A Remark on the Brylinski Conjecture for Orbifolds

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Outline

A metric g on an oriented manifold M of dimension n gives rise to the isomorphisms

$$\flat : \ \mathfrak{X}^* \to \Omega^*$$

and

$$\phi : \ \mathfrak{X}^* \ni X \mapsto \iota_X \theta \in \Omega^{n-*},$$

where θ is a volume form associated with g. Their composition, $\phi\flat^{-1} : \ \Omega^* \ni \xi \mapsto \iota_{\flat^{-1}(\xi)}\theta \in \Omega^{n-*}$, is therefore an isomorphism, usually denoted as $\star$.

Using $\star$ we can define the codifferential

$$\delta : \ \Omega^k \ni \xi \mapsto (-1)^{k+1} \star d \star \xi,$$

and the space of harmonic forms

$$\mathcal{H}^* = \ker d \cap \ker \delta.$$

The main result in the Hodge theory is the following

Theorem (Hodge)

On a compact manifold M there is an orthogonal decomposition

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus d\Omega^{k-1}(M) \oplus \delta\Omega^{k+1}(M).$$

In particular each cohomology class contains exactly one harmonic representant.

Let now M be a symplectic manifold of dimension $2n$ with symplectic structure ω. Nondegeneracy of ω gives the isomorphism

$$\flat_s : \ \mathfrak{X}^* \to \Omega^*$$

while the volume form $\nu = \frac{\omega^n}{n!}$ gives the isomorphism

$$\phi_s : \ \mathfrak{X}^* \ni X \mapsto \iota_X \nu \in \Omega^{n-*}.$$

Again we compose them to obtain the symplectic star operator

$$\star_s \xi = \ \phi_s \flat_s^{-1} \xi = \iota_{\flat_s^{-1}(\xi)} \nu.$$

Again, we define the codifferential

$$\delta_s : \ \Omega^k \ni \xi \mapsto (-1)^{k+1} \star_s d \star_s \xi$$

and the space of symplectically harmonic forms

$$\mathcal{H}_s^* = \ker d \cap \ker \delta_s.$$

Due to ω being nonsymetric, Hodge's result doesn't hold in symplectic case. Still, Brylinski showed, that for compact Kähler manifold, every cohomology class contains at least one symplectically harmonic form, and conjectured that

Conjecture (Brylinski)

On compact symplectic manifold every cohomology class contains at least one symplectically harmonic representant.

Soon, Mathieu proved:

Theorem (Mathieu)

On any symplectic manifold (M, ω), the following conditions are equivalent:

1. *every cohomology class contains symplectically harmonic representant,*
2. (M, ω) *satisfies Hard Lefschetz Property, ie.*

$$L^k : H^{n-k}(M) \to H^{n+k}(M)$$

is surjective.

Idea of proof

Space $\Omega^*(M)$ admits a structure of $sl(2)$-module by the representation

- $Y \rightsquigarrow L: \ \Omega^k \ni \xi \mapsto \omega \wedge \xi \in \Omega^{k+2}$,
- $X \rightsquigarrow \Lambda: \ \Omega^k \ni \xi \mapsto \star_s L \star_s \xi \in \Omega^{k-2}$,
- $A \rightsquigarrow H: \ \Omega^k \ni \xi \mapsto (n-k)\xi \in \Omega^k$,

Definition

Form ξ is called primitive iff $\Lambda\xi = 0$.

Idea of proof, continued

Primitive forms $\mathcal{P}^*(M) \subset \Omega^*(M)$ are important for two reasons.

1. *Lefschetz decomposition*

$$\Omega^*(M) = \mathcal{P}^*(M) \oplus L\mathcal{P}^*(M) \oplus L^2\mathcal{P}^*(M) \oplus \dots,$$

2. $[d, \Lambda] = \delta_s$.

From the latter we see, that every closed, primitive form is indeed harmonic. Together with the former it proves the theorem.

