WHITNEY TOWERS IN 4-MANIFOLDS

Peter Teichner

14th Andrzej Jankowski Memorial Lecture

Gdansk, June 2012

New results joint with Jim Conant (University of Tennessee, Knoxville) and Rob Schneiderman (Lehman College, New York City)

Six Papers available, a survey appeared in the

- Some History
- Main Questions
- Freedman's Classification
- Whitney Towers

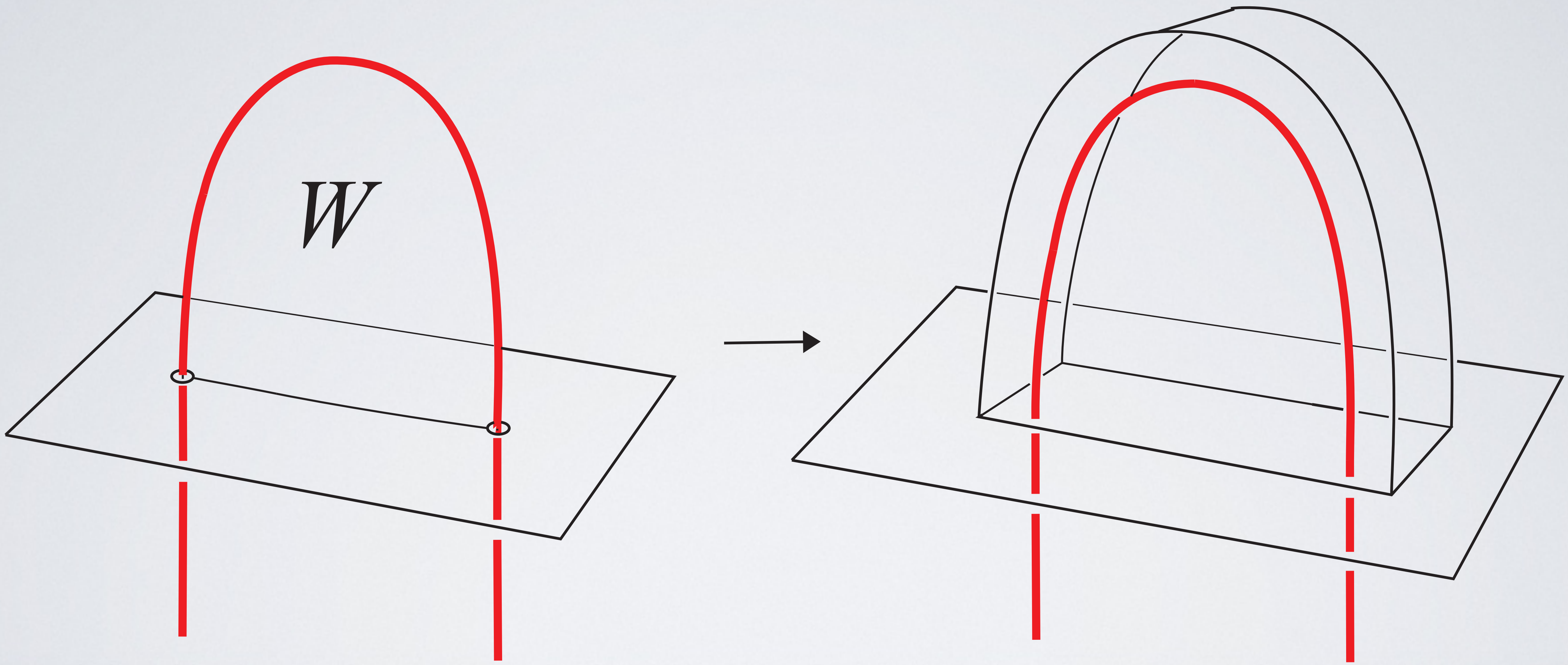
CLASSIFICATION OF SURFACES

Theorem: Closed connected 2-manifolds are classified by their intersection form on the first homology group (over Z/2). Every unimodular form arises exactly once.

The same classification holds for homotopy respectively diffeomorphism types.

WHITNEY'S TRICK

also known as a Whitney move



Add time as the 4-th dimension: red arc becomes a disk.

