Tatra Mt. Math. Publ. **28** (2004), 71–73

TATRA MOUNTAINS
Mathematical Publications

ON NON-MEASURABLE UNIONS OF SECTIONS OF A BOREL SET

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ABSTRACT. Under Continuum Hypothesis we prove that for any Borel set on the plane with sections of measure zero and the union of all sections of positive outer measure there is a family of sections with non-measurable union.

Assume that we have a family of Lebesgue null subsets of the real line with the union of positive outer measure. We will investigate the problem when there is a subfamily with the non-measurable union. Under the Continuum Hypothesis there is a family of null sets such that all unions of subfamilies are either null or equal to the whole real line. We just take $A_\alpha = \{x_\beta : \beta < \alpha\}$ for $\alpha < \omega_1$ where $\mathbb{R} = \{x_\alpha : \alpha < \omega_1\}$. However it is consistent that all such families satisfy the thesis. So in general we have to consider some additional conditions for families of null sets. It was proved that for families of pairwise disjoint sets (see [B]), point-finite (i.e., each point belongs to finitely many elements of the family, see [BCGR]) the thesis holds. Point-countable families problem is independent from ZFC.

In this paper we will consider families which are families of all sections of a Borel set of the plane.

J. Cichoń, M. Morayne, R. Rałowski and C. Ryll-Nardzewski (see [CMRR]) proved that there is a subset A of the Cantor set C such that $A + C$ is non-measurable. J. Cichoń asked the following question (private communication): Assume that we have a perfect null set D with $D + D$ of positive measure. Does there exist $A \subset D$ such that $A + D$ is non-measurable? Since $A + D = \bigcup_{x \in A} B_x$, where $B = -^{-1}[-D] \cap (D \times \mathbb{R})$, the question is a special case of our general problem. We solve the problem under additional set-theoretic assumptions such as the Continuum Hypothesis.

The following fact is well known.

2000 Mathematics Subject Classification: 03E15.
Keywords: non-measurable unions, Borel set.

Fakt 1. Assume the Continuum Hypothesis (or Martin's Axiom). Then for each null Borel set C on the plane there are sets X, Y of full outer measure such that $X \times Y \cap C = \emptyset$.

THEOREM 1. *Assume the Continuum Hypothesis (or Martin's Axiom). Then for each Borel set* $B \subset \mathbb{R}^2$ *such that* B_x *is null for each* x *and* $\bigcup_{x \in \mathbb{R}} B_x$ *is of positive outer measure there is* $A \subset \mathbb{R}$ *such that* $\bigcup_{x \in A} B_x$ *is not measurable.*

Proof. We can assume that $\bigcup_{x \in \mathbb{R}} B_x = \mathbb{R}$. There is a measurable function $f\colon \mathbb{R} \to \mathbb{R}$ such that $\{(f(x), x)\colon x \in \mathbb{R}\} \subset B$ (see Theorem 18.1 in [K]). We define $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (f(x), y)$. Then let $C = F^{-1}[B]$. C is Lebesgue measurable and $C_a = B_{f(a)}$ for each a, so C is null. Now we apply the fact above and we get $X, Y \subset \mathbb{R}$ of full outer measure such that $(X \times Y) \cap C = \emptyset$. We claim that $\bigcup_{x \in f[X]} B_x$ is non-measurable. Firstly observe that for $a \in X$ $(f(a), a) \in B$, so $a \in B_{f(a)}$. Thus $X \subset \bigcup_{x \in f[X]} B_x$. Next for $y \in Y$ and $a \in X$ $(a, y) \notin C$, so $(f(a), y) \notin B$. Thus $Y \cap \bigcup_{x \in f[X]} B_x = \emptyset$. So $\bigcup_{x \in f[X]} B_x$ is non-measurable.

The thesis of Fact 1 was proved to be independent (see [BS]). So there is a chance that Theorem 1 is independent in ZFC. The similar fact and the similar proof operate for category so we have the following Theorem. □

THEOREM 2. *Assume the Continuum Hypothesis. Then for each Borel set* $B \subset \mathbb{R}^2$ *such that* B_x *is meager for each* x *and* $\bigcup_{x \in \mathbb{R}} B_x$ *is non-meager there is* $A \subset \mathbb{R}$ *such that* $\bigcup_{x \in A} B_x$ *does not have the Baire property.*

REFERENCES

[BGJS] BARTOSZYŃSKI, R.—GOLDSTERN, M.—JUDAH, H.—SHELAH, S.: *All meager filters may be null*, Proc. Amer. Math. Soc. **117** (1993), 515–521.

[BCGR] BRZUCHOWSKI, J.—CICHOŃ, J.—GRZEGOREK, E.—RYLL-NARDZEWSKI, C.: *On the existence of nonmeasurable unions*, Bull. Acad. Pol. Sci. Ser. Sci. Math. **27** (1979), 447–448.

[B] BUKOVSKÝ, L.: *Any partition into Lebesgue measure zero sets produces a non-measurable set*, Bull. Acad. Pol. Sci. Ser. Sci. Math. **27** (1979), 431–435.

[CMRR] CICHOŃ, J.—MORAYNE, M.—RAŁOWSKI, R.—RYLL-NARDZEWSKI, C.: *On complex unions of subsets of the Cantor set*, preprint.

[F] FREMLIN, D.: *Measure-additive coverings and measurable selectors*, Dissertationes Math. (Rozprawy Mat.) **260** (1987).

[K] KECHRIS, A.: *Classical Descriptive Set Theory*. Graduate Texts in Mathematics, Vol. 156, Springer-Verlag, New York, 1995.

Received October 28, 2002

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