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SET THEORY

Some Remarks on the Baire Order of Functions Continuous Almost Everywhere

by

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Summary. We show that Baire order of the family of functions on the real line continuous $\mathcal{J}$-almost everywhere is 1 or ω_1 where $\mathcal{J}$ is arbitrary σ-ideal (not necessarily uniform). This solves a problem of R. D. Mauldin.

Let $\mathcal{J}$ be a σ-ideal on a separable metric space Y. Let $\Phi(Y,\mathcal{J})$ be the family of all real-valued functions defined on Y whose set of points of dicontinuity belongs to $\mathcal{J}$.

Let $\Phi(Y,X)$ be the family of all real-valued functions defined on Y which are continuous at each point of X.

$\Phi(Y,X)$ is also of the form $\Phi(Y,\mathcal{J})$. Simply let $\mathcal{J}=\{A\subset Y : A\cap X=\emptyset\}$.

If Φ is a family of real-valued functions on a set X then $B_0(\Phi)$ will denote Φ and for each ordinal $\alpha>0$, $B_\alpha(\Phi)$ will denote the family of all pointwise limits of sequences from $\bigcup_{\gamma<\alpha} B_\gamma(\Phi)$. Obviously, $B_{\omega_1}(\Phi)=B_{\omega_1+1}(\Phi)$. The first ordinal α for which $B_\alpha(\Phi)=B_{\alpha+1}(\Phi)$ will be called the Baire order of Φ. Thus the Baire order of every family is $\leqslant\omega_1$.

Borel order of a set X is a smallest ordinal α such that $\Sigma^0_{\alpha+1}(X)=\Sigma^0_\alpha(X)$ ($\Sigma^0_1(X)=$ open sets in X).

Denote by $\mathcal{N}$ the σ-ideal of Lebesgue measure zero sets on the real line; by $\mathcal{M}$ the σ-ideal of meagre sets on the real line. Kuratowski [5] in 1924 showed that Baire order of $\Phi(\mathbb{R},\mathcal{M})$ is 1. Mauldin [6] showed that Baire order of $\Phi(\mathbb{R},\mathcal{N})$ is ω_1. M. Balcerzak and D. Rogowska [2] showed that if $\mathcal{J}$

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is uniform (i.e. for each $x \in Y$ $\{x\} \in \mathcal{J}$) then Baire order of $\Phi(\mathbb{R}, \mathcal{J})$ is 1 or ω_1 which solved a problem of R. D. Mauldin from [2].

In [7] R. D. Mauldin asked a question whether there is a subset M of the real line such that Baire order of $\Phi(\mathbb{R}, M)$ is 2. The same question was posed also for any α with $2 \leqslant \alpha < \omega_1$. In this paper we show that for each Polish space Y and for each σ-ideal $\mathcal{J}$, Baire order of $\Phi(Y, \mathcal{J})$ is $\leqslant 1$ or ω_1 which solves a problem stated in [2]. In particular, we give negative aswer to both mentioned questions of R. D. Mauldin.

THEOREM 1.1. *Let $2 \leqslant \alpha < \omega_1$. If there is a Borel set B in Y such that for each G_δ-set G in Y with $Y \setminus G \in \mathcal{J}$, $(B \cap G) \notin \Sigma^0_{\alpha+1}(Y)$ then Baire order of $\Phi(Y, \mathcal{J})$ is $\geqslant \alpha$.*

Proof. We will show that f, the characteristic function of the set B, is not in $B_\alpha(\Phi(Y, \mathcal{J}))$ but is in $B_{\omega_1}(\Phi(Y, \mathcal{J}))$ because is Borel. Assume that $f \in B_\alpha(\Phi(Y, \mathcal{J}))$. There is a countable family of functions $\{f_n : n \in \omega\} \subset \Phi(Y, X)$ such that $f \in B_\alpha(\{f_n : n \in \omega\})$. Let G_n be the set of points of continuity of function f_n. Since the set of points of continuity of any real function is G_δ-set then $G = \bigcap_n G_n$ is a G_δ-set in Y. Then $f|G$ belongs to Baire class α on G (now in normal sense) because $f_n|G$ are continuous. Hence $(f|G)^{-1}(\{1\})$ is in $\Sigma^0_{\alpha+1}(G)$ thus is in $\Sigma^0_{\alpha+1}(Y)$. However, $(f|G)^{-1}(\{1\}) = B \cap G$, which is a contradiction.

LEMMA 1.2. *Let X be countable, dense in itself subset of a Polish space Y. For each $\alpha < \omega_1$ there is a Borel subset $B \subset Y$ such that for each G_δ-set $G \subset Y$ with $X \subset G$ we have that $G \cap B$ is not $\Sigma^0_\alpha(Y)$.*

Proof. Assume first that $Y = \mathbb{R}$ and $X = \mathbb{Q}$. Define $F : \omega^\omega \to 2^\omega$ such that $F(f)(n) = f(n)(\text{mod } 2)$. Then for each $y \in 2^\omega$ we have that $F^{-1}(y) \notin J$ where J is the σ-ideal generated by compact sets on ω^ω (so called property (M), see [1]). Then by [1] Borel sets cannot be approximated by elements from given additive class α modulo elements from σ-ideal generated by compact sets in $\mathbb{R} \setminus \mathbb{Q}$.