CLASSIFICATION OF HIGH DIMENSIONAL MANIFOLDS

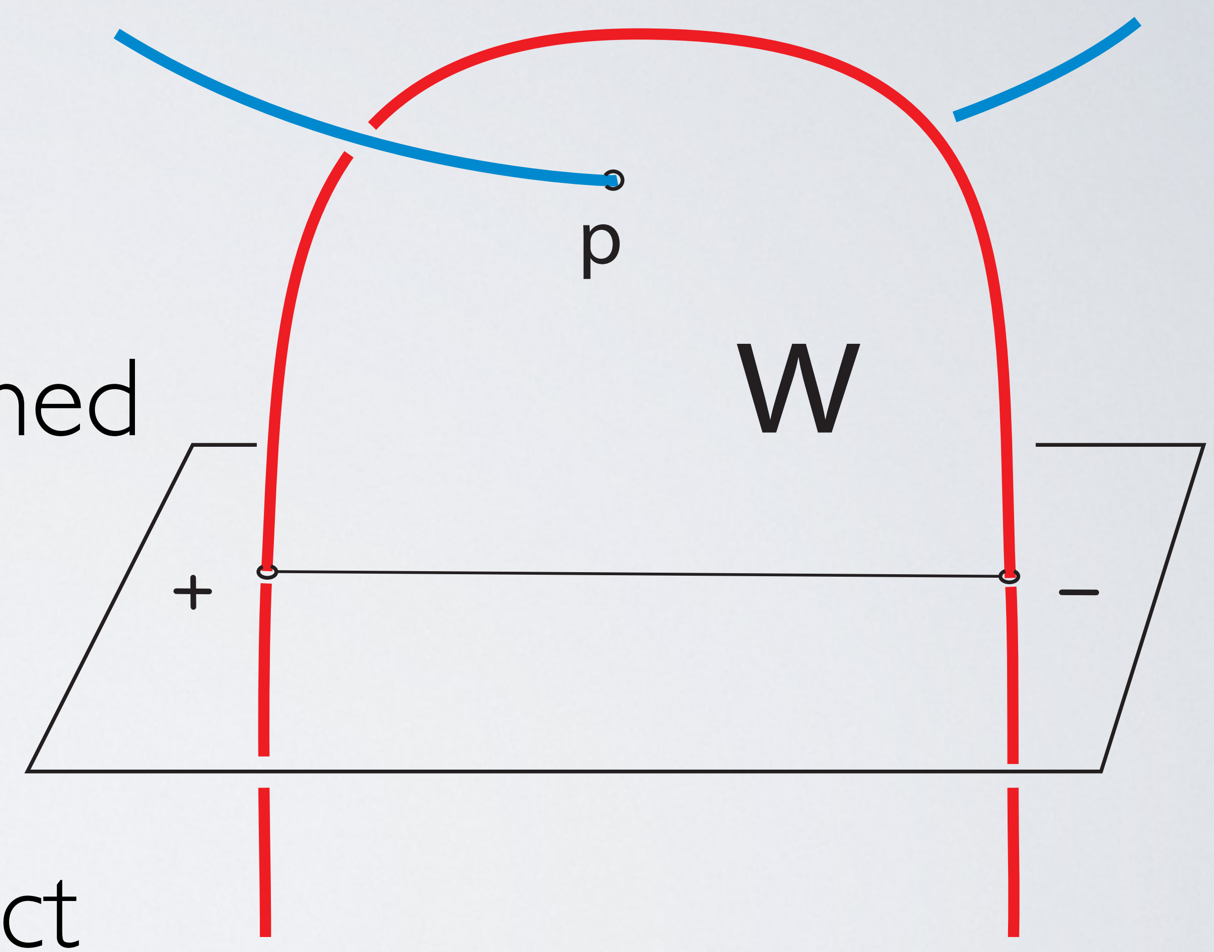
Get Smale's h-cobordism theorem which implies

Theorem: For n>2, closed (n-1)-connected 2n-manifolds are classified by their intersection form (and quadratic refinement) on the middle n-th homotopy = homology group.

Every unimodular quadratic form arises! Homotopy and diffeomorphism classification are very similar.

THE MAIN PROBLEM IN DIMENSION 4

Whitney disks may be assumed embedded and framed; they look as in our picture:

