Double pants decompositions of 2-surfaces

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joint work with Sergey Natanzon

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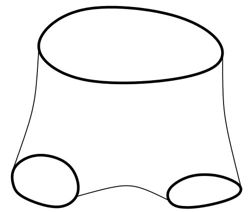
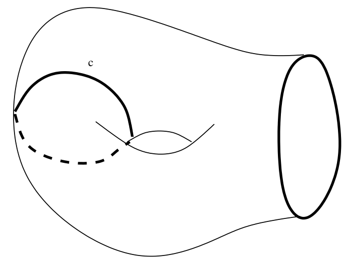
1. Pants decompositions

$S = S_{g,n}$ 2-surface, genus g, n holes;



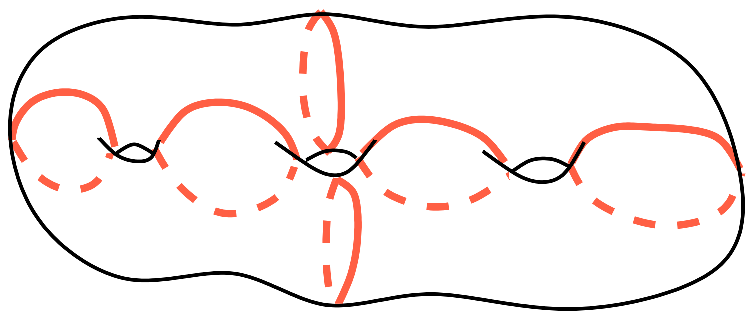


P pants decomposition of $S =$

$=$ dec. into pairs of pants

$$P = \langle c_1, \ldots, c_{3g-3+n} \rangle$$

- $c_i \cap c_j = \emptyset$
- maximal system of curves



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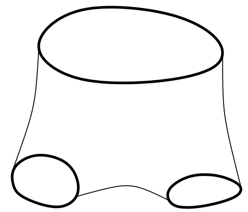

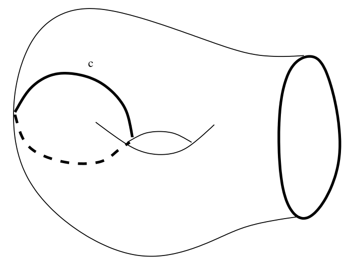

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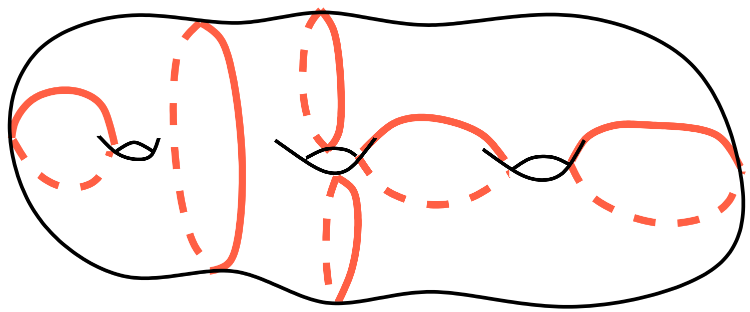
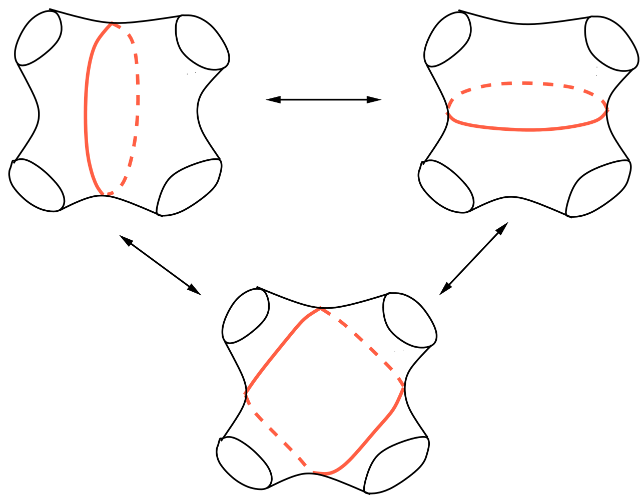


Flip:



- Flips are sufficient to get to any "combinatorial type" of dec.

- Flips do not act transitively on all pants decompositions:

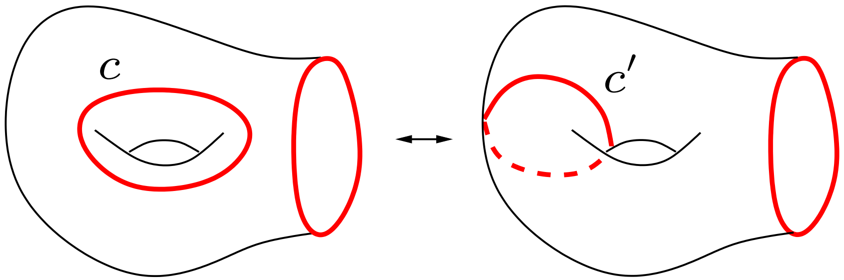
$$P \to \mathcal{L}(P) \subset H_1(S, \mathbb{Z})$$
$$\mathcal{L}(P) = \langle h(c_1), \ldots, h(c_{3g-3+n}) \rangle$$

Lagrangian plane.