Let X, Y be arbitrary as in Lemma 1.2. Then X is homeomorphic to $\mathbb{Q}$ [4]. By Lavrentiev's Theorem [4] there is homeomorphism $h : W \to U$ where W is a G_δ-subset in Y and U is a G_δ-subset in $\mathbb{R}$ with $h[X] = \mathbb{Q}$. For given $\alpha < \omega_1$ we find B with the required properties for the pair $\mathbb{Q}, \mathbb{R}$ and then $h^{-1}[B]$ has the required properties for Y, X.

COROLLARY 1.3. *Assume that $X \subset Y$ contains a subset dense in itself and Y is a Polish space. Then Baire order of $\Phi(Y, X)$ is ω_1.*

In the case, when X is scattered (i.e. X does not contain a subset dense in itself), Baire order is $\leqslant 1$ [7]. Hence this solves a problem of Mauldin from [7].

Theorem 1.1 can be partially reversed.

THEOREM 1.4. *Let $\alpha < \omega_1$. If for each Borel set B there is a G_δ-set G in Y with $Y \setminus G \in \mathcal{J}$ and $(B \cap G) \in \Sigma^0_{\alpha+1}(Y)$ then Baire order of $\Phi(Y, \mathcal{J})$ is $\leqslant \alpha + 1$.*

Proof. Assume that $f \in B_{\omega_1}(\Phi(Y, \mathcal{J}))$. There is a countable family of functions $\{f_n : n \in \omega\} \subset \Phi(Y, \mathcal{J})$ such that $f \in B_{\omega_1}(\{f_n : n \in \omega\})$.

Let G_n be the set of points of continuity of function f_n. $G = \bigcap_n G_n$ is a relative G_δ-set in Y. Then $f|G$ is Borel function on G because $f_n|G$ are continuous. Let $\{U_k : k \in \omega\}$ be a countable basis for the real line. Let G_k be a G_δ-sets such that $(f|G)^{-1}(U_k) \cap G_k \in \Sigma^0_{\alpha+1}(Y \cap G)$ and $Y \setminus G_k \in \mathcal{J}$. Let $G' = \bigcap_k G_k \cap G$. Then $(f|G')^{-1}(U_k) \in \Sigma^0_{\alpha+1}(Y \cap G')$ so $f|G'$ is of Baire class α. We can extend $f|G'$ to a function f' on Y of Baire class $\alpha + 1$ (see [4]).

Let $f' = \lim g_n$ where g_n are of Baire class $\leqslant \alpha$ on Y. Let $Y \setminus G' = \bigcup_n F_n$ where F_n increasing sequence of a relative F_σ-sets in Y. Define

$$g'_n(x) = \begin{cases} g_n(x) & \text{if } x \notin F_n \\ f(x) & \text{otherwise.} \end{cases}$$

It is easy to see that $g'_n \in B_\alpha(\Phi(Y, \mathcal{J}))$ and $\lim g'_n = f$.

The next result generalizes a result from [2]. It also contains Corollary 1.3.

THEOREM 1.5. *Let $\mathcal{J}$ be an arbitrary σ-ideal on a Polish space Y. Then Baire order of $\Phi(Y, \mathcal{J})$ is either $\leqslant 1$ or ω_1.*

Proof. Let $X = \{x \in Y : \{x\} \notin \mathcal{J}\}$.

Case 1. Assume that X contains dense in itself set A. Then by Lemma 1.2 and Theorem 1.1, Baire order is ω_1.

Case 2. Assume that X is scattered. Define $\mathcal{J}_0 = \{A \cup B : A \in \mathcal{J} \wedge B \subset X\}$. Then $\mathcal{J}_0$ is uniform.

Case 2a. Assume that Baire order of $\Phi(Y, \mathcal{J}_0)$ is $\leqslant 1$. Let $f \in B_{\omega_1}(\Phi(Y, \mathcal{J}))$ then $f \in B_1(\Phi(Y, \mathcal{J}_0))$. Thus $f = \lim f_n$ where $f_n \in \Phi(Y, \mathcal{J}_0)$. Let G be G_δ-set with $Y \setminus G \in \mathcal{J}_0$ and f_n are continuous at each point of G. Let $X \setminus G = \bigcap_n H_n = \{x_n : n \in \omega\}$ where H_n are open sets. $f_n|(G \setminus H_n) \cup \{x_0 \dots x_n\}$ are continuous and can be extended to continuous functions $f'_n : G \cup H_n \to \mathbb{R}$ because $(G \setminus H_n) \cup \{x_0 \dots x_n\}$ is a closed subset in $G \cup H_n$. Define $f''_n : Y \to \mathbb{R}$;

$$f''_n(x) = \begin{cases} f'_n(x) & \text{if } x \in G \cup H_n \\ f_n(x) & \text{otherwise.} \end{cases}$$

Now let $\{y_k\}$ be such that $\lim y_k = x \in G \cup H_n$. If $\{y_k\} \subset G \cup H_n$ then $\lim_k f''_n(y_k) = \lim_k f'_n(y_k) = f'_n(x) = f''_n(x)$. $\{y_k\} \subset Y \setminus (G \cup H_n)$. Then $x \in G \setminus H_n$, so $\lim_k f''_n(y_k) = \lim_k f_n(y_k) = f_n(x) = f''_n(x)$.

