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Universal summands for families of measurable functions

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Abstract. (1) For any $\alpha < \omega_1$ there exists a Borel measurable function $g: \mathbb{R} \to \mathbb{R}$ such that $g + f$ is a Darboux function (is almost continuous in the sense of Stallings) for every $f \in B_\alpha$. This solves a problem of J. Ceder.

(2) There is a function g that is universally measurable and has the Baire property in restricted sense such that $g + f$ is Darboux for every Borel measurable function f.

(3) There is $g: \mathbb{R} \to \mathbb{R}$ such that $f + g$ is extendable for each $f: \mathbb{R} \to \mathbb{R}$ that is Lebesgue measurable (has the Baire property).

(4) For every $\alpha < \omega_1$, each $f \in B_\alpha$ is the sum of two extendable functions $f_1, f_2 \in B_\alpha$. This answers a question of A. Maliszewski.

1. Preliminaries

Our terminology is standard. By $\mathbb{R}$ and $\mathbb{I}$ we denote the set of all reals and the interval $[0, 1]$, respectively. If A is a planar set, we denote its x-projection by $\operatorname{dom}(A)$ and y-projection by $\operatorname{rng}(A)$. The symbol $|X|$ stands for the cardinality of a set X. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$. All notions and properties of Borel and projective sets that we use can be found in [AK].

For $A \subset X \times Y$ and $x \in X$ we denote by A_x the *x-section* of A, i.e., the set $A_x = \{y \in Y: (x, y) \in A\}$. Similarly, if $f: X \times Y \to Z$ and $x \in X$, then f_x denotes the *x-section* of f, i.e., $f_x(y) = f(x, y)$ for each $y \in Y$.

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Let C be a Cantor set (i.e., a non-empty nowhere dense perfect subset of $\mathbb{R}$). We say that $A \subset C \times X$ is a *"C-universal" set* for a class $\mathcal{P}$ of subsets of X iff $\mathcal{P} \subset \{A_x : x \in C\}$. Similarly, we say that $f: C \times X \to Y$ is a *"C-universal" function* for a family $\mathcal{F}$ of functions from X to Y iff $\mathcal{F} \subset \{f_x : x \in C\}$.

We shall deal mainly with real functions of one real variable. No distinction is made between a function and its graph. By C we denote the class of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. For $X = \mathbb{R}$ or $X = \mathbb{R}^2$ the symbols $\mathcal{L}(X)$, $\mathcal{N}(X)$, $\mathcal{K}(X)$ and $\mathcal{M}(X)$ stand for the families of Lebesgue measurable sets, measure zero sets, Baire sets and meager sets, respectively. Moreover, the symbols $\mathcal{L}(X : \mathbb{R})$ and $\mathcal{K}(X : \mathbb{R})$ denote the families of all Lebesgue measurable functions and of all functions with the Baire property from X to $\mathbb{R}$.

We shall consider the following classes of functions from X to Y:

- D – f is a *Darboux function* if $f(C)$ is connected whenever C is connected in X;
- Conn – f is a *connectivity function* if the graph of f restricted to C, denoted by $f{\restriction}C$, is connected in $X \times Y$ whenever $C \subset X$ is connected;
- ACS – f is an *almost continuous function* in the sense of Stallings, if U is an open subset of $X \times Y$ containing the graph of f, then U contains the graph of a continuous function $g: X \to Y$ [JS];
- Ext – f is an *extendable function* if there exists a connectivity function $g: X \times \mathbb{I} \to Y$ such that $f(x) = g(x, 0)$ for all $x \in X$ [JS].

Recall that for $X = Y = \mathbb{R}$ we have the following chain of proper inclusions

$$\mathrm{C} \subset \mathrm{Ext} \subset \mathrm{ACS} \subset \mathrm{Conn} \subset \mathrm{D} \subset \mathbb{R}^{\mathbb{R}}.$$

For a given family $\mathcal{F}$ of real functions we can examine the following question: *For which families $F \subset \mathbb{R}^{\mathbb{R}}$ does there exist a $\mathcal{F}$-universal summand, i.e., such $g \in \mathbb{R}^{\mathbb{R}}$ that $f + g \in \mathcal{F}$ for all $f \in F$?*

This question was considered by many authors. The first result in this direction was obtained by H. Fast in 1959.

Theorem 1.1. [HF]. *If F is a family of functions and $|F| \leq \mathfrak{c}$, then there exist a* D*-universal summand for F.*

The analogous theorem for the class of all almost continuous functions was obtained in 1974 by K. Kellum [KK]. Those results were generalized in 1994 by K. Ciesielski and A. Miller in [CMi] and, for extendable functions, in 1995 by K. Ciesielski and I. Recław [CR].

