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NOT EVERY γ-SET IS STRONGLY MEAGER

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ABSTRACT. We present two constructions of γ-sets which are large in sense of category.

1. INTRODUCTION

Most of the notation used in this paper is standard. By reals we mean the space 2^ω with the operation $+ \mod 2$. By rationals we mean the canonical dense subset of 2^ω,

$$\mathbb{Q} = \{x \in 2^\omega : \forall^\infty n \; x(n) = 0\}.$$

Finally, let $\mathcal{N}$ denote the ideal of measure zero subsets of 2^ω, with respect to the standard product measure in this space. $\mathbf{0}$ and $\mathbf{1}$ denote constant functions equal to 0 and 1 respectively.

Let us recall the following definitions:

Definition 1.1. (1) *A family $\mathcal{J} \subseteq P(X)$ is an ω-cover of X if for every finite set $F \subseteq X$ there exists $B \in \mathcal{J}$ such that $F \subseteq B$,*

(2) *A topological space X is a γ-set if for every $\mathcal{J}$, open ω-cover of X, there exists a family $\{D_n : n \in \omega\} \subseteq \mathcal{J}$ such that $X \subseteq \bigcup_m \bigcap_{n>m} D_n$,*

(3) *A set $X \subseteq \mathbb{R}$ is strongly meager if for every null set $G \subset \mathbb{R}$, $X + G \neq \mathbb{R}$.*

Galvin and Miller in [1] constructed a γ-set of reals of size continuum under Martin's Axiom. They showed that for every γ-set $X \subseteq \mathbb{R}$ and every meager set F, $X + F$ is meager. They asked whether the same is true for null sets. We give negative answer to this question.

Note that γ-sets are always meager, that is, if X is a γ-set then $X \cap P$ is meager in P for every perfect set $P \subseteq 2^\omega$ ([2]).

Since every γ-set is a strong measure zero set (see [2]), by a result of Laver it is consistent that every γ-set is countable so a sum with a null set is null.

On the other hand, for an ideal $\mathcal{I}$, Pawlikowski defined the cardinal coefficients $\mathsf{add}_t(\mathcal{I}) = \min\{|X| : \forall F \in \mathcal{I} \; X + F \notin \mathcal{I}\}$. He showed in [4], that $\mathsf{add}(\mathcal{N}) = \min\{\mathsf{add}_t(\mathcal{N}), \mathfrak{b}\}$. It is consistent that $\mathsf{add}(\mathcal{N}) < \mathfrak{p}$. Then $\mathsf{add}_t(\mathcal{N}) = \mathsf{add}(\mathcal{N}) < \mathfrak{p}$. So every subset of size $\mathsf{add}_t(\mathcal{N})$ is a γ-set, since all sets of size $< \mathfrak{p}$ are γ-sets ([3]), but there is a set X of size $\mathsf{add}_t(\mathcal{N})$ and a null set G with $X + G \notin \mathcal{N}$.

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2. Main results

In this section we present two constructions of γ-sets.

Theorem 2.1. *Assume* $\mathfrak{p} = 2^{\aleph_0}$. *Then there is a* γ*-set* $X \subseteq 2^\omega$ *which is not strongly meager.*

PROOF For $n \in \omega$ let $k_n = \sum_{i=0}^n 2^i$. Let $A_n = \{x \in 2^\omega : x[k_n, k_{n+1}) = \mathbf{0}\}$ for $n \in \omega$. Note that each set A_n is clopen and has measure measure $2^{-(n+1)}$. So $G = \bigcap_m \bigcup_{n>m} A_n$ is null.

We will construct a set $X' = \{x_\alpha : \alpha < 2^\omega\}$ such that

(1) $\forall \alpha\ x_\alpha \in 2^\omega$,
(2) $\forall \alpha\ \forall^\infty n\ x_{\alpha+1}(n) \leq x_\alpha(n)$,
(3) $X = X' \cup \mathbb{Q}$ is a γ-set,
(4) $\forall z \in 2^\omega\ \exists \alpha\ \exists^\infty n\ z[k_n, k_{n+1}) = x_\alpha[k_n, k_{n+1})$,
(5) $\forall \alpha\ \exists^\infty n\ x_\alpha[k_n, k_{n+1}) = \mathbf{1}$.

Note that if z and x_α are like in (4) then $z + x_\alpha \in G$. Therefore, condition (4) implies that $X + G = 2^\omega$.

Let $\mathcal{S} = \{\ddagger \in \in^\omega : \exists^\infty \backslash\ \ddagger[||\backslash, ||\backslash+\infty) = \mathbf{1}\}$. For $x \in 2^\omega$ let $[x]^\star = \{z \in \mathcal{S} : \forall^\infty n\ x(n) \geq z(n)\}$.