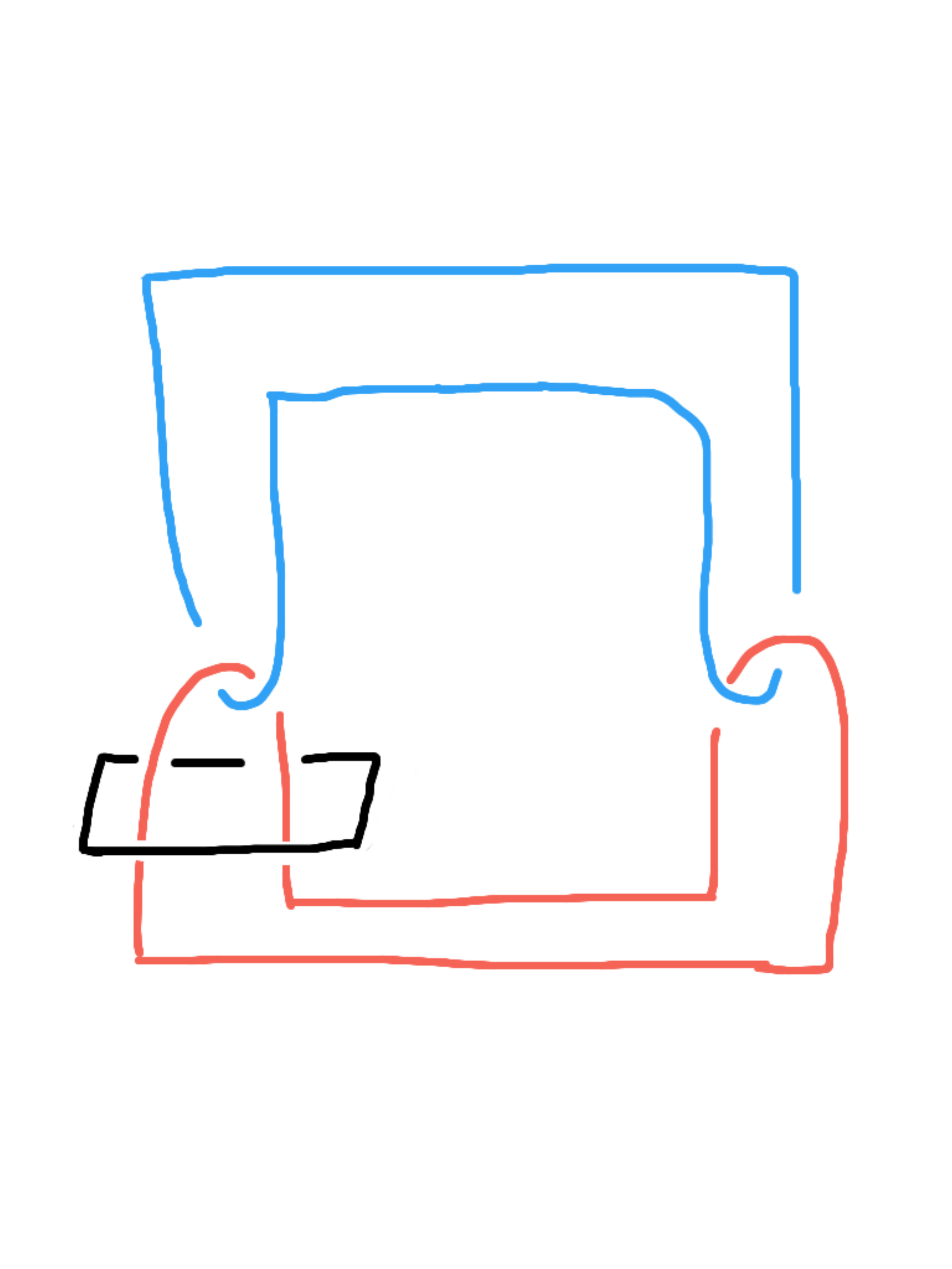


But other sheets can intersect them!

LINK ON THE BOUNDARY

A neighborhood of the Whitney disk W is a 4-ball (3-ball x time) and its boundary is a 3-sphere.

The three disks in 4-ball have their boundary in this 3-sphere, the (ugly) Borromean rings:



MILNOR'S CONTRIBUTIONS

The Borromean rings are not slice, i.e. they don't bound 3 disjoint disks in the 4-ball. Milnor invariant **μ(123)=1**. **That makes 4-manifolds special and interesting!**

Nevertheless, the homotopy type of 1-connected closed 4-manifolds is completely determined by their intersection form on second homotopy group.

- Which forms λ are realized ?

- Are the 4-manifolds unique ?

ROKHLIN'S THEOREM

For a smooth closed spin 4-manifold the signature is divisible by 16. In particular, the even definite form E8 is not realized smoothly.

Freedman-Kirby: If c is a characteristic surface in a closed 4-manifold M with intersection form λ and quadratic refinement τ then (mod 16) have

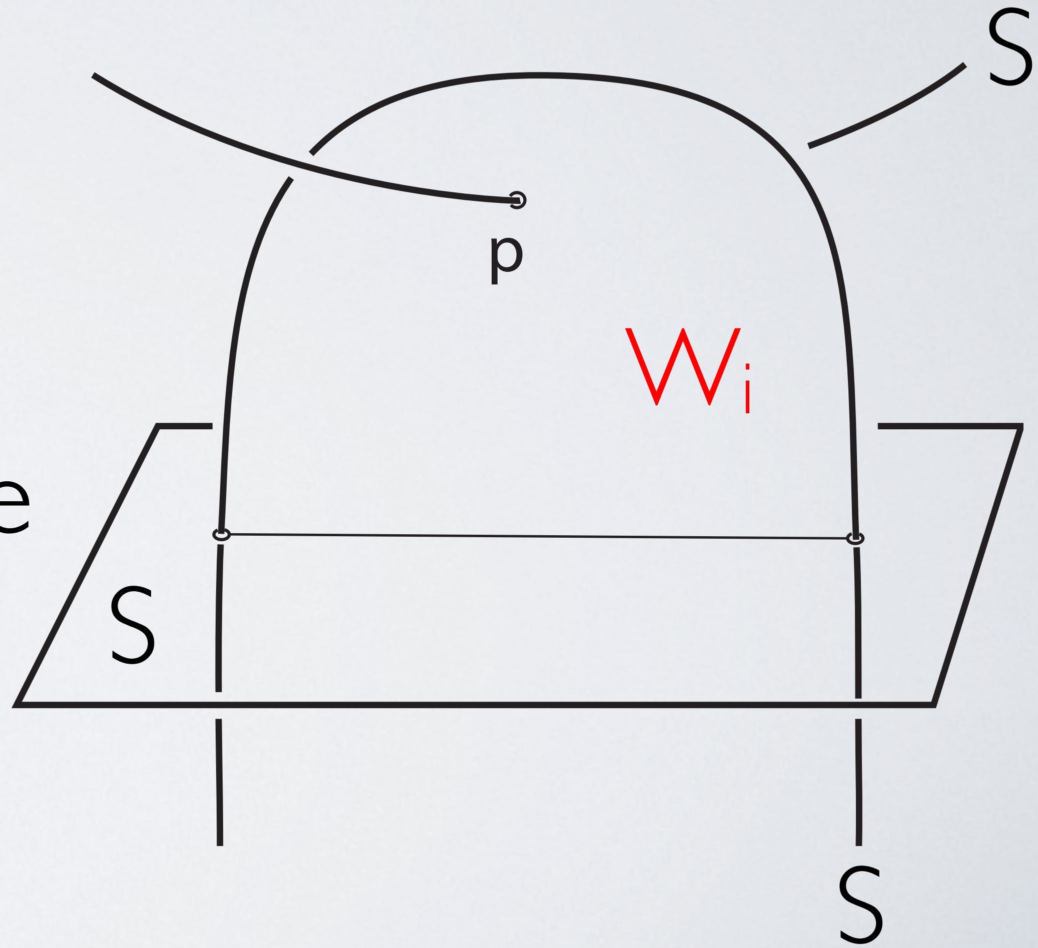
signature $\lambda = 8\,\mathrm{KS}(M) + 8\,\tau(c) + \lambda(c, c)$

HIGHER ORDER INTERSECTIONS

Freedman-Quinn-Stong: Geometric formula for τ :

$$\tau(c) = \tau_1(S, W_i) := \sum_i \#\{S \pitchfork W_i\} \qquad \text{mod } 2$$

Assume S is a characteristic sphere in the 4-manifold, representing the element c, such that all self- intersections of S are paired by Whitney disks W_i.