Then f''_n are continuous at each point of $G \cup H_n$ and $\lim f''_n = f$, so $f \in B_1(\Phi(Y, \mathcal{J})$.

Case 2b. Assume that Baire order of $\Phi(Y, \mathcal{J}_0)$ is ω_1. Then for each $\alpha < \omega_1$ there is Borel set B such that for each G_δ-set G with $Y \setminus G \in \mathcal{J}_0$ and $B \cap G \notin \Sigma^0_{\alpha+1}(I)$. Since if $Y \setminus G \in \mathcal{J}$ then $Y \setminus G \in \mathcal{J}_0$, by Theorem 1.1 we get that Baire order of $\Phi(Y, \mathcal{J})$ is ω_1.

In the next few results we will consider Baire order of $\Phi(Y, X)$ when Y is not necessarily Polish.

LEMMA 1.6. *Let A be a countable, dense in itself set and C an analytic set. Assume that A is not a relative G_δ-set in $C \cup A$. Then for each $\alpha < \omega_1$ there is a Borel set $B \subset C$ such that for each G_δ-set H with $A \subset H$, $H \cap B \notin \Sigma^0_\alpha$.*

Proof. We can assume that $A = \mathbb{Q}$ (comp. proof of Lemma 1.2). By Kechris, Saint Raymond result [3] there is a set $F \subset C$ closed in $\mathbb{R} \setminus \mathbb{Q}$ homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$. Let $\alpha < \omega_1$. Then by Lemma 1.2 there is a Borel sets $B \subset F$ such that for each σ-compact set $H \subset F$ $B \setminus H \notin \Sigma^0_\alpha$, so for each σ-compact set $H \subset \mathbb{R} \setminus \mathbb{Q}$ $B \setminus H \notin \Sigma^0_\alpha$.

Next theorem is a corollary from Lemma 1.6 and Theorem 1.1.

THEOREM 1.7. *Let X be a set containing a countable dense in itself subset A, and let Y be an analytic set. Assume that A is not a relative G_δ-set in Y. Then Baire order of $\Phi(Y, X)$ is ω_1.*

COROLLARY 1.8. *Let Y be Borel set and $X \subset Y$. Then Baire order of $\Phi(Y, X)$ is either $\leqslant 1$ if X is countable, relative G_δ in Y or ω_1, otherwise.*

Proof. If Y is discrete then Baire order of $\Phi(Y, X)$ is 0. If X is countable and relative G_δ-set in Y then Baire order of $\Phi(Y, X)$ is 1. Let $X = \{x_n : n \in \omega\}$ and $Y \setminus X = \bigcup_n F_n$ where F_n denotes an increasing sequence of relative F_σ-sets in Y. Let f be any function on Y.

Let g_n be a sequence of continuous functions on Y such that for each $k \leqslant n$ $g_n(x_k) = f(x_k)$.

Define

$$g'_n(x) = \begin{cases} f(x) & \text{if } x \in F_n \\ g_n(x) & \text{otherwise.} \end{cases}$$

Then $g'_n \in \Phi(Y, X)$ and $f = \lim g'_n$.

The case when X is countable and not relative G_δ in Y is solved by Theorem 1.7.

Assume that X is uncountable and let $f : B \to Y$ be continuous one-to-one function from a closed subset of ω^ω onto Y. Then there is countable, dense in itself subset $A' \subset f^{-1}(X)$ which is not G_δ in B. Thus $f(A')$ is not G_δ in Y and by Theorem 1.7 we have that Baire order of $\Phi(Y, X)$ is ω_1.

Next fact shows that for irregular sets we can obtain each possible ordinal as a Baire order.

FACT 1.9. *Assume continuum hypothesis (CH). For each $\alpha < \omega_1$ there is a set X such that Baire order of $\Phi(X, X)$ is α.*

Proof. Observe that $\Phi(X, X)$ is just the family of all continuous function on X. By Kunen result [8] under CH there is a set of Borel order α. By classical results (see [3]) Baire order of $\Phi(X, X)$ is α.

PROBLEMS.

(1) *Let $B \subset \mathbb{R}$ be a Borel set and $\mathcal{J}$ an arbitrary σ-ideal on B. What are possible Baire orders of $\Phi(B, \mathcal{J})$?*

(2) *For which α do there exist (in ZFC) sets $Y, X \subset \mathbb{R}$ such that Baire order of $\Phi(Y, X)$ is α?*

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