- Flips preserve $\mathcal{L}(P)$.

How to get out of $\mathcal{L}(P)$?

- Classical approach: add S-moves:

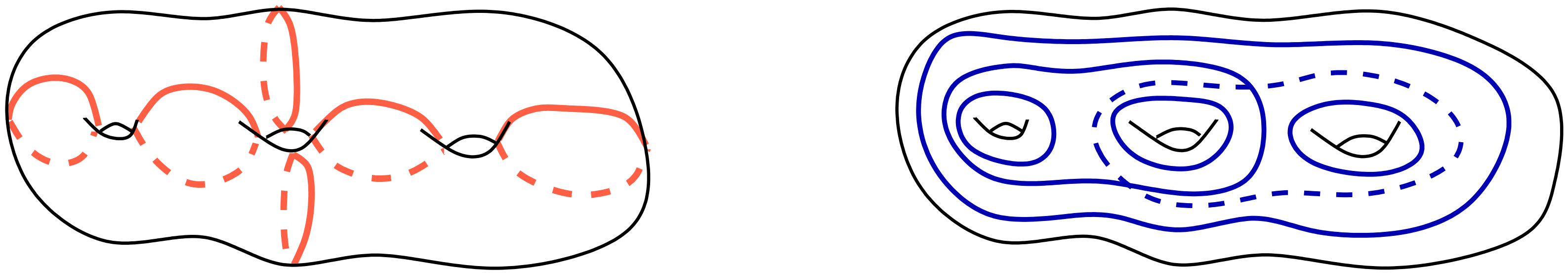


$$|c \cap c'| = 1.$$

Theorem.(Hatcher, Thurston' 80)
Flips and S-moves act transitively on pants dec.

2. Double pants decompositions

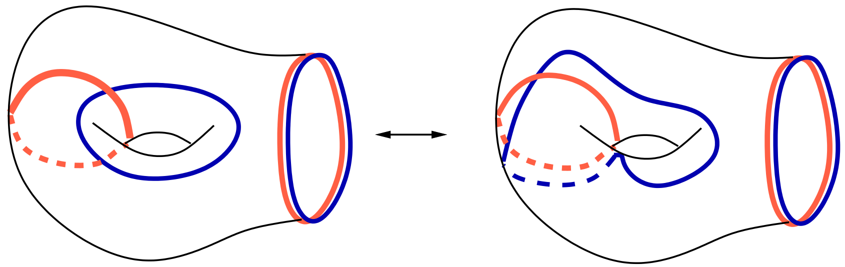
$DP = (P_a, P_b) \quad \langle \mathcal{L}(P_a), \mathcal{L}(P_b) \rangle = H_1(S, \mathbb{Z}).$



Operations:

1. Flips in P_a, P_b

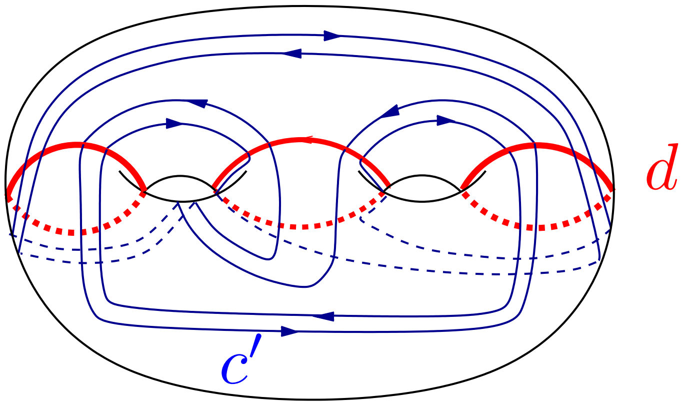
2. Handle-twists:



Prop. 1. Flips and handle-twists act transitively on the pairs of Lagrangian planes.

Problem: Do flips and handle-twists act transitively on DP's?

Natural way to solve: Prop. 1 + ~~Transitivity of flips on $\mathcal{L}(P)$~~



A. Hatcher: $S \hookrightarrow \mathbb{R}^3$ defines an inner handlebody S_+.

- If all $c \in P$ are contractible in S_+ then $flip(c)$ is...
- c' is not contr. in S_+ (linked with d).

Problem: Do flips and handle-twists act transitively on DP's?