FREEDMAN'S CLASSIFICATION

Any unimodular form is realized as the intersection form of a closed simply-connected topological 4–manifold.

Even forms determine the manifold uniquely, any odd form is realized by exactly two 4–manifolds, distinguished by KS or τ .

Classification for other fundamental open, except:

- Infinite cyclic [Freedman-Quinn]

- Finite cyclic [Hambleton-Kreck]

- solvable Baumslag-Solitar groups [H-K-T]:

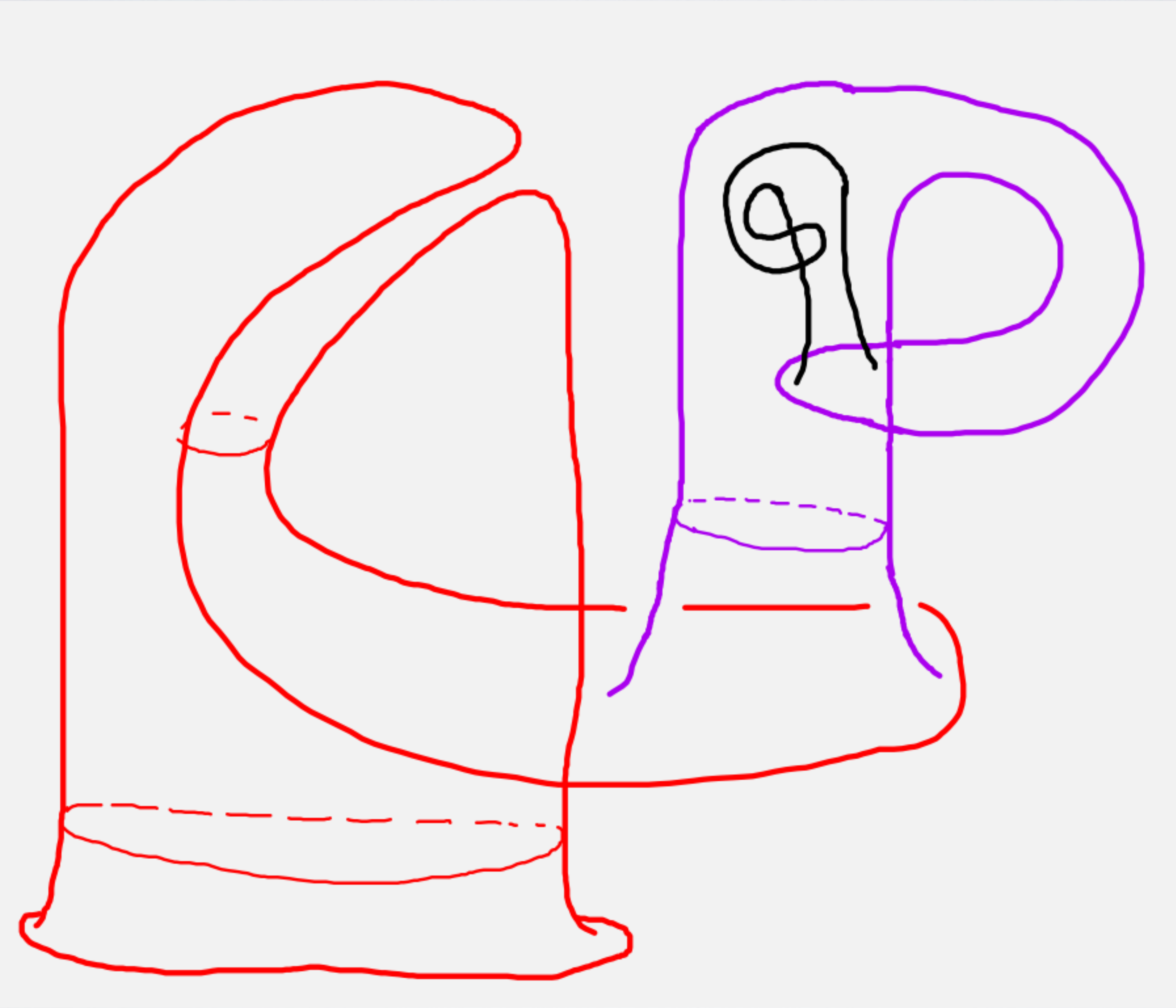
$$B(k) := \{a, b \,|\, aba^{-1} = b^k\}$$

SMOOTH 4-MANIFOLDS

Very exciting long story about relation to Gauge theory, started by Simon Donaldson. Briefly:

- The only definite forms that are realized smoothly are diagonalizable.
- 11/8-conjecture (Furuta's10/8-Theorem) predicts smooth realizability for even forms.
- Most 4-manifolds have (infinitely many) distinct smooth structures, including Euclidean 4-space! Open: 4-sphere.