D-universal summands for families of Borel measurable functions were studied by J. Ceder in [JC]. He proved that for every $\alpha < \omega_1$ and for any countable family

$F \subset B_\alpha$ there exists a D-universal summand for F that is Borel measurable. Moreover, he asked whether the analogous theorem is valid for every $F \subset B_\alpha$ with $|F| \leq \mathfrak{c}$, i.e., whether there is a Borel measurable D-summand for the class B_α. We shall answer this question in the affirmative. Moreover, we shall prove the analogous theorem for ACS-summands.

By Fast's Theorem, there exist D-universal summands for the family B of all Borel measurable functions. It is easy to observe that every such summand does not belong to B. We shall show that such a summand can be universally measurable and with the Baire property in restricted sense.

Finally, it is easy to observe that there is no D-universal summand for the family $\mathbb{R}^{\mathbb{R}}$. On the other hand, there are D-universal summands for big "regular" families of functions. In particular, there are ACS-universal summands for the family $\mathcal{L}(\mathbb{R} : \mathbb{R})$ of all Lebesgue measurable functions and for the family $\mathcal{K}(\mathbb{R} : \mathbb{R})$ of all functions with the Baire property [TN]. We shall prove that there are Ext-universal summands for $\mathcal{L}(\mathbb{R} : \mathbb{R})$ and $\mathcal{K}(\mathbb{R} : \mathbb{R})$.

2. D-universal summands

Let $A(X)$ be one of the classes: Σ^0_α for $\alpha < \omega_1$ or Σ^1_n for $n < \omega$. Then let $\underline{\mathrm{M}}A(X) = \{f: X \to [0,1]: (\forall c \in [0,1))\ f^{-1}((c,1]) \in A(X)\}$. J. Cichoń and M. Morayne proved the following theorem:

Theorem 2.1. [CM]. *There is $f \in \underline{\mathrm{M}}A(C \times C)$ that is "C-universal" for the class $\underline{\mathrm{M}}A(C)$.*

From this theorem we can easily deduce the following two lemmas.

Lemma 2.2. *Let C be a Cantor set. For each $\alpha < \omega_1$ there is a function $f: C \times \mathbb{R} \to \mathbb{R}$ of the Baire class $\alpha + 2$ that is "C-universal" for the class B_α.*

Lemma 2.3. *Let C be a Cantor set. For each $n \in \omega$ there is a $\sigma(\Sigma^1_n)$-measurable function $f: C \times \mathbb{R} \to \mathbb{R}$ that is "C-universal" for the class of all Σ^1_n-measurable functions.*

Theorem 2.4. *For every $\alpha < \omega_1$ there is a function $h \in B_{\alpha+2}$ such that for each function $g \in B_\alpha$ the set $\{x \in \mathbb{R}: h(x) = g(x)\}$ is dense in $\mathbb{R}$.*

Proof. Let $\{U_n : n \in \omega\}$ be a basis of $\mathbb{R}$. Let $\{P_n : n \in \omega\}$ be a family of pairwise disjoint perfect sets with $P_n \subset U_n$ for each n. Fix $n \in \omega$. By Lemma 2.2, there exists $f_n : P_n \times \mathbb{R} \to \mathbb{R}$ that is "P_n-universal" for the class B_α and that belongs to the class $B_{\alpha+2}$. Let $h_n : P_n \to \mathbb{R}$ be the diagonal of f_n, i.e., $h_n(x) = f_n(x,x)$ for $x \in P_n$. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} h_n(x) & \text{for } x \in P_n, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to observe that $h \in B_{\alpha+2}$. Moreover, for each $n \in \omega$ and $g \in B_\alpha$ there is $x \in P_n \subset U_n$ with $f_n(x,x) = g(x)$, so $h(x) = g(x)$. ∎

Analogously we can prove the following theorem.