Lemma 2.2. *Let* $\mathcal{J}$ *be open* ω*-cover of* $\mathbb{Q}$ *and* $x \in \mathcal{S}$. *Then there is a sequence* $D_n \in \mathcal{J}$ *and* $y \in [x]^\star$ *such that* $\mathbb{Q} \cup [y]^\star \subseteq \bigcup_m \bigcap_{n>m} D_n$.

PROOF Using the fact that $\mathcal{J}$ is open ω-cover of $\mathbb{Q}$ we can find a sequence $\langle l_n : n \in \omega\rangle$ of natural numbers and a sequence $\langle D'_n : n \in \omega\rangle$ of elements of $\mathcal{J}$ such that for every n,

$$\forall z \in 2^\omega \left(z[n, l_n) = \mathbf{0} \rightarrow z \in D'_n\right).$$

Without loss of generality we can assume that there exists a set $Z \subseteq \omega$ such that

$$x[k_n, k_{n+1}) = \begin{cases} \mathbf{1} & \text{if } n \in Z \\ \mathbf{0} & \text{if } n \notin Z \end{cases}.$$

Choose $Y \subseteq \omega$ and $Z' \subseteq Z$ such that

$$\bigcup_{n \in Z'} [k_n, k_{n+1}) \cap \bigcup_{n \in Y} [n, l_n) = \emptyset.$$

Define

$$y[k_n, k_{n+1}) = \begin{cases} \mathbf{1} & \text{if } n \in Z' \\ \mathbf{0} & \text{if } n \notin Z' \end{cases}.$$

It is clear that $y \in [x]^\star$. Suppose that $z \in [y]^\star$. Note that

$$\forall^\infty n \in Y\ z[n, l_n) = \mathbf{0},$$

which means that $z \in D'_n$ for all except finitely many $n \in Y$. Thus, in order to finish the proof it is enough to define $D_n = D'_{y(n)}$, where $y(n)$ is the n-th element of Y. □

Lemma 2.3. *Suppose that* $\{x_\alpha : \alpha < \kappa < \mathfrak{p}\} \subseteq 2^\omega$ *is a sequence such that* $x_\alpha \in [x_\beta]^\star$ *for* $\alpha > \beta$. *Then* $\bigcap_{\alpha<\kappa} [x_\alpha]^\star \neq \emptyset$.

PROOF Define $Y_\alpha = \{n : x_\alpha[k_n, k_{n+1}) = \mathbf{1}\}$. Then $Y_\alpha \subseteq^\star Y_\beta$ if $\alpha > \beta$. Since $\kappa < \mathfrak{p}$, there is Y such that $Y \subseteq^\star Y_\alpha$ for each $\alpha < \kappa$. Define

$$x[k_n, k_{n+1}) = \begin{cases} \mathbf{1} & \text{if } n \in Y \\ \mathbf{0} & \text{if } n \notin Y \end{cases}$$

It is clear that $x \in \bigcap_{\alpha<\kappa}[x_\alpha]^\star$. □

Let $\{\mathcal{J}_\alpha : \alpha < 2^\omega\}$ be an enumeration of all ω-covers of $\mathbf{Q}$, and let $\{z_\alpha : \alpha < 2^\omega\}$ be enumeration of all elements of 2^ω.

Assume that the set $X_\alpha = \{x_\beta : \beta < \alpha\}$ has been already constructed. Assume that $\mathcal{J}_\alpha$ is ω-cover of $X_\alpha \cup \mathbf{Q}$. Since $|\alpha| < \mathfrak{p}$ and all sets of size $< \mathfrak{p}$ are γ-sets, we can choose $\mathcal{J}'_\alpha = \{U_n : n \in \omega\}$, ω-subcover of this such that

$$X_\alpha \cup \mathbf{Q} \subseteq \bigcup_m \bigcap_{n>m} U_n.$$

If α is limit then apply 2.3 to get a real $x'_\alpha \in \bigcap_{\beta<\alpha}[x_\beta]^\star$. If α is not limit let $x'_\alpha = x_{\alpha-1}$.

Next apply 2.2 to x'_α and $\mathcal{J}'_\alpha$ and x'_α to get a real y_α. Finally let x_α be such that

(1) $\exists^\infty n\ z_\alpha[k_n, k_{n+1}) = x_\alpha[k_n, k_{n+1})$,
(2) $\exists^\infty n\ x_\alpha[k_n, k_{n+1}) = \mathbf{1}$,
(3) $\forall^\infty n\ x_\alpha(n) \le y_\alpha(n)$.