SYMMETRIC WHITNEY TOWER

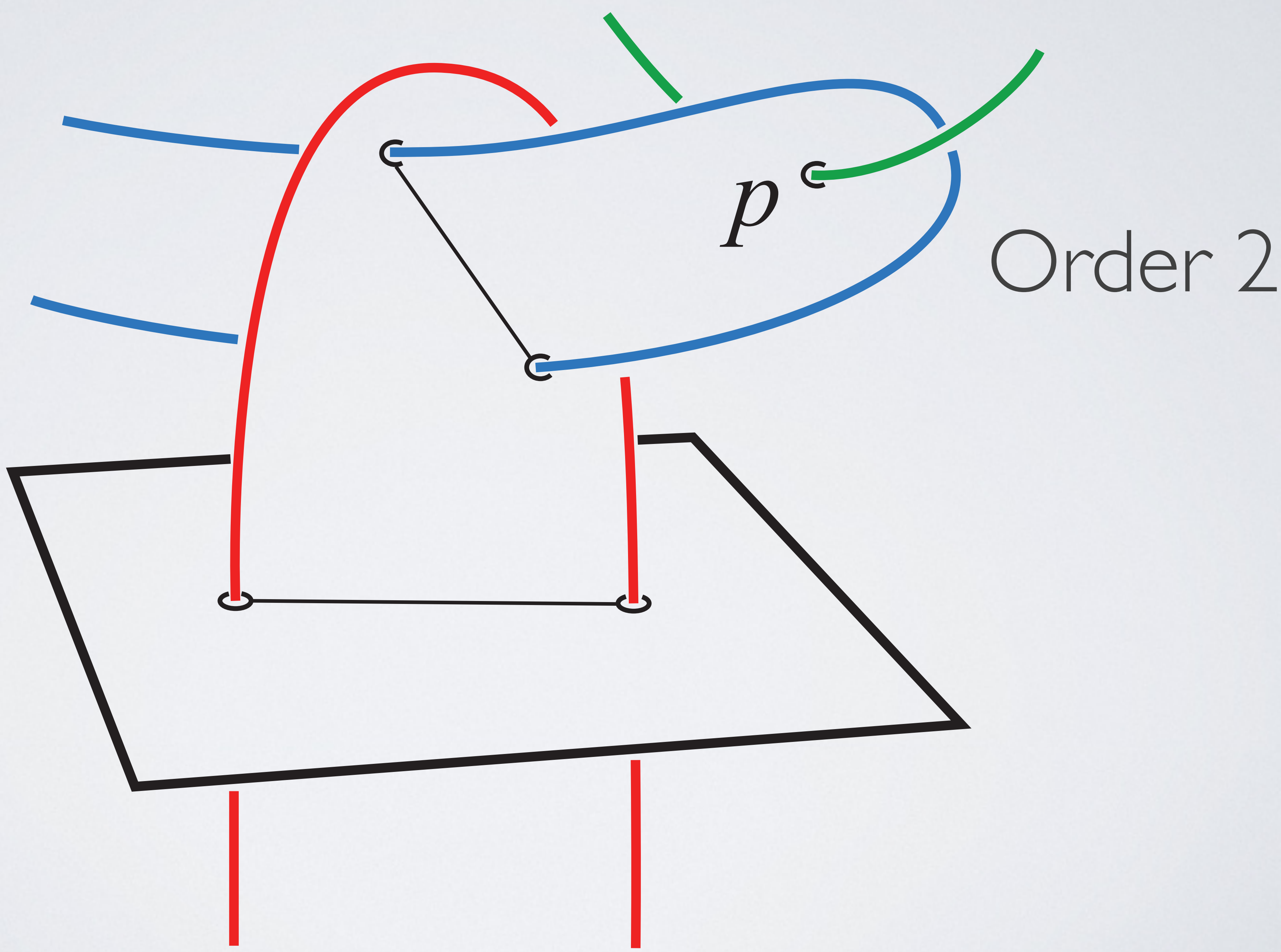


WITH COCHRAN & ORR

- Levine-Tristran signatures vanish if knot bounds a symmetric Whitney tower of height 2
- Casson-Gordon signatures vanish if knot bounds a symmetric Whitney tower of height 3
- There are von Neumann signatures obstructing inductive existence of height n symmetric Whitney towers for all n.

Cochran-Harvey-Leidy: All iterated quotients are infinitely generated groups with lots of torsion.

SIMPLIFY TO HIGHER ORDER WHITNEY DISKS



THAT LEAD TO HIGHER ORDER WHITNEY TOWERS

COMPUTATION OF $W_n(m)$

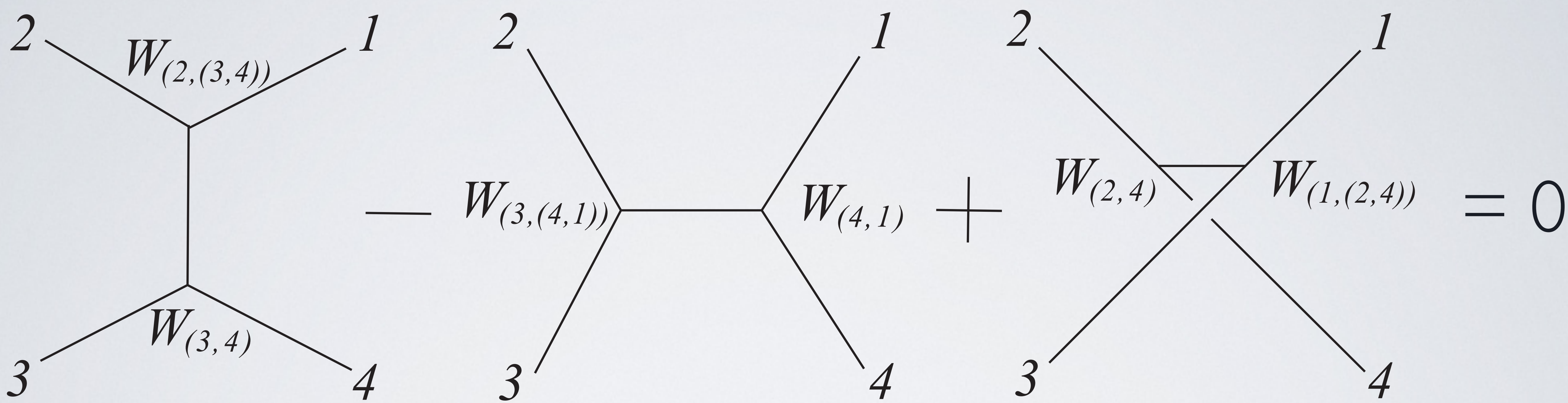
Group of m-component (framed) links in 3-sphere, bounding Whitney tower of order exactly n in 4-ball.

