3-dimensional affine space forms and hyperbolic geometry

William M. Goldman

Department of Mathematics University of Maryland

26 March 2010
Second W. Killing and W. Weierstrass Colloquium
Braniewo, Poland

Euclidean manifolds

- When can a group G act on $\mathbb{R}^n$ with quotient $M^n = \mathbb{R}^n/G$ a (Hausdorff) manifold?
- G acts by Euclidean isometries $\Longrightarrow$ G finite extension of a subgroup of *translations* $G \cap \mathbb{R}^n \cong \mathbb{Z}^k$(Bieberbarch 1912);
- A Euclidean isometry is an *affine transformation*

$$\vec{x} \stackrel{\gamma}{\longmapsto} A\vec{x} + \vec{b}$$

$$A \in \mathrm{GL}(n, \mathbb{R}), \vec{b} \in \mathbb{R}^n,$$

where the *linear part* $\mathbb{L}(\gamma) = A$ is *orthogonal.* $(A \in \mathrm{O}(n))$

- Only finitely many topological types in each dimension.
- Only one *commensurability* class.

Euclidean manifolds

- When can a group G act on $\mathbb{R}^n$ with quotient $M^n = \mathbb{R}^n/G$ a (Hausdorff) manifold?
- G acts by Euclidean isometries $\Longrightarrow$ G finite extension of a subgroup of *translations* $G \cap \mathbb{R}^n \cong \mathbb{Z}^k$(Bieberbarch 1912);
- A Euclidean isometry is an *affine transformation*

$$\vec{x} \stackrel{\gamma}{\longmapsto} A\vec{x} + \vec{b}$$

$$A \in \mathrm{GL}(n, \mathbb{R}), \vec{b} \in \mathbb{R}^n,$$

where the *linear part* $\mathbb{L}(\gamma) = A$ is *orthogonal.* $(A \in \mathrm{O}(n))$

- Only finitely many topological types in each dimension.
- Only one *commensurability* class.

Euclidean manifolds

- When can a group G act on $\mathbb{R}^n$ with quotient $M^n = \mathbb{R}^n/G$ a (Hausdorff) manifold?
- G acts by Euclidean isometries $\Longrightarrow$ G finite extension of a subgroup of *translations* $G \cap \mathbb{R}^n \cong \mathbb{Z}^k$(Bieberbarch 1912);
- A Euclidean isometry is an *affine transformation*

$$\vec{x} \stackrel{\gamma}{\longmapsto} A\vec{x} + \vec{b}$$

$$A \in \mathrm{GL}(n, \mathbb{R}), \vec{b} \in \mathbb{R}^n,$$

  where the *linear part* $\mathbb{L}(\gamma) = A$ is *orthogonal.* $(A \in \mathrm{O}(n))$
- Only finitely many topological types in each dimension.
- Only one *commensurability* class.

Euclidean manifolds

- When can a group G act on $\mathbb{R}^n$ with quotient $M^n = \mathbb{R}^n/G$ a (Hausdorff) manifold?
- G acts by Euclidean isometries $\Longrightarrow$ G finite extension of a subgroup of *translations* $G \cap \mathbb{R}^n \cong \mathbb{Z}^k$ (Bieberbarch 1912);
- A Euclidean isometry is an *affine transformation*

$$\vec{x} \xrightarrow{\gamma} A\vec{x} + \vec{b}$$

$$A \in \mathrm{GL}(n, \mathbb{R}), \vec{b} \in \mathbb{R}^n,$$

  where the *linear part* $\mathbb{L}(\gamma) = A$ is *orthogonal.* $(A \in \mathrm{O}(n))$
- Only finitely many topological types in each dimension.
- Only one *commensurability* class.

Euclidean manifolds

- When can a group G act on $\mathbb{R}^n$ with quotient $M^n = \mathbb{R}^n/G$ a (Hausdorff) manifold?
- G acts by Euclidean isometries $\Longrightarrow$ G finite extension of a subgroup of *translations* $G \cap \mathbb{R}^n \cong \mathbb{Z}^k$(Bieberbarch 1912);
- A Euclidean isometry is an *affine transformation*

$$\vec{x} \stackrel{\gamma}{\longmapsto} A\vec{x} + \vec{b}$$

$$A \in \mathrm{GL}(n, \mathbb{R}), \vec{b} \in \mathbb{R}^n,$$

where the *linear part* $\mathbb{L}(\gamma) = A$ is *orthogonal.* $(A \in \mathrm{O}(n))$

- Only finitely many topological types in each dimension.
- Only one *commensurability* class.