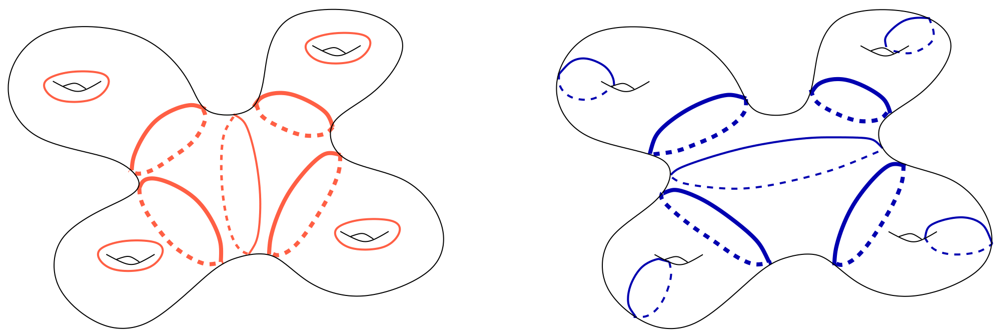
New strategy: use Hatcher-Thurston's theorem:

Need: $(P_a, P_b) \longrightarrow (P'_a, P'_b)$.
HT treats P_a
We take care of P_b

Price: restrictions on DP.

3. Admissible double pants decompositions

- Standard DP:
  g common handles.



- Admissible DP:
  may be transformed to a standard one by flips.

Question: How wide is the class of admissible DP's?

- Have no examples of non-admissible DP's.

Theorem 1.

Flips and handle-twists act transitively on admissible DP's.

Idea of proof.

1. Sufficient to prove transitivity on standard DP's.

2. For two standard DP's with the same set of handles: is easy.

3. Need to bring any set of g handles to any other: $(P_a, P_b) \to (P'_a, P'_b)$

• $P_a \to P'_a$ by Hatcher-Thurston's thm.

• $P_b \to P''_b$ "accompyning decomposition" such that (P'_a, P''_b) is standard and may be obtained from (P_a, P_b) by flips and handle-twists.

4. do Step 3 only for pairs of "adjacent" standard DP's.

Example: if $g = 2$ and (P_a, P_b) and (P'_a, P'_b) are two "adjacent" standard DP's, then P_a may be transformed to P'_a in 2 flips.

Theorem 2. A groupoid generated by flips and twists contains a group isomorphic to mapping class group.

Idea of proof: There exists an admissible DP such that no element of MCG fixes this double pants decomposition.

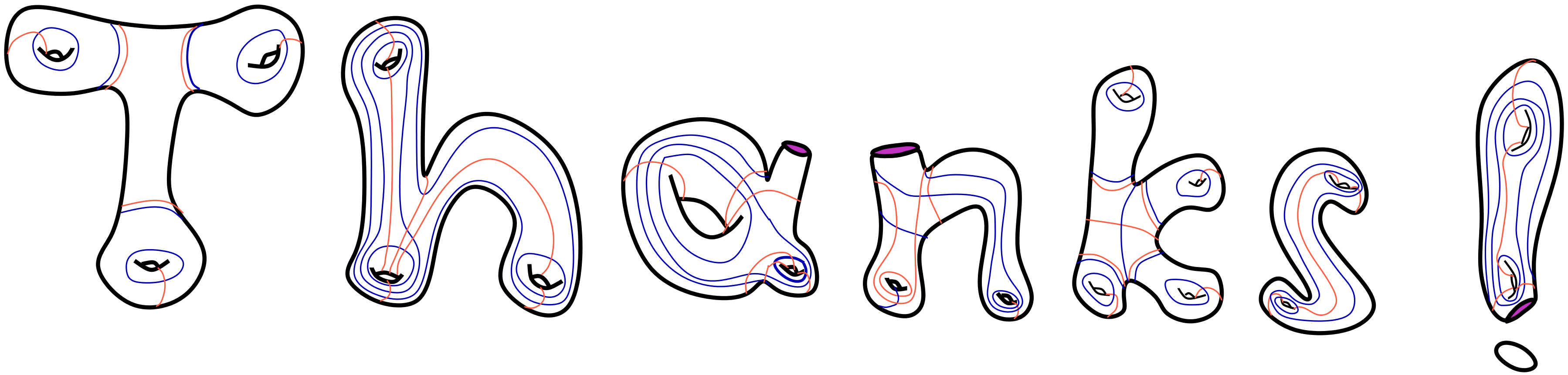
4. What's next to do?

- Combinatorics of "double-curve complex" (or exchange graph);

- Variables: for each $c_i \in DP$ define $f(c_i)$
  so that $(DP, \{f(c_i)\})$ define the whole geometric structure on S.

  Example: $f(c_i)$ is a length of c_i,
  then $(DP, \{f(c_i)\})$ defines a hyperbolic surface.

- Application: moduli spaces and their compactifications.


Thanks!