number m of link components

order n

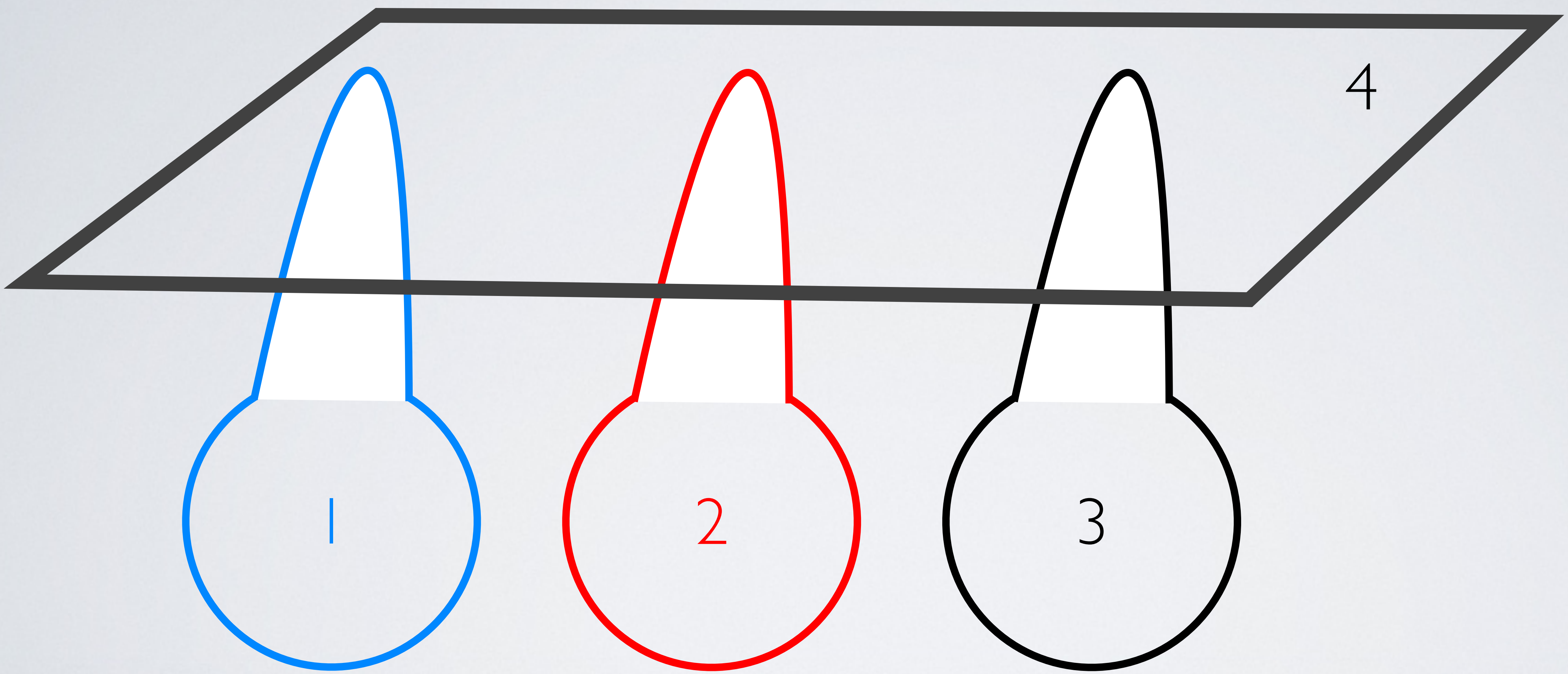
	1	2	3	4	5
0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^6$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{15}$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z} \oplus \mathbb{Z}_2^6$	$\mathbb{Z}^4 \oplus \mathbb{Z}_2^{10}$	$\mathbb{Z}^{10} \oplus \mathbb{Z}_2^{15}$
2	0	$\mathbb{Z}$	$\mathbb{Z}^6$	$\mathbb{Z}^{20}$	$\mathbb{Z}^{50}$
3	0	$\mathbb{Z}_2^2$	$\mathbb{Z}^6 \oplus \mathbb{Z}_2^8$	$\mathbb{Z}^{36} \oplus \mathbb{Z}_2^{20}$	$\mathbb{Z}^{126} \oplus \mathbb{Z}_2^{40}$
4	0	$\mathbb{Z}^3$	$\mathbb{Z}^{28}$	$\mathbb{Z}^{146}$	$\mathbb{Z}^{540}$
5	0	$\mathbb{Z}_2^{e_2}$	$\mathbb{Z}^{36} \oplus \mathbb{Z}_2^{e_3}$	$\mathbb{Z}^{340} \oplus \mathbb{Z}_2^{e_4}$	$\mathbb{Z}^{1740} \oplus \mathbb{Z}_2^{e_5}$
6	0	$\mathbb{Z}^6$	$\mathbb{Z}^{126}$	$\mathbb{Z}^{1200}$	$\mathbb{Z}^{7050}$