Euclidean manifolds

- When can a group G act on $\mathbb{R}^n$ with quotient $M^n = \mathbb{R}^n/G$ a (Hausdorff) manifold?
- G acts by Euclidean isometries $\Longrightarrow$ G finite extension of a subgroup of *translations* $G \cap \mathbb{R}^n \cong \mathbb{Z}^k$(Bieberbarch 1912);
- A Euclidean isometry is an *affine transformation*

$$\vec{x} \xrightarrow{\gamma} A\vec{x} + \vec{b}$$

$$A \in \mathrm{GL}(n, \mathbb{R}), \vec{b} \in \mathbb{R}^n,$$

where the *linear part* $\mathbb{L}(\gamma) = A$ is *orthogonal.* $(A \in \mathrm{O}(n))$

- Only finitely many topological types in each dimension.
- Only one *commensurability* class.

Euclidean manifolds

- When can a group G act on $\mathbb{R}^n$ with quotient $M^n = \mathbb{R}^n/G$ a (Hausdorff) manifold?
- G acts by Euclidean isometries $\Longrightarrow$ G finite extension of a subgroup of *translations* $G \cap \mathbb{R}^n \cong \mathbb{Z}^k$(Bieberbarch 1912);
- A Euclidean isometry is an *affine transformation*

$$\vec{x} \stackrel{\gamma}{\longmapsto} A\vec{x} + \vec{b}$$

$$A \in \mathrm{GL}(n, \mathbb{R}), \vec{b} \in \mathbb{R}^n,$$

where the *linear part* $\mathbb{L}(\gamma) = A$ is *orthogonal.* $(A \in \mathrm{O}(n))$

- Only finitely many topological types in each dimension.
- Only one *commensurability* class.

Euclidean manifolds

- When can a group G act on $\mathbb{R}^n$ with quotient $M^n = \mathbb{R}^n/G$ a (Hausdorff) manifold?
- G acts by Euclidean isometries $\Longrightarrow$ G finite extension of a subgroup of *translations* $G \cap \mathbb{R}^n \cong \mathbb{Z}^k$(Bieberbarch 1912);
- A Euclidean isometry is an *affine transformation*

$$\vec{x} \xmapsto{\gamma} A\vec{x} + \vec{b}$$

$$A \in \mathrm{GL}(n, \mathbb{R}), \vec{b} \in \mathbb{R}^n,$$

where the *linear part* $\mathbb{L}(\gamma) = A$ is *orthogonal.* $(A \in \mathrm{O}(n))$

- Only finitely many topological types in each dimension.
- Only one *commensurability* class.

Euclidean manifolds

- When can a group G act on $\mathbb{R}^n$ with quotient $M^n = \mathbb{R}^n/G$ a (Hausdorff) manifold?
- G acts by Euclidean isometries $\implies$ G finite extension of a subgroup of *translations* $G \cap \mathbb{R}^n \cong \mathbb{Z}^k$(Bieberbarch 1912);
- A Euclidean isometry is an *affine transformation*

$$\vec{x} \xrightarrow{\gamma} A\vec{x} + \vec{b}$$

$$A \in \mathsf{GL}(n, \mathbb{R}), \vec{b} \in \mathbb{R}^n,$$

where the *linear part* $\mathbb{L}(\gamma) = A$ is *orthogonal.* $(A \in \mathsf{O}(n))$

- Only finitely many topological types in each dimension.
- Only one *commensurability* class.

Euclidean manifolds

- When can a group G act on $\mathbb{R}^n$ with quotient $M^n = \mathbb{R}^n/G$ a (Hausdorff) manifold?
- G acts by Euclidean isometries $\implies$ G finite extension of a subgroup of *translations* $G \cap \mathbb{R}^n \cong \mathbb{Z}^k$(Bieberbarch 1912);
- A Euclidean isometry is an *affine transformation*

$$\vec{x} \stackrel{\gamma}{\longmapsto} A\vec{x} + \vec{b}$$

$$A \in \mathsf{GL}(n, \mathbb{R}), \vec{b} \in \mathbb{R}^n,$$

  where the *linear part* $\mathbb{L}(\gamma) = A$ is *orthogonal.* $(A \in \mathsf{O}(n))$
- Only finitely many topological types in each dimension.
- Only one *commensurability* class.