Theorem 2.5. *For each $n \in \omega$ there is a $\sigma(\Sigma^1_n)$-measurable function $h : \mathbb{R} \to \mathbb{R}$ such that for every Σ^1_n-measurable $g : \mathbb{R} \to \mathbb{R}$ the set $\{x \in \mathbb{R} : h(x) = g(x)\}$ is dense in $\mathbb{R}$.*

Corollary 2.6. *For each $\alpha < \omega_1$ there is $k \in B_{\alpha+2}$ that is a D-universal summand for the class B_α.*

Proof. Put simply $k = -h$, where h is defined in Theorem 2.4. Then k is a D-universal summand for the class B_α. Indeed, fix $g \in B_\alpha$. Then for each $r \in \mathbb{R}$, $g - r \in B_\alpha$. Thus $\{x \in \mathbb{R} : r = g(x) + k(x)\} = \{x \in \mathbb{R} : h(x) = g(x) - r\}$ is dense in $\mathbb{R}$, so $g + k$ is Darboux. ∎

Observe that if k is D-universal summand for the class B_α, $\alpha > 0$, then $k \notin B_\alpha$. Indeed, if there were $k \in B_\alpha$ such that $f + k$ were Darboux for each $f \in B_\alpha$, then the function $h \in \mathbb{R}^\mathbb{R}$ defined by $h(x) = -k(x)$ for $x \neq 0$ and $h(0) = -k(0) + 1$ belongs to B_α, but $k + h$ fails to be Darboux. We are unable to determine whether a D-universal summand for the class B_α can be found in the class $B_{\alpha+1}$.

Theorem 2.5 yields the following result.

Corollary 2.7. *For each $n \in \omega$ there is a D-universal summand for the class of Σ^1_n-measurable functions that is $\sigma(\Sigma^1_n)$-measurable.*

Recall that every $\sigma(\Sigma^1_1)$-measurable function is universally measurable and has the Baire property in restricted sense (see, e.g., [AK, Theorem 21.10, p. 155]) and every Borel measurable function is Σ^1_1-measurable. Thus we obtain the following corollary.

Corollary 2.8. *There is a universally measurable and with the Baire property in restricted sense function* $k:\mathbb{R}\to\mathbb{R}$ *such that for each Borel function* $g:\mathbb{R}\to\mathbb{R}$, $k+g$ *is Darboux.*

3. ACS-universal summands

Recall that if f intersects all closed subsets K of $\mathbb{R}^2$ with $\operatorname{dom}(K)$ being a non-degenerate interval and $\operatorname{rng}(K)=\mathbb{R}$, then f is almost continuous [KK]. In this paper every such set is called a *blocking set.*

Theorem 3.1. *For each* $\alpha<\omega_1$ *there is a Borel function* $k:\mathbb{R}\to\mathbb{R}$ *such that it is an* ACS*-universal summand for the class* B_α.

Proof. Let $\{U_n: n\in\omega\}$ be a basis of $\mathbb{R}$ and let $\{P_n: n\in\omega\}$ be a family of pairwise disjoint perfect sets with $P_n\subset U_n$ for each n. Fix $n\in\omega$. Let $\varphi_n: P_n\to P_n\times P_n$, $\varphi_n=(\varphi_n',\varphi_n'')$, be a homeomorphism, $k_n: P_n\times\mathbb{R}\to\mathbb{R}$ be a Borel measurable, "P_n-universal" function for the class B_α (see Lemma 2.2) and let $D_n\subset P_n\times\mathbb{R}\times\mathbb{R}$ be a closed "P_n-universal" set for closed sets in $\mathbb{R}\times\mathbb{R}$. (See e.g., [AK, Theorem 22.3, p. 168].)

Define

$$W_n=\{(p,y)\in P_n\times\mathbb{R}\colon (\varphi_n''(p),p,y+k_n(\varphi_n'(p),p))\in D_n\}$$

and observe that W_n is Borel measurable, since it is the inverse image of the closed set D_n under the Borel map $(p,y)\mapsto(\varphi_n''(p),p,y+k_n(\varphi_n'(p),p))$. Additionally, all sections of W_n are closed, so σ-compact. Thus Arsenin–Kunugui's Theorem ([AK, Theorem 18.18, p. 127.]) implies that W_n has a Borel uniformization, i.e., there exists $u_n:\operatorname{dom}(W_n)\to\mathbb{R}$ with $u_n\subset W_n$. Let $u:\mathbb{R}\to\mathbb{R}$ be any Borel extension of all u_n, $n\in\omega$.

We shall show that u is an ACS-universal summand for the class B_α. So, fix $f\in B_\alpha$ and a blocking set $F\subset\mathbb{R}^2$. Then there are $n\in\omega$ and $q,r\in P_n$ such that $P_n\subset\operatorname{dom}(F)$, $(k_n)_q=f$ and $(D_n)_r=F$. Then for $p=\varphi_n^{-1}(q,r)$ we have $k_n(\varphi_n'(p),p)=f(p)$ and $(D_n)_{(\varphi_n''(p),p)}=F_p$. Since F_p is non-empty, $u(p)\in F_p-f(p)$. Consequently, $(p,u(p)+f(p))\in F$. Thus $u+f$ intersects each blocking sets, so it is almost continuous. ∎

4. Ext-universal summands

Lemma 4.1. *There exists a family $\mathcal{E}$ of pairwise disjoint perfect sets such that* $|\{E \in \mathcal{E}\colon E \subset A\}| = \mathfrak{c}$ *for each* $A \in \mathcal{L}(\mathbb{R}) \setminus \mathcal{N}(\mathbb{R})$.