KEY: OUR 4-DIMENSIONAL JACOBI IDENTITY

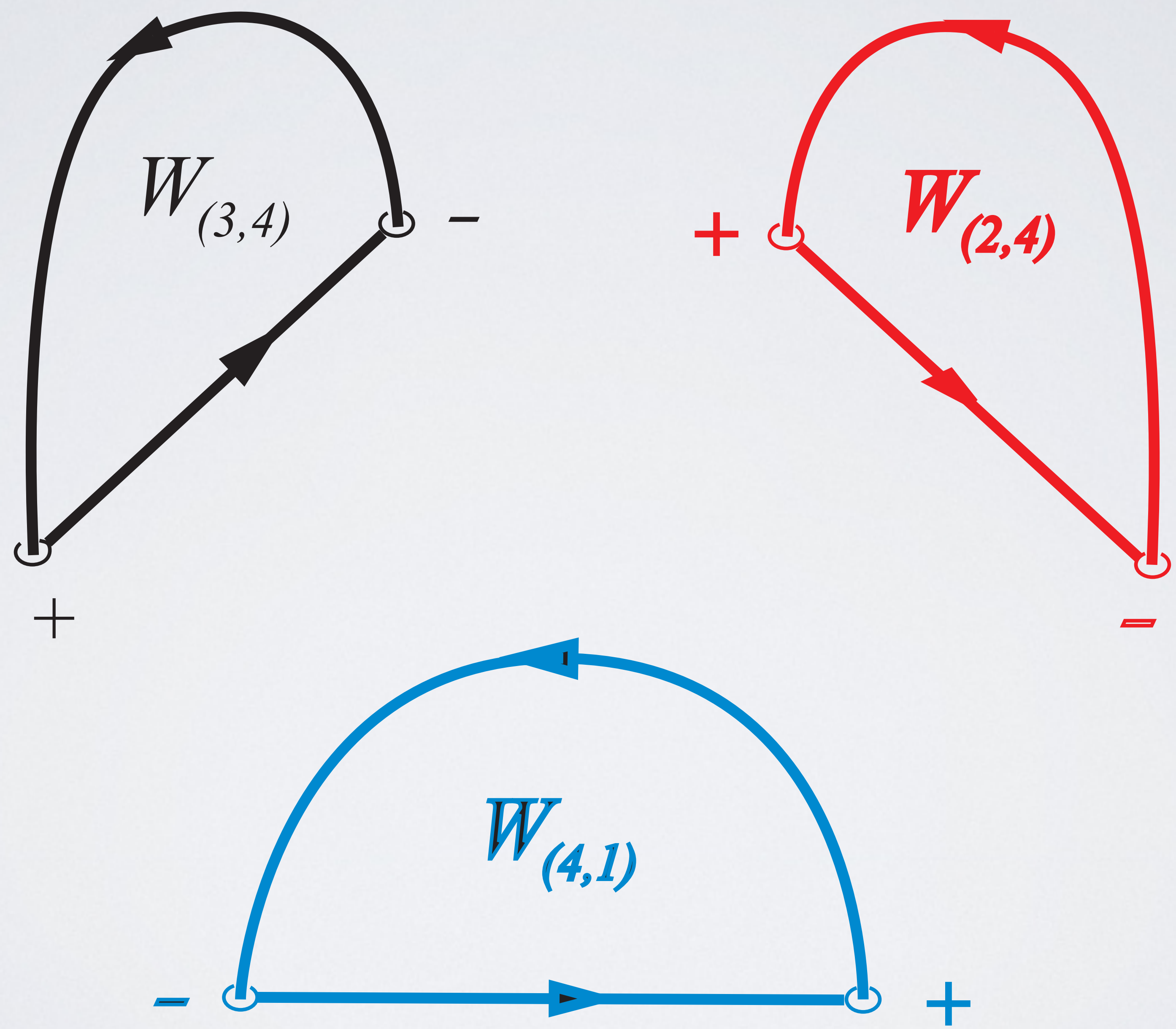


Proof is an exercise in visualization:

