mathscr{J}$ such that $D \subseteq C$. Then the pair $(\mathscr{A}, \mathscr{J})$ has CRP.*

Definition 4. We say that the sequence $f_n : X \to \mathbb{R}$ is quasinormally converging to f $(f_n \overset{QN}{\to} f)$ if $\exists_{\varepsilon_n \searrow 0} \forall_{x \in X} \exists_k \forall_{n \geq k} |f_n(x) - f(x)| < \varepsilon_n$.

It is known (see [BRR]) that $f_n \overset{QN}{\to} f$ iff $X = \bigcup_k X_k$ and $f_n \rightrightarrows f$ on X_k for each k.

We say that a property is satisfied a.e. (almost everywhere) if it is satisfied on $X \backslash D$ for a set $D \in \mathscr{J}$.

Lemma 2. *Let X be a normal topological space and $\mathscr{J} \subseteq B(X)$ such that $B(X)/\mathscr{J}$ is weakly distributive and CCC. Then*

(a) *If $f_n \to f$ then $f_n \overset{QN}{\to} f$ a.e.*
(b) *If $f_n \to f$ and $f_{nk} \to f_n$ then there is a subsequence k_n such that $f_{nk_n} \overset{QN}{\to} f$ a.e.*

Proof (due to [BRR] or [W]). (a) Let $A_{nk} = \{x : \forall_{j \geq n} |f(x) - f_j(x)| < 1/k\}$ and $\mathscr{A}_n = \{A_{nk} : k \in \omega\}$. Then the $\mathscr{A}_n$ are predense, so as in the proof of Theorem 2 there is predense $\mathscr{C}$ such that for every $C \in \mathscr{C}$, $f_n \rightrightarrows f$ on C. By CCC there is a countable $\mathscr{C}' \subseteq \mathscr{C}$ such that for every $C \in \mathscr{C}'$, $f_n \rightrightarrows f$ on C and $X \backslash \bigcup \mathscr{C}' \in \mathscr{J}$.

(b) From (a) we get that there are countable $\mathscr{A}_n$ such that $f_{nk} \rightrightarrows f_n$ on C for every $C \in \mathscr{A}_n$, and a countable $\mathscr{B}$ for every $C \in \mathscr{B}$, $f_n \rightrightarrows f$ on C, and $(X \backslash (\bigcap_n \bigcup \mathscr{A}_n \cap \bigcup \mathscr{B})) \in \mathscr{J}$. So there is a countable $\mathscr{C}$ such that for every $C \in \mathscr{C}$, $f_n \rightrightarrows f$ on C, $f_{nk} \rightrightarrows f_n$ on C, and $X \backslash \bigcup \mathscr{C} \in \mathscr{J}$. We can choose by induction k_n such that for every $C \in \mathscr{C}$, $f_{nk_n} \rightrightarrows f$ on C.

Corollary 3. *Let X be a normal topological space. Let $\mathscr{A}$ be a σ-algebra, $\mathscr{J}$ a σ-ideal such that $\mathscr{A}/\mathscr{J}$ is weakly distributive and CCC, and for every $C \in \mathscr{A} \setminus \mathscr{J}$ there is a $D \in B(X) \backslash \mathscr{J}$ such that $D \subseteq C$. Then for every $\mathscr{A}$-measurable function $f : X \to \mathbb{R}$ there is a $B \in \mathscr{J}$ and a sequence $f_n : X \to \mathbb{R}$ of continuous functions such that $f_n : X \backslash B \overset{QN}{\to} \mathbb{R}$.*

Proof. First by Lemma 1 we get that there is a set $F \in \mathscr{J}$ such that $f|X \backslash F$ is B-measurable because a countable union of Borel functions on Borel sets is a Borel function. As in the proof of Theorem 2 we show by induction with respect to the Baire class of the Borel function $f : X \to \mathbb{R}$ that there is a sequence $f_n : X \to \mathbb{R}$ of continuous functions and $B \in \mathscr{J}$ such that $f_n | X \backslash B \overset{QN}{\to} \mathbb{R}$. f is a pointwise limit of functions f_n of lower Baire class. By induction we have that f_n is a pointwise limit of continuous functions a.e. and by Lemma 2 we get that f is also a quasi-normal limit of continuous functions a.e.

Now we will consider an ideal of the form $\mathscr{J} = [X]^{<\kappa}$ where κ is a cardinal. Let $X \subseteq \mathbb{R}$ such that $|X| = \kappa$.

Fact 1. (a) *The pair $(P(X), [X]^{<\kappa})$ does not have CRP if $\kappa = 2^\omega$ (Sierpiński and Zygmund* [SZ]).