This finishes the construction and the proof of the theorem. □

The set constructed above is a γ-set but it contains a subset which is not a γ-set. In the next theorem we will show how to build a set which is a hereditarily γ-set and is not strongly meager. The construction is a slight modification of Todorcevic's construction of a hereditarily γ-set from [1].

Theorem 2.4. *Assume $\diamondsuit$. Then there exists a hereditarily γ-set which is not strongly meager.*

PROOF For a tree $p \subseteq 2^{<\omega}$ let $[p]$ be the set of branches of p. Similarly, for a finite set $U \subseteq 2^{<\omega}$ let

$$[U] = \{x \in 2^\omega : \exists s \in U\ x\mathsf{dom}(s) = s\}.$$

Let $\{k_n : n \in \omega\}$ and G be the sequence and the set defined at the beginning of the proof of 2.1. Let $\mathcal{P}$ be the collection of all perfect trees p such that there exists a sequence $\{U_n : n \in \omega\}$ such that

(1) $U_n \subseteq 2^{[k_n,k_{n+1})}$ for all n,
(2) $\exists^\infty n\ U_n = 2^{[k_n,k_{n+1})}$,
(3) $[p] = \bigcap_{n\in\omega}[U_n]$.

By induction on levels we will build an Aronszajn tree consisting of elements of $\mathcal{P}$ ordered by inclusion. The γ-set we are looking for will be a selector from the elements of this tree.

For perfect trees p, q and a set $R \subseteq 2^n \cap q$ define

$$p \le_R q \Rightarrow p \cap 2^n = R \ \&\ p \subseteq q.$$

If $R = q \cap 2^n$ we write $p \le_n q$ instead of $p \le_{q\cap 2^n} q$.

Let $\{z_\alpha : \alpha < \omega_1\}$ be enumeration of 2^ω.

We will build by induction a partial ordering $\prec$ on ω_1, $\{p_\alpha : \alpha \in \omega_1\}$ and $\{x_\alpha : \alpha \in \omega_1\}$ such that

(1) $T = (\omega_1, \prec)$ is an Aronszajn tree and for limit α, $\bigcup_{\beta<\alpha} T_\beta = \alpha$,
(2) $p_\alpha \in \mathcal{P}$ for all α,
(3) $\forall \alpha, \beta \left(\alpha \prec \beta \iff p_\alpha \subseteq p_\beta\right)$,
(4) $x_\alpha \in [p_\alpha]$ for all α,
(5) $\exists^\infty n\ x_\alpha[k_n, k_{n+1}) = z_\alpha[k_n, k_{n+1})$,
(6) if $\mathcal{J}$ is an "appropriate" ω-cover then there exists α and a sequence $\langle D_n : n \in \omega\rangle$ of elements of $\mathcal{J}$ such that for all $p \in T_\alpha$ $[p] \subseteq \bigcup_m \bigcap_{n>m} D_n$,
(7) if $\beta > \alpha$ and $q \in T_\alpha$, $R \subseteq q \cap 2^m$ then there exists $p \in T_\beta$ such that $p \leq_R q$.

Note that the condition (7) guarantees that the construction will not terminate after countably many steps. Condition (6) is rather vague, but with the right interpretation of the word "appropriate", together with the condition (4) it will guarantee that the set $X = \{x_\alpha : \alpha < \omega_1\}$ is a hereditary γ-set. Finally (5) yields that $X + G = 2^\omega$.

For a set $Y \subseteq 2^\omega$ and a perfect tree p let $p(Y)$ be the tree representing the closure of $[p] \cap Y$.

We use $\diamondsuit$ to construct oracle sequences $\{\mathcal{J}_\alpha, X_\alpha, \langle p^\beta : \beta < \alpha\rangle : \alpha < \omega_1\}$ such that for any $Y \subseteq X$ and an open ω-cover of Y, $\mathcal{J}$, there are stationary many α's such that

(1) $\mathcal{J}_\alpha = \mathcal{J}$,
(2) $X_\alpha = \{x_\beta : \beta < \alpha\} \cap Y$,
(3) $p^\beta = p_\beta(Y)$ for $\beta < \alpha$.

Note that these sequences are easy to obtain from an ordinary $\diamondsuit$ sequence by coding $\mathcal{J}$'s and Y's by subsets of ω_1 (even ω in case of $\mathcal{J}$).

We will build the tree T (or $\prec$) by induction on levels. If $\alpha = \beta + 1$ and T_β is already constructed then T_α is any extension of T_β satisfying the requirements.