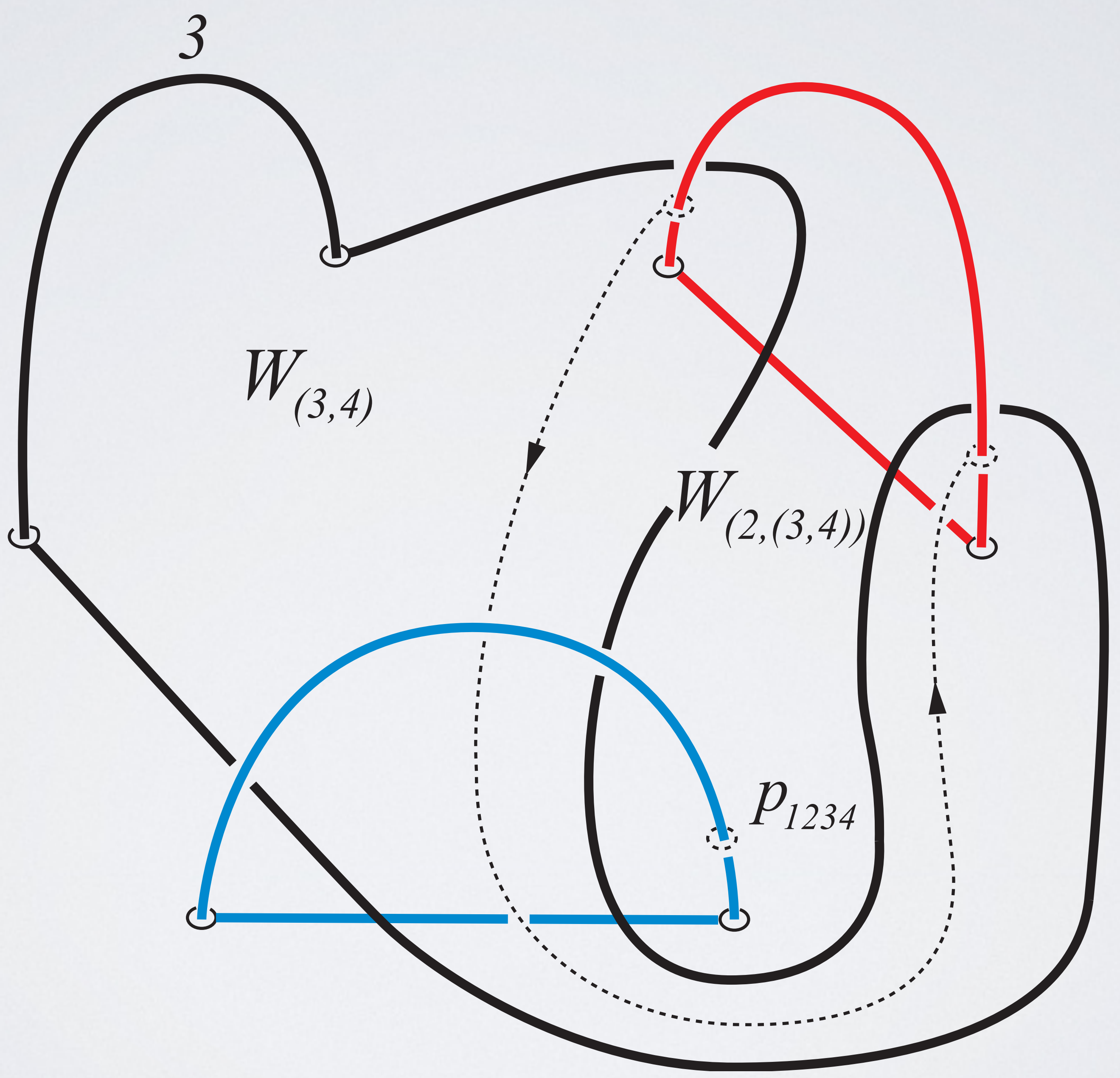
START WITH
FOUR SMALL SPHERES



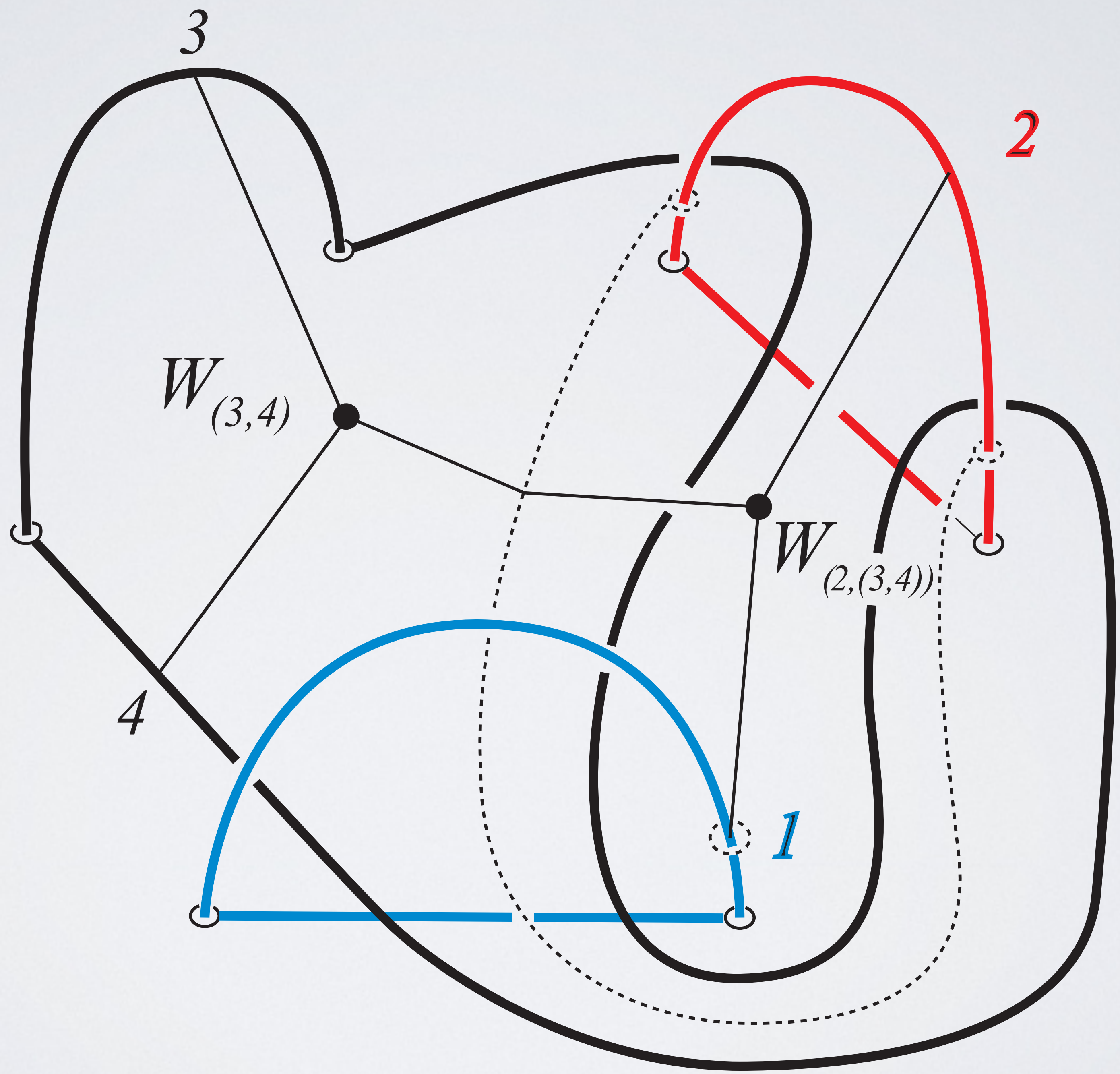
PICK THREE WHITNEY DISKS



MOVE WHITNEY ARCS



GET A WHITNEY TOWER OF ORDER 2



REMOVE INTERSECTIONS