Horizontal lift of symmetric connections to the bundle of volume forms $\mathcal{V}$

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Notations

Throughout the talk we assume that $i, k, \ldots = 1, 2, 3, \ldots, n$ and $\alpha, \beta, \ldots = 0, 1, 2, \ldots, n$. Moreover, the Einstein summation convention will be used with respect to these systems of indices.

Construction of the bundle of volume forms $\mathcal{V}$

Let:

1) M be an orientable n-dimensional manifold,
2) $\mathcal{V}$ be a bundle of volume forms over M,
3) $\pi : \mathcal{V} \to M$ be a projection of the bundle.

We consider two local charts (U, x^i), (U', x'_i) on M, where $U \cap U' \neq \emptyset$, and a volume form ω on M. Assume that form ω is given by

$$\omega = v(x)dx^1 \wedge \ldots \wedge dx^n$$

and

$$\omega = v'(x')dx^{1'} \wedge \ldots \wedge dx^{n'}$$

in the charts (U, x^i) and (U', x'_i), respectively, and $v > 0$, $v' > 0$ are smooth functions on M.

Construction of the bundle of volume forms $\mathcal{V}$

Let functions $x^{i'} = x^{i'}(x)$ be orientation-preserving transitions functions on M. Then the transitions functions on $\mathcal{V}$ are given by

$$\upsilon' = \bar{\mathcal{I}} \cdot \upsilon, \quad x^{i'} = x^{i'}(x),$$

where $\bar{\mathcal{I}}$ is the Jacobian of the map $x^{i'} = x^{i'}(x)$. Now, we introduce a new coordinate system $(x^0, x^1, \ldots, x^n)$ on $\mathcal{V}$, where $x^0 = \ln \upsilon$. Then the transition functions in the terms of these coordinate system are

$$x^{0'} = x^0(x) + \ln \bar{\mathcal{I}}(x), \quad x^{i'} = x^{i'}(x).$$

Earlier results for $\mathcal{V}$

Let $\nabla = (\Gamma_{ij}^k)$ be a symmetric connection on M.

1) (Dhooghe, 1995) Let $X = X^i \frac{\partial}{\partial x^i}$ be a vector field on M. Then

$$\bar{X} = -X^i \Gamma_{ik}^k \frac{\partial}{\partial x^0} + X^i \frac{\partial}{\partial x^i}$$

is globally defined vector field on $\mathcal{V}$, which is called the horizontal lift of X.

2) (Miernowski, Mozgawa, 2003) Let $g = (g_{ij})$ be a tensor of type $(0,2)$ on M. Then

$$\bar{g} = \begin{bmatrix} 1 & \Gamma_{ik}^k \\ \Gamma_{ik}^k & g_{ij} + \Gamma_{ik}^k \Gamma_{jt}^t \end{bmatrix}$$

is globally defined $(0,2)-$tensor field on $\mathcal{V}$, which is called the horizontal lift of g.

Earlier results for $\mathcal{V}$

3) (Miernowski, Mozgawa, 2003) Let g be a Riemannian metric on M. Then $\bar{g}$ is a Riemannian metric on $\mathcal{V}$ and

$$(\bar{g})^{-1} = \begin{bmatrix} g^{ij}\Gamma^k_{ik}\Gamma^t_{jt} & -g^{ij}\Gamma^k_{jk} \\ -g^{ij}\Gamma^k_{jk} & g^{ij} \end{bmatrix}.$$

4) (Miernowski, Mozgawa, 2003) Let $F = (F^i_j)$ be a tensor field of type $(1,1)$ on M. Then

$$\bar{F} = \begin{bmatrix} 1 & -F^t_j\Gamma^k_{tk} + \Gamma^k_{jk} \\ 0 & F^i_j \end{bmatrix}$$

is a tensor field of type $(1,1)$ on $\mathcal{V}$, which is called a horizontal lift of the tensor field F.

Earlier results for $\mathcal{V}$

5) (Gąsior, 2006) Let g be a Riemannian metric on M. Then the nonzero coefficients of a Levi-Civita connection $\widetilde{\nabla}$ for the horizontally lifted Riemannian metric $\bar{g}$ are given by formulas

$$\widetilde{\nabla}_n \frac{\partial}{\partial x^m} = \widetilde{\nabla}_m \frac{\partial}{\partial x^n} = \left(\Gamma^t_{mt|n} - \Gamma^t_{mn}\Gamma^k_{tk}\right) \frac{\partial}{\partial x^0} + \Gamma^s_{mn} \frac{\partial}{\partial x^s},$$

where $\nabla = (\Gamma^k_{ij})$ is Levi-Civity connection on (M, g).

