BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 45, No. 3, 1997

SET THEORY

Metric Spaces not Distinguishing Pointwise and Quasinormal Convergence of Real Functions

by

Ireneusz RECŁAW(*)

Presented by C. BESSAGA on January 31, 1997

Summary. We show that there is (in ZFC) an uncountable wQN-set and that it is consistent that there is no uncountable, metric QN-space.

We say that a sequence of functions $f_k : X \to \mathbb{R}$ converges quasinormally to f ($f_k \xrightarrow{QN} f$) if there is a sequence $\varepsilon_n \to 0$ such that for each x there is k_0 such that for each $k > k_0$, $|f(x) - f_k(x)| < \varepsilon_k$. It easier to imagine this notion if we know the following characterisation.

$f_k \xrightarrow{QN} f$ iff $X = \bigcup_n X_n$ and f_k converges uniformly to f on X_n for each n.

We will consider spaces on which this convergence and pointwise convergence of continuous functions coincide (or are close to).

A topological space X is a QN-space if for each sequence of continuous functions $f_k : X \to \mathbb{R}$, if $f_k \to 0$ then $f_k \xrightarrow{QN} 0$.

A topological space X is a wQN-space if for each sequence of continuous functions $f_k : X \to \mathbb{R}$ with $f_k \to 0$ there is a subsequence k_l such that $f_{k_l} \xrightarrow{QN} 0$

A set $X \subset (0,1)$ is a QN-set (a wQN-set) if X with subspace topology is a QN-space (a wQN-space).

All countable spaces are QN-spaces and, of course, $QN \subset wQN$.

In [2] the authors introduced notions defined above and showed several

---

1991 MS Classification: 04E15, 54G99.
Key words: quasinormal convergence, QN-set, σ-set.

(*) Some of the results of this paper were obtained when the author was visiting University of Scranton.

properties and constructed a few examples of such spaces. In particular, we showed that there is a uncountable QN-space with subspace which is not a QN-space. The real line is not wQN-space but under some additional set-theoretical assumptions, for example Continuum Hypothesis, there are uncountable subsets of the reals which are QN-spaces. However, we left open problem whether there are uncountable examples of such sets in ZFC. Below we solve this problem.

A subset X is a σ-space if every G_δ in X is F_σ in X.

THEOREM 1. *Every metric QN-space is σ-space.*

Proof. Let $w_n : \mathbb{R} \to [0,1]$ be a sequence of continuous function such that $w_n(x) = 0$ if $x \leqslant 0$ or $x \geqslant 1/n$, and $w_n(x) = 1$ if $\frac{1}{n+2} \leqslant x \leqslant \frac{1}{n+1}$. Let $G = \bigcap_k U_k$ where U_k are open. We define $f_k = \Sigma_i \frac{1}{2^i} w_k(\mathrm{dist}\,(x, U_i^c))$. $f_k \to 0$. Assume that X is a QN-space. Let $\varepsilon_k \to 0$ such that for each $x \in X$ there is k_0 such that for each $k > k_0$, $|f_k(x)| \leqslant \varepsilon_k$. Then $X = \bigcup_n A_n$ where $A_n = \{x : \forall_{k \geqslant n} |f_k(x)| \leqslant \varepsilon_k\}$. Since f_k are continuous, A_n are closed.

We show now that for each i and n, $A_n \cap U_i$ is closed. Fix i_0. There is k_0 such that for each $k \geqslant k_0$, $\varepsilon_k < \frac{1}{2^{i_0}}$. Let $x \in A_n$. $f_k(x) \geqslant \frac{1}{2^{i_0}} w_k(\mathrm{dist}\,(x, U_{i_0}^c))$ so for $k \geqslant k_0$ if $f_k(x) < \varepsilon_k$ then $\mathrm{dist}\,(x, U_{i_0}) \notin [\frac{1}{k+2}, \frac{1}{k+1}]$. Thus for $x \in A_n$, $\mathrm{dist}\,(x, U_{i_0}) = 0$ or $\mathrm{dist}\,(x, U_{i_0}) > \frac{1}{k_0+1}$. Then $A_n \cap U_{i_0} = \{x \in A_n : \mathrm{dist}\,(x, U_{i_0}) \geqslant \frac{1}{k_0+1}\}$ is closed. Hence $A_n \cap G$ is closed. Thus $\bigcup_n (A_n \cap G) = G$ is F_σ-set.