Proof. Let $\varphi\colon \mathbb{R} \to \mathbb{R}^2$ be a Borel isomorphism that maps the ideal $\mathcal{N}(\mathbb{R})$ onto $\mathcal{N}(\mathbb{R}^2)$. (See, e.g., [AK, Theorem 17.41. p. 116].) Let $\{G_\alpha\colon \alpha < \mathfrak{c}\}$ be the family of all Borel subsets of $\mathbb{R}$ that does not belong to $\mathcal{N}(\mathbb{R})$. For every $\alpha < \mathfrak{c}$, $\varphi(G_\alpha) \notin \mathcal{N}(\mathbb{R}^2)$, so $A_\alpha = \{x\colon |(\varphi(G_\alpha))_x| = \mathfrak{c}\}$ is of the size $\mathfrak{c}$. For each α choose $x_\alpha \in A_\alpha$ such that $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$, and put $B_\alpha = \{x_\alpha\} \times (\varphi(G_\alpha))_{x_\alpha}$. Since $\varphi^{-1}(B_\alpha)$ is a Borel set in $\mathbb{R}$ and $|\varphi^{-1}(B_\alpha)| = \mathfrak{c}$, there is a perfect set $C_\alpha \subset \varphi^{-1}(B_\alpha) \subset G_\alpha$. Note that $C_\alpha \cap C_\beta \subset \varphi^{-1}(B_\alpha \cap B_\beta) = \emptyset$ whenever $\alpha \neq \beta$. Finally decompose each C_α onto $\mathfrak{c}$ many perfect sets $E_{\alpha,\beta}$, $\beta < \mathfrak{c}$, and put $\mathcal{E} = \{E_{\alpha,\beta}\colon \alpha, \beta < \mathfrak{c}\}$. ∎

Theorem 4.2. *There exists an* Ext*-universal summand for the family* $\mathcal{L}(\mathbb{R} : \mathbb{R})$.

Proof. Let $B = \{f_\beta\colon \beta < \mathfrak{c}\}$ be the family of all Borel measurable functions from $\mathbb{R}$ to $\mathbb{R}$ and let $\{G_\alpha\colon \alpha < \mathfrak{c}\}$ be the family of all Borel sets $G \subset \mathbb{R}$ such that $\mathbb{R} \setminus G \in \mathcal{N}(\mathbb{R})$. Applying Lemma 4.1 choose a family of pairwise disjoint F_σ c-dense sets $E_\alpha \subset G_\alpha$, $\alpha < \mathfrak{c}$. Next divide each E_α onto $\mathfrak{c}$ many F_σ c-dense sets $E_{\alpha,\beta}$, $\beta < \mathfrak{c}$. By [CR, Corollary 3.4], [WG, Lemma 4] and [TN1, Lemma 2], for each pair $(\alpha,\beta) \in \mathfrak{c} \times \mathfrak{c}$ there exists $g_{\alpha,\beta} \in \mathrm{Ext}$ such that $\mathbb{R} \setminus E_{\alpha,\beta}$ is $g_{\alpha,\beta}$-negligible, i.e., every $f\colon \mathbb{R} \to \mathbb{R}$ with $f \restriction E_{\alpha,\beta} = g_{\alpha,\beta} \restriction E_{\alpha,\beta}$ is an extendable function. Define $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} g_{\alpha,\beta}(x) - f_\beta(x) & \text{for } x \in E_{\alpha,\beta},\ \alpha, \beta < \mathfrak{c}, \\ 0 & \text{otherwise.} \end{cases}$$

We shall verify that g is an Ext-universal summand for the class $\mathcal{L}(\mathbb{R} : \mathbb{R})$. For fixed $f \in \mathcal{L}(\mathbb{R} : \mathbb{R})$ choose $\alpha, \beta < \mathfrak{c}$ such that $A = \{x \in \mathbb{R}\colon f(x) \neq f_\beta(x)\} \in \mathcal{N}(\mathbb{R})$ and $G_\alpha \subset \mathbb{R} \setminus A$. Then $f(x) + g(x) = g_{\alpha,\beta}(x)$ for $x \in E_{\alpha,\beta}$, and therefore $f + g$ is an extendable function. ∎

Remark. Note that an Ext-universal summand for the family $\mathcal{L}(\mathbb{R} : \mathbb{R})$ cannot be measurable.

