Classification of Ricci-flat Kähler metrics on the cotangent bundles of compact rank-one symmetric spaces

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Braniewo, 2010

Introduction

Over the last ten years there has been considerable interest in a family of Ricci-flat Kähler metrics discovered by Stenzel with underlying manifold diffeomorphic to the tangent bundle of the rank-one symmetric space G/K.

M. Stenzel, "Ricci-flat metrics on the complexification of a compact rank one symmetric space", Manuscripta Math., **80** (1993), 151–163.

Cvetic, Gibbons, Lü and Pope have studied the harmonic forms on these metrics and found an explicit formula for the Stenzel metrics in terms of hypergeometric functions.

M. Cvetic, G.W. Gibbons, H. Lu and C.N. Pope, "Ricci-flat metrics, harmonic forms and brane resolutions", Commun. Math. Phys., **232** (2003), 457–500.

Introduction

Lee gave an explicit formula for the Stenzel metrics for classical spaces G/K but in another terms, using the approach of Patrizio and Wong.

T.-C. Lee, "Complete Ricci-flat Kähler metric on M_I^n, M_{II}^{2n}, M_{III}^{4n}", Pacific J. of Math., **185**:2 (1998), 315–326.

Exploiting the fact that the Stenzel metrics are of cohomogeneity one with respect to an action of the Lie group G on $T(G/K)$, Dancer and Strachan gave a much more elementary and concrete treatment in the case when the homogeneous space G/K is the round sphere $\mathbb{S}^n = SO(n+1)/SO(n)$.

A.S. Dancer, I.A.B. Strachan, "Einstein metrics on tangent bundles of spheres", preprint, arXiv:math.DG/0202297 v1 (2002).

Introduction

If G/K is the standard sphere $\mathbb{S}^2$, the Stenzel metric coincides with the well known Eguchi-Hanson metric.

Let $M = G/K$ be a rank-one symmetric space of dimension $n > 2$ with a semi-simple compact connected Lie group G and a connected closed subgroup K, i.e.

$$G/K \in \{\mathbb{S}^n, \mathbb{C}\mathrm{P}^n, \mathbb{H}\mathrm{P}^n, \mathbb{C}\mathrm{aP}^2\}.$$

The tangent bundle $T(G/K)$ is a symplectic manifold with the symplectic structure Ω which comes from the canonical symplectic structure on the cotangent bundle $T^*(G/K)$ using a homogeneous metric g_M on $M = G/K$ to identify these two bundles.

The main RESULT

- All complete Ricci-flat Kähler G-invariant metrics $(\mathbf{g}, J, \Omega)$ on the tangent bundle TM with the fixed Kähler form Ω are classified.
- It is proved that the set of the equivalent classes $\{[(\mathbf{g}^{\gamma_a}, J^{\gamma_a}, \Omega)]\}$ of these metrics can be parameterized by positive numbers $a > 0$.
- Each class is an orbit of some infinite dimensional group of symplectomorphisms – the homomorphic image of the additive group of even functions $C^\infty_+(\mathbb{R}, \mathbb{R})$.
- This group of symplectomorphisms on $(T(G/K), \Omega)$is constructed for all reductive (not only compact and rank one) spaces G/K.

The classification is based on

- results of our paper where all (not only global) G-invariant Kähler structures (J, Ω) were described

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I.V. Mykytyuk, "Kähler structures on the tangent bundle of rank one symmetric spaces", *Sbornik: Mathematics* **192**:11 (2001), 1677–1704.

- on the idea of Stenzel to use a global holomorphic trivialization of the canonical line bundle $\Lambda^{n,0}(G^{\mathbb{C}}/K^{\mathbb{C}})$ to reduce the non-linear partial differential equation governing the Ricci form to a simple first-order ordinary differential equation for the function γ_a.

In the case when $\dim M = 2$ ($M = \mathbb{S}^2$) constructed metrics are Ricci-flat but our classification possibly is not complete.

The construction

- ⋆ The canonical G-equivariant diffeomorphism $G^{\mathbb{C}}/K^{\mathbb{C}} \to T(G/K)$ supplies the tangent bundle $T(G/K)$ with the canonical complex structure J_c^K (Mostow, 1955).
- ⋆ The pair (J_c^K, Ω) is a Kähler structure (R. Szőke, 1998)

1. We describe all G-invariant global Kähler structures (J, Ω) on TM using the previous results (Mykytyuk, 2001).

The construction

2. We show that each structure (J^{γ_a}, Ω) admits an alternative description in terms of the Kähler reduction(Guillemin, Sternberg,1982):

   the Kähler structure (J^{γ_a}, Ω) on $T(G/K)$ is a reduced Kähler structure canonically associated with some Kähler structure $(\tilde{J}, \tilde{\Omega})$ on $TG = T^*G$, where $\tilde{\Omega}$ is the canonical symplectic structure on T^*G.

