The Harish-Chandra isomorphism for reductive symmetric superspaces

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partly joint work with
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Motivation

Motivated by random matrix theory, ZIRNBAUER (1996) considers

$$X_0 = G_0/K_0 \hookrightarrow X = G/K$$

where

(G, K, θ)	symmetric superpair of complex Lie supergroups
(G_0, K_0, θ)	Riemannian real form of underlying symmetric pair

ZIRNBAUER thus embeds ten of CARTAN's infinite series of RSS

Class		G_Λ/H_Λ	M_B	M_F
$A\vert A$	$\mathrm{Gl}(m\vert n)$	A	A
$A\mathrm{I}\vert A\mathrm{II}$	$\mathrm{Gl}(m\vert 2n)/\mathrm{Osp}(m\vert 2n)$	$A\mathrm{I}$	$A\mathrm{II}$
$A\mathrm{II}\vert A\mathrm{I}$	$\mathrm{Gl}(m\vert 2n)/\mathrm{Osp}(m\vert 2n)$	$A\mathrm{II}$	$A\mathrm{I}$
$A\mathrm{III}\vert A\mathrm{III}$	$\mathrm{Gl}(m_1+m_2\vert n_1+n_2)/\mathrm{Gl}(m_1\vert n_1)\times\mathrm{Gl}(m_2\vert n_2)$	$A\mathrm{III}$	$A\mathrm{III}$
$BD\vert C$	$\mathrm{Osp}(m\vert 2n)$	BD	C
$C\vert BD$	$\mathrm{Osp}(m\vert 2n)$	C	BD
$C\mathrm{I}\vert D\mathrm{III}$	$\mathrm{Osp}(2m\vert 2n)/\mathrm{Gl}(m\vert n)$	$C\mathrm{I}$	$D\mathrm{III}$
$D\mathrm{III}\vert C\mathrm{I}$	$\mathrm{Osp}(2m\vert 2n)/\mathrm{Gl}(m\vert n)$	$D\mathrm{III}$	$C\mathrm{I}$
$BD\mathrm{I}\vert C\mathrm{II}$	$\mathrm{Osp}(m_1+m_2\vert 2n_1+2n_2)/\mathrm{Osp}(m_1\vert 2n_1)\times\mathrm{Osp}(m_2\vert 2n_2)$	$BD\mathrm{I}$	$C\mathrm{II}$
$C\mathrm{II}\vert BD\mathrm{I}$	$\mathrm{Osp}(m_1+m_2\vert 2n_1+2n_2)/\mathrm{Osp}(m_1\vert 2n_1)\times\mathrm{Osp}(m_2\vert 2n_2)$	$C\mathrm{II}$	$BD\mathrm{I}$

PROGRAMME: Develop the harmonic analysis on $X = G/K$.
First, understand the invariant differential operators $D(X)^G$.

Super CHEVALLEY's restriction theorem

Invariant symbols: $\mathbb{C}[T^*X]^G \cong S(\mathfrak{p}^*)^{\mathfrak{k}}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the θ-eigenspaces.

Thus, let $(\mathfrak{g}, \mathfrak{k}, \theta)$ *symmetric superpair* of complex Lie superalgebras. We assume that $(\mathfrak{g}, \mathfrak{k})$ is

- *strongly reductive* ($\Rightarrow \mathfrak{g} = \bigoplus\{$simple basic class. ideals $\neq A(1|1)\}$)
- of *even type*, $\exists$ even Cartan subspace: $\mathfrak{a} \subset \mathfrak{p}_{0,\text{semi-simple}}$, $\mathfrak{a} = \mathfrak{z}_{\mathfrak{p}}(\mathfrak{a})$.

Let

$\Sigma_i = \Sigma(\mathfrak{g}_i : \mathfrak{a}), W_0 = W(\mathfrak{g}_0 : \mathfrak{a})$	roots, even Weyl group
$\bar{\Sigma}_1$	$\lambda \in \Sigma_1,\ \lambda, 2\lambda \notin \Sigma_0$
$W_\lambda = \langle \exp \mathbb{R} T_\lambda \rangle \subset \mathrm{GL}(\mathfrak{a})$	$\langle h', T_\lambda(h) \rangle = \lambda(h')\lambda(h)$

Theorem (A, HILGERT, ZIRNBAUER 2010)

The image $I(\mathfrak{a}^*)$ of the restriction map $S(\mathfrak{p}^*)^{\mathfrak{k}} \to S(\mathfrak{a}^*)$ is

$$I(\mathfrak{a}^*) = S(\mathfrak{a}^*)^W \quad \textit{where} \quad W = \left\langle W_0 \cup \textstyle\bigcup_{\lambda \in \bar{\Sigma}_1} W_\lambda \right\rangle .$$

'Group' case: SERGEEV (1999), KAC (1984), GORELIK (2004).

Super Harish-Chandra isomorphism

Let $I(\mathfrak{a}) \cong I(\mathfrak{a}^*)$ be the image of the 'restriction' $S(\mathfrak{p})^{\mathfrak{k}} \to S(\mathfrak{a})$. Choose a positive system $\Sigma^+ \subset \Sigma$ and let $\mathfrak{n} = \bigoplus\{\text{positive root spaces}\}$.

For $D \in \mathfrak{U}(\mathfrak{g})^{\mathfrak{k}}$, define $D_{\mathfrak{a}} \in \mathfrak{U}(\mathfrak{a}) = S(\mathfrak{a})$ by $D - D_{\mathfrak{a}} \in \mathfrak{n}\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{k}$ and $\Gamma(D) = e^{-\varrho} D_{\mathfrak{a}} e^{\varrho} \in S(\mathfrak{a})$ where $\varrho = \frac{1}{2}\operatorname{str}_{\mathfrak{n}} \operatorname{ad}|_{\mathfrak{a}}$.

Theorem (A 2010)

Γ is a surjective algebra morphism $\mathfrak{U}(\mathfrak{g})^{\mathfrak{k}} \to I(\mathfrak{a})$ with kernel $(\mathfrak{U}(\mathfrak{g})\mathfrak{k})^{\mathfrak{k}}$; it induces an algebra isomorphism $D(X)^G \to I(\mathfrak{a})$.

'Group' case: Kac (1984), Gorelik (2004).

Main ingredient: Spherical superfunction $\phi_\lambda(a) = \int_{K/M} L_a^* H^*(e^{\lambda-\varrho})$, $H : G \to \mathfrak{a}$ Iwasawa $\mathfrak{a}$-projection.

Differences: $\phi_\lambda(1) = 0$; $\phi_{w\lambda} = |\det w|^{-1} \cdot \phi_\lambda$ for $w \in W_\lambda$, $\lambda \in \bar{\Sigma}_1$; 'shifted' $\mathbf{c}$-function asymptotics.

Examples

Let $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{k} \oplus \mathfrak{k}, \mathfrak{k})$ (group type), $\mathfrak{k} = \mathfrak{osp}(2|2)$. Then

$$I(\mathfrak{a}) = \mathbb{C}[L] \qquad (L = \text{super-Laplacian})$$

Let $\mathfrak{g} = \mathfrak{osp}(2|2)$, 'special' involution. Then

$$I(\mathfrak{a}) = \mathbb{C}[z, w]/(z^3 - w^2)$$

is the ring of regular functions on a singular curve.

Thanks!