The horizontal lift of a symmetric connection

Let $\nabla = (\Gamma_{ij}^k)$ be a symmetric connection and $\nabla_1 = (\Phi_{ij}^k)$ be a connection on M. Then an operator $\bar{\nabla}_1$ whose nonzero coefficients are given by

$$\bar{\nabla}_{1\;i}\frac{\partial}{\partial x^j} = \bar{\nabla}_{1\;j}\frac{\partial}{\partial x^i} = \left(\Gamma_{it|j}^t - \Phi_{ij}^r\Gamma_{rt}^t\right)\frac{\partial}{\partial x^0} + \Phi_{ij}^k\frac{\partial}{\partial x^k}$$

is a linear connection $\mathcal{V}$, which will be called the horizontal lift of the connection ∇_1 with respect to the connection∇.

A curvature tensor on $\mathcal{V}$

Let $\bar{R} = (\bar{R}^{\delta}_{\alpha\beta\gamma})$ be the curvature tensor of the horizontal lift of the connection ∇_1 with respect to the connection ∇. Then the nonzero coefficients of the tensor $\bar{R}$ are giving by following formulas

$$\bar{R}^s_{ikj} = R^s_{ikj},$$
$$\bar{R}^0_{ikj} = -\Gamma^t_{rt} R^r_{ikj} + 2\Phi^d_{ik}\Gamma^t_{[jt|d]} + 2\Phi^d_{jk}\Gamma^t_{[dt|i]} + 2\Gamma^t_{[it|kj]},$$

where $R = (R^s_{ijk})$ is the curvature tensor of the connection ∇_1 on M.

A curvature tensor on $\mathcal{V}$

Let ∇ be a Riemannian connection on the Riemannian manifold (M, g) and ∇_1 be a connection on the manifold M. Then the nonzero coefficients of the curvature tensor $\bar{R} = (\bar{R}^{\delta}_{\alpha\beta\gamma})$ of the horizontally lifted connection $\bar{\nabla}_1$ with respect to the connection $\nabla = (\Gamma^k_{ij})$ are giving by

$$\bar{R}^s_{ijk} = R^s_{ijk},$$
$$\bar{R}^0_{ijk} = -\Gamma^t_{rt} R^r_{ijk},$$

where $R = (R^s_{ijk})$ is the curvature tensor of the connection ∇_1 on M.

Ricci tensor on $\mathcal{V}$

Let ∇ be a symmetric connection, ∇_1 be a connection on the manifold M and $(\bar{R}_{\alpha\beta})$ be the coefficients of a Ricci tensor $\bar{R}$ of the horizontally lifted connection $\bar{\nabla}_1$ with respect to connection ∇. Then the nonzero coefficients of $\bar{R}$ are given by the formulas

$$\bar{R}_{ik} = R_{ik},$$

where (R_{ik}) are the coefficients of Ricci tensor R of the connection ∇_1 on the manifold M.

A scalar curvature on $\mathcal{V}$

Let g be a Riemannian metric on the manifold M and let $\bar{g}$ be a horizontally lifted Riemannian metric on $\mathcal{V}$. If $\bar{K}$ is a scalar curvature of the horizontally lifted connection $\bar{\nabla}_1$ with respect to the symmetric connection ∇, then

$$\bar{K} = \frac{n-1}{n+1} K,$$

where K is a scalar curvature of the connection ∇ on M.

$\pi-$conjugate connections (A. P. Norden, K. Radziszewski)

Let $\nabla = (\Gamma^k_{ij})$ be the linear connection and let π be a non-singular tensor field of the type $(0,2)$ on M. The connection $\nabla^* = (G^i_{ks})$ which is given by

$$G^i_{ks} = \pi^{pi}\nabla_k \pi_{ps} + \Gamma^i_{ks}$$

is said to be a $\pi-$conjugate connection with respect to the connection ∇.

$\pi-$conjugate connections on $\mathcal{V}$

Let ∇_2 be a $\pi-$conjugate connection with respect to a connection ∇_1 on manifold M. Let $\bar{\nabla}_1$ and $\bar{\nabla}_2$ be horizontally lifted connections with respect to a connection ∇ on $\mathcal{V}$. Then $\bar{\nabla}_2$ is a $\bar{\pi}-$conjugate connection with respect to a horizontally lifted connection $\bar{\nabla}_1$, where $\bar{\pi}$ is horizontal lift of π with respect to a connection ∇.