For an arbitrary pseudogroup of local diffeomorphisms Γ on a manifold M, the Γ-invariant forms constitute a subcomplex $\Omega^*_\Gamma(M) \subset \Omega^*(M)$. The key observation is that for Γ-invariant symplectic form $\omega \in \Omega^2_\Gamma(M)$, the subcomplex $\Omega^*_\Gamma(M)$ is an $sl(2)$-submodule as well, and Mathieu's Theorem translates to

Theorem

The following conditions are equivalent:

1. *every Γ-invariant cohomology class contains symplectically harmonic representant,*
2. *ring $H^*_\Gamma(M)$ satisfies Hard Lefschetz Property.*

- Let M be a manifold with foliation $\mathfrak{F}$ and transversally symplectic form ω.
- Foliation $\mathfrak{F}$ is described by a family of submersions $\{p_i : M \supset U_i \to V_i \subset \mathbb{R}^{2n}\}_i$ and a Haefliger's cocycle $\{h_{i,j} : V_i \cap V_j \to V_i \cap V_j\}_{i,j}$ satisfying $f_i = h_{i,j} f_j$.
- If we now take the transverse manifold $T = \sqcup_i V_i$ and the pseudogroup Γ generated by Haefliger's cocycle, then $\Omega^*_B(M, \mathfrak{F})$ is chain isomorphic to $\Omega^*_\Gamma(T)$.

Now applying the “invariant Mathieu’s theorem” we obtain

Theorem

For the foliated, transversally symplectic manifold $(M, \mathfrak{F}, \omega)$ *the following conditions are equivalent:*

1. *every basic cohomology class contains transversally symplectic harmonic representant,*
2. *ring* $H_B^*(M)$ *satisfies Hard Lefschetz Property.*

It is well known fact, that every orbifold X can be realised as a space of leaves of a foliated manifold $(M, \mathfrak{F})$, and there is a chain isomorphism $\Omega^*(X) \cong \Omega^*_B(M, \mathfrak{F})$.
Now we can apply the foliated result to obtain

Theorem

On any symplectic orbifold X, the following conditions are equivalent:

1. *every cohomology class contains symplectically harmonic representant,*
2. *X satisfies Hard Lefschetz Property.*

Let us now consider orientable foliated manifold $(M, \mathfrak{F})$ with any metric g and transversally symplectic form ω. We can mix $\flat$ along the leaves and $\flat_s$ transversally into isomorphism

$$\flat_m : \ \mathfrak{X}^* \ni X \mapsto \flat(X^l) + \flat_s(X^t) \in \Omega^*.$$

Volume form $\nu \wedge \theta$ induces an isomorphism ϕ_m, and again we obtain a star operator $\star_m = \phi_m \flat_m^{-1} : \ \Omega^*(M) \to \Omega^{p+2n-*}(M)$.

- This operator was considered by Pak, who studied the Mathieu's type theorem on basic forms for transversally symplectic flows.
- He stated that the theorem holds with additional, geometric condition on the flow - tensness.

To obtain a similar result for a foliation of arbitrary dimension, we have to ensure that $[d, \Lambda] = \pm\delta_m$
It can be checked that on basic forms $[d, \Lambda] = \delta_s$ and $\delta_m \xi = (-1)^{p(k-1)}(\delta_s \xi + \iota_{\flat_m^{-1}(\kappa)} \xi)$, where κ is the mean curvature form of $\mathfrak{F}$.
The proof will follow for foliations with vanishing mean curvature, i.e. the taut foliations.

For the special case of Riemannian foliations on compact manifolds, a number of results concerning mean curvature form have been obtained. In particular:

Ton. tense foliation $\Rightarrow$ (tautness $\Leftrightarrow$ minimalizability of leaves),

Dom. Riemannian foliation on compact manifold is tense,

Masa minimalizability $\Leftrightarrow H_B^{\operatorname{codim}\mathfrak{F}}(M, \mathfrak{F}) \neq 0$.

Recently, Yau and Tseng introduced a new approach to symplectic Hodge theory. Their new cohomology has a number of interesting properties: it admits Lefschetz decomposition, each class has unique harmonic representative, and therefore may be better suited for study of symplectic manifolds. It might be interesting to check, whether this cohomology may be foliated preserving those properties.