Analogously we can prove the following result.

Theorem 4.3. *There exists an* Ext-*universal summand for the family of all functions possessing the Baire property.*

Problem 4.4. *Let* $0 < \alpha < \omega_1$*. Does there exist a Borel measurable* Ext-*universal summand for the class* B_α*?*

Finally, note that Fast's theorem implies an old result of Lindenbaum.

Theorem 4.5. [AL]. *Every function from* $\mathbb{R}$ *to* $\mathbb{R}$ *can be expressed as the sum of two Darboux functions.*

Indeed, let g be a D-universal summand for the family $\{f, 0\}$. Then $g = g + 0$ and $g + f$ are Darboux, and $f = (g + f) - g$. Similarly, Ceder's theorem implies that every Borel measurable functions can be expressed as the sum of two Borel measurable Darboux functions, and, by Kellum's theorem, every functions is the sum of two almost continuous functions. Note that Theorem 3.1 implies that every Borel measurable function can be written as the sum of two Borel measurable almost continuous functions. This solves Problem 1.3 from [AM]. However, applying [CR, Corollary 3.4], we can easily obtain a more exact result.

Theorem 4.6. *For every* $\alpha < \omega_1$ *and for each* $f \in B_\alpha$ *there are two extendable functions* $f_1, f_2 \in B_\alpha$ *with* $f = f_1 + f_2$*. Similarly, every* $f \in \mathcal{L}(\mathbb{R} : \mathbb{R})$ $(f \in \mathcal{K}(\mathbb{R} : \mathbb{R}))$ *can be written as the sum of two extendable functions* $f_1, f_2 \in \mathcal{L}(\mathbb{R} : \mathbb{R})$ $(f_1, f_2 \in \mathcal{K}(\mathbb{R} : \mathbb{R}))$.

Proof. Fix $f \in B_\alpha$. If $\alpha = 1$, then f is the sum of two Darboux functions $f_1, f_2 \in B_1$ [BCK], and since $\mathrm{Ext} \cap B_1 = \mathrm{D} \cap B_1$ [BHL], f_1 and f_2 are extendable. So assume that $\alpha \geq 2$. Let $\hat{f}\colon \mathbb{R} \to \mathbb{R}$ be the function constructed in [CR, Corollary 3.4]. It is easy to observe that $\hat{f} \in B_2$. Since $\hat{f}$ is dense in $\mathbb{R}^2$, there exists an F_σ set $E \subset \mathbb{R}$ such that every $g\colon \mathbb{R} \to \mathbb{R}$ with $\hat{f}\restriction E = g\restriction E$ is extendable. (See [HR].) Let $h\colon \mathbb{R} \to \mathbb{R}$ be a homeomorphism such that $h(E) \cap E = \emptyset$. (See [WG].) Then $\hat{g} = \hat{f} \circ h^{-1}$ is an extendable function of the Baire class two and every $g\colon \mathbb{R} \to \mathbb{R}$ with $g\restriction h(E) = \hat{g}\restriction h(E)$ is an extendable function. (See [TN1, Lemma 2].) Consequently the functions

$$f_1(x) = \begin{cases} \hat{f}(x) & \text{for } x \in E, \\ f(x) - \hat{g}(x) & \text{for } x \in h(E), \\ f(x) & \text{otherwise,} \end{cases} \qquad f_2(x) = \begin{cases} f(x) - \hat{f}(x) & \text{for } x \in E, \\ \hat{g}(x) & \text{for } x \in h(E), \\ 0 & \text{otherwise} \end{cases}$$

are extendable and belong to the class B_α, and $f = f_1 + f_2$. ∎

Note that in the analogous way we can prove that for every $\alpha > 1$ and for each countable family of B_α functions there exists an Ext-universal summand in the class B_α.

Remark. Recently Sławomir Solecki proved the following result*: *For every* $1 \leq \alpha < \omega_1$ *there exists* $f\colon \mathbb{R} \to \mathbb{R}$ *in* B_α *such that* $f + g \in \mathrm{Ext}$ *for any* $g\colon \mathbb{R} \to \mathbb{R}$ *in* $\bigcup_{\gamma<\alpha} B_\gamma$.

This theorem improves our results. In particular, it answers Problem 1.

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