A Killing vector field

Let (M, g) be a Riemannian manifold. If X is a vector field and ∇ is a symmetric locally volume preserving connection on M then the horizontally lifted vector field $\bar{X}$ is a Killing vector field on $(\mathcal{V}, \bar{g})$ if and only if X is a Killing vector field on M.

Infinitesimal transformations

Let $\bar{\nabla}_1$ be the horizontal lift of the connection ∇_1 with respect to the symmetric connection ∇ on M and let $\bar{X}$ be the horizontal lift of the vector field X on $\mathcal{V}$. Then

1) $\bar{X}$ is an infinitesimal affine transformation of the horizontally lifted connection $\bar{\nabla}_1$ if and only if X is an infinitesimal affine transformation of the connection ∇_1 on M,
2) $\bar{X}$ is a fibre-preserving infinitesimal transformation on $\mathcal{V}$,
3) $\bar{X}$ is never a conformal infinitesimal transformation on $\mathcal{V}$,
4) $\bar{X}$ is never the infinitesimal projective transformation on $\mathcal{V}$.

Horizontal lift of the tensor field of type $(1,1)$

Let F be a tensor field of type $(1,1)$ which define the $F(3,1)-$structure on a manifold M and let ∇ be a linear connection on M. Then the horizontally lifted tensor field $\bar{F}$ defines the $\bar{F}(3,1)-$structure on the bundle of volume forms $\mathcal{V}$.

Horizontal lift of the tensor field of type $(1,1)$

Let F be a tensor field of type $(1,1)$ which define the $F(3,1)-$structure on manifold the M and let ∇ be a linear volume-preserving connection on M. Then the horizontally lifted tensor field $\bar{F}$ defines an integrable $\bar{F}(3,1)-$structure on $\mathcal{V}$ if and only if the $F(3,1)-$structure is integrable on M.

Some special substructure of F-structure on M (Singh, K. D., Singh, R., 1977)

1) Let F be a tensor field of type $(1,1)$ which define the $F(3,\varepsilon)-$structure on a manifold M, where $\varepsilon = \pm 1$.
2) Let $A = 1 - \varepsilon F^2$.
3) On $F(3,\varepsilon)-$manifold always exists a Riemannian metric g satisfying a condition $g(X,Y) = g(FX,FY) + g(AX,Y)$, (Ishihara, S., Yano, K., 1965). This metric is called the Ishihara-Yano metric.
4) We define the tensor field G of type $(0,2)$ by the following form $G(X,Y) = g(FX,Y)$, (Yano, 1963).

Some special substructure of F-structure on M (Singh, K. D., Singh, R., 1977)

Let ∇ be a metric connection of the Ishihara-Yano metric g on the manifold M. Then the tensor field F of type $(1,1)$ define

1) FK-structure if and only if $\nabla_{FX}(F) = 0$,
2) FAK-structure if and only if $dS(FX, FY, FZ) = 0$, where $S(X,Y) = -S(Y,X) = g(FX, Y)$,
3) FNK-structure if and only if $\nabla_{FX}(G)(FY, FZ) - \varepsilon\nabla_{FY}(G)(FX, FZ) = 0$,
4) FQK-structure if and only if

$$2\nabla_{FX}(G)(FY, FZ) + (1-\varepsilon)\nabla_{F^2X}(G)(F^2Y, FZ) = (1+\varepsilon)\left[\nabla_{F^2Z}(G)(FX, F^2Y) - \nabla_{F^2Y}(G)(FZ, F^2X)\right],$$

5) FH-structure if and only if $N(FX, FY) = 0$.

Some special substructure of F-structure on $\mathcal{V}$

Let F be a tensor field of type $(1,1)$ which defines a $F(3,\varepsilon)-$structure and ∇ be the Levi-Civita connection on a Riemannian manifold (M,g). Let $\bar{g}$ be the horizontally lifted Riemannian metric and let $\widetilde{\nabla}$ be the Levi-Civita connection on the Riemannian manifold $(\mathcal{V},\bar{g})$. Then the horizontal lift $\bar{F}$ of the tensor field F defines

1) the $\bar{F}QK-$structure on $\mathcal{V}$ if and only if the tensor field F defines the $FQK-$structure on M,
2) the $\bar{F}K$, $\bar{F}AK$, $\bar{F}NK-$structure on $\mathcal{V}$ if and only if the tensor F defines the FK, FAK, $FNK-$structure on the M, respectively,
3) the $\bar{F}H-$structure on $\mathcal{V}$ if and only if the tensor field F defines a $FH-$structure on M.

The end

Thank you for your attention.