(b) *The pair $(P(X), [X]^{<\kappa})$ has CRP if $\omega < \kappa < 2^\omega$ under MA (Shinoda* [Sh]).

The following theorem gives us independence of the result (b).

Theorem 4 (Gruenhage). *Let V satisfy $2^\omega = \kappa$ and let $\lambda \geq \kappa$ be a cardinal with $\lambda^\omega = \lambda$. If λ Cohen or random reals are added to V then in the extension*

$\exists F : \mathbb{R} \to \mathbb{R}$ such that $F|X$ is not Borel whenever X is an uncountable subset of $\mathbb{R}$.

Proof. It suffices to prove the proposition with 2^ω in place of $\mathbb{R}$. We will verify the Cohen reals case; the random reals case is analogous, using the measure algebra of $2^{\lambda\times\omega}$ in place of $\mathrm{Fn}(\lambda \times \omega, 2)$.

Let $P = \mathrm{Fn}(\lambda \times \omega, 2)$, and let G be a generic filter. Then $V[G]$ satisfies $2^\omega = \lambda$. For each $x \in 2^\omega \cap V[G]$, there is a countable set $C_x \subseteq \lambda$ such that $x \in V[G \cap \mathrm{Fn}(C_x \times \omega, 2)]$. One can define a one-to-one function $\Theta : \omega^\omega \to \lambda$ such that $\forall x \;\; \Theta(x) \notin C_x$. Let $f_{\Theta(x)} = \bigcup G(\Theta(x), \cdot) : \omega \to 2$, and let $F(x) = f_{\Theta(x)}$. Suppose $F|X$ is Borel and X is uncountable. $F|X$ extends to some $F' : Y \to 2^\omega$, where Y is Borel. Then there exists a countable $E \subseteq \lambda$ such that $F' \in V[G \cap \mathrm{Fn}(E \times \omega, 2)]$. Choose $x \in X$ such that $\Theta(x) \notin E$. Then $F', x \in V[G \cap \mathrm{Fn}((E \cup C_x) \times \omega, 2)]$, but $F'(x) = F(x) = f_{\Theta(x)} \notin V[G \cap \mathrm{Fn}((E \cup C_x) \times \omega, 2)]$ since $\Theta(x) \notin E \cup C_x$. This is a contradiction.

Of course this theorem implies that the pair $(P(X), [X]^{\leq 2\omega}1)$ does not have CRP for every $X \subseteq \mathbb{R}$ of size ω_1. It is also worthwhile to see the following facts.

Fact 2. *If $X \subseteq \mathbb{R}$ is a Sierpiński set or a Lusin set of size ω_1 then there is a set $Y \subseteq \mathbb{R}$ such that for every one-to-one function $f : X \to Y$, f is not continuous on any uncountable subset of X.*

Proof. Let $Y \subset \mathbb{R}$ be a universally null set of size ω_1. Then let f be a one-to-one function $f : X \to Y$ and suppose that f is continuous on an uncountable subset of X. Then this subset in universally null; but a Sierpiński set does not contain an uncountable subset of measure zero.

In the case of a Lusin set Y is an uncountable λ-set.

Fact 3. *If $L \subseteq \mathbb{R}^2$ is a Lusin set or Sierpiński set in $\mathbb{R}^2$ then for every function $f : p_1(L) \to p_2(L)$ such that* graph$(f) \subseteq L$, *f is not continuous on any uncountable subset of $p_1(L)$.*

Proof. The graph of any continuous function is of first category and measure zero in $\mathbb{R}^2$, so its intersection with L is countable.

For the case of the σ-algebra $B(X)$ we recall the following results.

Theorem 5 (Sierpiński [S]). *If there is a Lusin set $L \subseteq \mathbb{R}$ then there is a set X of size $|L|$ such that $(B(X), [X]^{<\omega_1})$ does not have CRP.*

Proof. As in the proof of Theorem 1 we take the Luzin function $g : \omega^\omega \to \omega^\omega$. Let us take a Lusin set $L \subseteq \omega^\omega$ and let X be $g[L]$. Then $\mathscr{J} \cap X = [X]^{\leq\omega}$ for the ideal $\mathscr{J}$ from Theorem 1. So $g^{-1}|X$ is not continuous on any uncountable set.

Theorem 6 (Cichoń and Morayne [CM]). *It is consistent that, for every set $X \subseteq \mathbb{R}$ of size 2^ω, $(B(X), [X]^{<2^\omega})$ has CRP.*

Acknowledgment

The author would like to express his thanks to J. B. Brown, R. Beaudoin, G. Gruenhage, and J. Jasiński for fruitful discussions.

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