THEOREM 2. *It is consistent that there is no uncountable metric QN-space.*

Proof. A. Miller showed (see [5]) that it is consistent that there is no uncountable σ-space of the reals. Hence in this model there is no uncountable QN-set. Suppose that in this model there is an uncountable, metric QN-space X. Assume first that X is separable. X is zero-dimensional (see Corollary 4.6, [2]) so X is homeomorphic to a subset of the reals. It is a contradiction. Thus let us assume that X is not separable. Let X_k be a maximal subset of X such that for each $x, y \in X_k$, $\mathrm{dist}\,(x, y) \geqslant 1/(k+1)$. Then $\bigcup_k X_k$ is dense in X so there is a k such that X_k is uncountable. X_k is closed subset of X so it is also QN-space (see Theorem 4.1 [2]). Let $f : X_k \to \mathbb{R}$ be any map onto a uncountable subset of the real line. f is continuous so $f[X_k]$ is also QN-space. It is a contradiction.

COROLLARY 3. *Every subspace of a metric QN-space is a QN-space.*

Proof. See Corollary 5.5 [2].

In particular, it simplifies the proof of Theorem 6.4, from [2] that under $\mathbf{p} = \mathbf{c}$ there is a wQN-set which is not QN-set. In [GM], under this assumption that there is a γ-set (so wQN-set) such that it contains a countable set which is not relative G_δ in this set so the set is not σ-set.

In [4] the authors introduced the definition of $S_1(\Gamma,\Gamma)^*$. We say that $\mathcal{U}$ is γ-cover if it is infinite and for each x in X the set $\{U \in \mathcal{U} : x \notin U\}$ is finite.

A set $X \subset \mathbb{R}$ is in $S_1(\Gamma,\Gamma)^*$ if for given any $\{\mathcal{U}_n : n \in \omega\}$ a sequence of γ-covers of X, there exists $\{V_n \in \mathcal{U}_n : n \in \omega\}$ and a countable $Y \subset X$ such that $\{V_n : n \in \omega\}$ is γ-cover of $X \setminus Y$.

THEOREM 4. *$S_1(\Gamma,\Gamma)^* \subset wQN$.*

P r o o f. Let f_k be a sequence of continuous functions on X converging to 0. Let $\mathcal{U}_n = \{f_k^{-1}(-1/n, 1/n) : k \in \omega \setminus n\}$. It is easy to see that $\mathcal{U}_n$ are γ-covers.

If $X \in S_1(\Gamma,\Gamma)^*$ then there is a sequence $(V_n : n \in \omega)$ with $V_n \in \mathcal{U}_n$ and a countable set $Y \subset X$ such that $X \setminus Y \subset \bigcup_m \bigcap_{n>m} V_n$. $V_n = f_{k_n}^{-1}(-1/n, 1/n)$. We can assume that k_n is increasing. Then $\forall_{x \in X\setminus Y} \forall_n^\infty |f_{k_n}(x)| < 1/n$. Now we choose subsequence k_{n_l} such that $\forall_{x\in Y} \forall_l^\infty |f_{k_{n_l}}(x)| < 1/l$, so $\forall_{x \in X} \forall_l^\infty |f_{k_{n_l}}(x)| < 1/l$.

In [4], the authors showed that if $\mathbf{b} = \omega_1$ then there is an uncountable set of reals in $S_1(\Gamma,\Gamma)^*$.

If $\mathbf{b} > \omega_1$ then every set of size ω_1 is QN-space (see [2]).

COROLLARY 5. *There is an uncountable wQN-set.*

INSTITUTE OF MATHEMATICS, UNIVERSITY OF GDAŃSK, WITA STWOSZA 57, 80-952 GDAŃSK, POLAND
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SCRANTON, SCRANTON, PA 18510 USA
E-mail: matir@paula.univ.gda.pl

REFERENCES

[1] L. B u k o v s k y, *Thin sets related to trigonometric series*, Israel Mathematical Conference Proceedings, **6** (1993) 107–118.

[2] L. B u k o v s k y, I. R e c ł a w, M. R e p i c k y, *Spaces not distinguishing pointwise and quasinormal convergence of real functions*, Topol. Appl., **41** (1991) 25–40.

[3] F. G a l v i n, A. W. M i l l e r, *γ-sets and other singular sets of real numbers*, Topol. Appl., **17** (1984) 145–155.

[4] W. J u s t, A. W. M i l l e r, M. S c h e e p e r s, P. J. S z e p t y c k i, *The combinatorics of open covers (II)*, preprint.

[5] A. W. M i l l e r, *Special subsets of the real line*, in: *Handbook of Set Theoretic Topology*, ed.: K. K u n e n, J. E. V a u g h a n, North-Holland, Amsterdam, (1984) 201–235.