Suppose that α is a limit ordinal. We look for a sequence $\{D_n : n \in \omega\} \subseteq \mathcal{J}_\alpha$ such that

(i) $\forall y \in X_\alpha\ \forall^\infty n\ y \in D_n$,
(ii) $\forall \beta < \alpha\ \forall m\ \forall R \subseteq p_\beta \cap 2^m\ \exists \delta \left(p_\delta \leq_R p_\beta\ \&\ \forall^\infty n\ [p^\delta] \subseteq D_n\right)$.

If such a sequence $\{D_n : n \in \omega\}$ does not exist then T_α is an arbitrary extension of $\bigcup_{\beta<\alpha} T_\beta$.

Otherwise we fix a sequence $\{D_n : n \in \omega\}$ satisfying the above conditions then for every $\beta < \alpha$ and every $R \subseteq p_\beta \cap 2^m$ we build a chain $\{\delta_n : n \in \omega\} \subseteq \alpha$ and $\{l_n : n \in \omega\}$ such that

(1) $\delta_n \prec \delta_{n+1}$ for all n,
(2) $p_{\delta_{n+1}} \leq_{l_n} p_{\delta_n}$ for all n,
(3) $\bigcap_{n\in\omega} p_{\delta_n} \in \mathcal{P}$,
(4) $p_{\delta_0} \leq_R p_\beta$,
(5) $\forall^\infty n\ [p^{\delta_0}] \subseteq D_n$.

The branch $\{\delta_n : n \in \omega\}$ will be extended on level α by, say, ρ and the corresponding set is $p_\rho = \bigcap_{n\in\omega} p_{\delta_n}$. Note that in this way we extend only countably many branches.

Observe that condition (3) will follow from (2) if only the sequence l_n is increasing fast enough. Conditions (4) and (5) follow from the condition (ii) above. This concludes the construction. It remains to show that X is a hereditary γ-set.

Suppose that $Y \subseteq X$ and let $\mathcal{J}$ be an ω-cover of Y. Let α be a limit ordinal such that

(1) $\mathcal{J}_\alpha = \mathcal{J}$,
(2) $X_\alpha = \{x_\beta : \beta < \alpha\} \cap Y$,
(3) $p^\beta = p_\beta(Y)$ for $\beta < \alpha$.

To finish the proof it is enough to check that there exists a sequence $\{D_n : n \in \omega\} \subseteq \mathcal{J}_\alpha$ satisfying conditions (i) and (ii). In this case, according to the construction, $Y \subseteq \bigcup_m \bigcap_{n>m} D_n$.

Let $\{\langle \beta_n, R_n\rangle : n \in \omega\}$ be enumeration of the set

$$\{\langle \beta, R\rangle : \beta < \alpha \ \&\ \exists m\ R \subseteq p_\beta \cap 2^m\}.$$

It is enough to construct by induction a sequence $\{D_n : n \in \omega\} \subseteq \mathcal{J}_\alpha$ such that

(iii) $\forall i < n\ x_{\beta_i} \in D_n$,
(iv) $\forall i < n\ \exists \delta\ p_\delta \leq_{R_i} p_{\beta_i}\ \&\ [p^\delta] \subseteq D_n$.

We describe how to construct set D_n. For each $i < n$ and $s \in R_i$ choose $x^i_s \in Y$ (if possible) such that $s \subseteq x^i_s$. Let D_n be an element of $\mathcal{J}_\alpha$ such that

$$\{x_{\beta_i} : i < n\} \cup \{x^i_s : i < n, s \in R_i\} \subseteq D_n.$$

Such D_n exists since $\mathcal{J}_\alpha$ is an ω-cover of Y.

We need to verify condition (iv). Fix $i < n$ and choose m so large that $[x^i_s m] \subseteq D_n$ for all $s \in R_i$. If x^i_s does not exist let x_s be any element of $[p_{\beta_i}]$ extending s. Let $R = \{x^i_s m : s \in R_i\} \cup \{x_s m : s \in R_i\}$. By inductive hypothesis there exists δ such that $p_\delta \leq_R p_{\beta_i}$. Note that $[p^\delta] \subseteq \bigcup_{s\in R_i}[x^i_s m]$. Thus $[p^\delta] \subseteq D_n$, which finishes the construction and the proof. □

We can show the same theorems for the algebraic structure of the real line. Let us take a standard function $f : 2^\omega \to [0,1]$, $f(x) = \sum_{i\in\omega} x(i)/2^i$. f is continuous so $f(X)$ is a γ-set, where X is as in 2.1 or 2.4. Let $H = \bigcap_m \bigcup_{n>m} f(A_n)$. It is easy to see that $[0,1] \subseteq f(X) + H$. Let $G = H + \mathbb{Q}$ then $f(X) + G = \mathbb{R}$. □

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