⋆ The G-invariant Kähler structures, constructed by Stenzel, are structures of the type (J_c^K, Ω'_a), where $\Omega'_a = -i\partial\bar{\partial} f_a$ for some strictly plurisubharmonic function $f_a(v) = f_a(\|v\|)$ (in this case the canonical complex structure is fixed). Here $\|v\| = \sqrt{g_M(v,v)}$ denotes the norm of a vector $v \in TM$.

The construction

3. We prove that the Kähler structures (J^{γ_a}, Ω) and $(J_c^K, 2\Omega'_a)$ are diffeomorphic.

Stenzel's construction

Stenzel fixed the canonical complex structure J_c^K and looked for a G-invariant Kähler potential $f(r)$ (a strictly plurisubharmonic function) such that the Ricci form of the metric corresponding to the pair $(J_c^K, -i\partial\bar\partial f)$ is zero:

$$\mathrm{Ric}(f) \stackrel{\mathrm{def}}{=} -i\,\partial\bar\partial \ln\det \frac{\partial^2 f}{\partial z_j \partial \bar z_p} = 0.$$

Such a smooth potential function exists. This function is a unique smooth solution of the equation

$$\left((f'(r))^n\right)' = a \cdot r^{n-1}/S(r), \quad (a > 0),$$

where

$$S(r) = S^c, \quad \Omega^n = S^c \cdot \varepsilon_n \Theta_h \wedge \bar\Theta_h,$$

Stenzel's construction

and Θ_h – a $G^{\mathbb{C}}$-invariant non-vanishing holomorphic form of maximal rank on the complex homogeneous space $G^{\mathbb{C}}/K^{\mathbb{C}}$ (i.e. the canonical bundle $\Lambda^{(n,0)}(G^{\mathbb{C}}/K^{\mathbb{C}})$, $n = \dim G^{\mathbb{C}}/K^{\mathbb{C}}$ is holomorphically trivial).
We proved that $S^c(w) = \det E^c_w$, where

$$E^c_w : \mathfrak{m} \to \mathfrak{m}, \quad E^c_w = \left.\frac{\mathrm{ad}_w}{\cos \mathrm{ad}_w \sin \mathrm{ad}_w}\right|_{\mathfrak{m}}, \; w \in \mathfrak{m},$$

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}.$$

and $T_o(G/K) = \mathfrak{m}, o = \{K\} \in G/K$.

Our approach

We fix the canonical symplectic structure Ω and looked for a G-invariant Kähler structure (J, Ω) such that the Ricci form of the metric g corresponding to this pair is zero.

Theorem 1. *For any G-invariant Kähler structure (J, Ω) on the tangent bundle $T(G/K)$, $\dim G/K \geqslant 3$, there exists a unique pair of diffeomorphisms $(\Phi^{\chi}, \Psi^{\gamma})$, where $\chi \in C^{\infty}_{+}(\mathbb{R}, \mathbb{R})$ and $\gamma \in C^{\infty}_{\sharp}(\mathbb{R}, \mathbb{R}^{+})$, such that $(\Phi^{\chi}\Psi^{\gamma})_{*}J^{K}_{c} = J$.*

$C^{\infty}_{+}(\mathbb{R}, \mathbb{R})$ – the set of all smooth even real-valued functions on $\mathbb{R}$.
$C^{\infty}_{\sharp}(\mathbb{R}, \mathbb{R}^{+})$ – the set of all smooth positive even function $\gamma : \mathbb{R} \to \mathbb{R}^{+}$ such that $\frac{d}{dr}(r\gamma(r)) > 0$ for all $r \in \mathbb{R}$.

Theorem 2. *For G-invariant Kähler structure (J, Ω, g) on the tangent bundle $T(G/K)$, $\dim G/K \geqslant 3$, where $(\Psi^\gamma)_* J_c^K = J$, $\gamma \in C_\sharp^\infty(\mathbb{R}, \mathbb{R}^+)$ the metric g is Ricci-flat if and only if*

$$(y^n(r))' = a \cdot r^{n-1}/S(r), \quad (a > 0), \quad y(r) = r\gamma(r).$$

For any $a > 0$ there exists a unique smooth solution $y_a(r)$ such that the function $\gamma_a(r) = y_a(r)/r \in C_\sharp^\infty(\mathbb{R}, \mathbb{R}^+)$, where

$$\gamma_a(r) = \frac{1}{r}\left((a/2^{n-1})\int_0^r \sinh^{n-1}(2t)\cosh^d(2t)dt\right)^{1/n},$$

$d \in \{0, 1, 3, 7\}$.

Theorem 3. *The Kähler metric corresponding to the Kähler structure* $(J^{\gamma_a} = \Psi_*^{\gamma_a} J_c^K, \Omega)$ *is complete. The set* $\{(J^{\gamma_a}, \Omega), a > 0\}$ *consists of non-equivalent Kähler structures and any* G*-invariant Kähler structure on* $T(G/K)$, $\dim G/K \geqslant 3$, *with the Kähler form* Ω *and with the complete Ricci-flat metric is equivalent to some structure from this set.*