Euclidean manifolds

- When can a group G act on $\mathbb{R}^n$ with quotient $M^n = \mathbb{R}^n/G$ a (Hausdorff) manifold?
- G acts by Euclidean isometries $\implies$ G finite extension of a subgroup of *translations* $G \cap \mathbb{R}^n \cong \mathbb{Z}^k$(Bieberbarch 1912);
- A Euclidean isometry is an *affine transformation*

$$\vec{x} \xrightarrow{\gamma} A\vec{x} + \vec{b}$$

$$A \in \mathrm{GL}(n, \mathbb{R}), \vec{b} \in \mathbb{R}^n,$$

  where the *linear part* $\mathbb{L}(\gamma) = A$ is *orthogonal.* $(A \in \mathrm{O}(n))$
- Only finitely many topological types in each dimension.
- Only one *commensurability* class.

Complete affine manifolds

- A *complete affine manifold* M^n is a quotient $\mathbb{R}^n/G$ where G is a discrete group of affine transformations.
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \mathrm{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \Longrightarrow$ go to ∞ in every orbit Gx).
    - More precisely, the map

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

      is a proper map (preimages of compacta are compact).
    - Unlike Riemannian isometries, discreteness does not imply properness.
- Equivalently this structure is a geodesically complete torsionfree affine connection on M.

Complete affine manifolds

- A *complete affine manifold* M^n is a quotient $\mathbb{R}^n/G$ where G is a discrete group of affine transformations.
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \text{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \Longrightarrow$ go to ∞ in every orbit Gx).
    - More precisely, the map

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

      is a proper map (preimages of compacta are compact).
    - Unlike Riemannian isometries, discreteness does not imply properness.
- Equivalently this structure is a geodesically complete torsionfree affine connection on M.

Complete affine manifolds

- A *complete affine manifold* M^n is a quotient $\mathbb{R}^n/G$ where G is a discrete group of affine transformations.
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \mathrm{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \Longrightarrow$ go to ∞ in every orbit Gx).
    - More precisely, the map

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

      is a proper map (preimages of compacta are compact).
    - Unlike Riemannian isometries, discreteness does not imply properness.
- Equivalently this structure is a geodesically complete torsionfree affine connection on M.

Complete affine manifolds

- A *complete affine manifold* M^n is a quotient $\mathbb{R}^n/G$ where G is a discrete group of affine transformations.
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \mathrm{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \implies$ go to ∞ in every orbit Gx).
    - More precisely, the map

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

      is a proper map (preimages of compacta are compact).
    - Unlike Riemannian isometries, discreteness does not imply properness.
- Equivalently this structure is a geodesically complete torsionfree affine connection on M.

Complete affine manifolds

- A *complete affine manifold* M^n is a quotient $\mathbb{R}^n/G$ where G is a discrete group of affine transformations.
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \mathrm{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \Longrightarrow$ go to ∞ in every orbit Gx).
    - More precisely, the map

$$G \times X \longrightarrow X \times X$$
$$(g, x) \longmapsto (gx, x)$$

      is a proper map (preimages of compacta are compact).
    - Unlike Riemannian isometries, discreteness does not imply properness.
- Equivalently this structure is a geodesically complete torsionfree affine connection on M.

Complete affine manifolds

- A *complete affine manifold* M^n is a quotient $\mathbb{R}^n/G$ where G is a discrete group of affine transformations.
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \mathrm{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \implies$ go to ∞ in every orbit Gx).
    - More precisely, the map

$$G \times X \longrightarrow X \times X$$
$$(g, x) \longmapsto (gx, x)$$

      is a proper map (preimages of compacta are compact).
    - Unlike Riemannian isometries, discreteness does not imply properness.
- Equivalently this structure is a geodesically complete torsionfree affine connection on M.

Complete affine manifolds

- A *complete affine manifold* M^n is a quotient $\mathbb{R}^n/G$ where G is a discrete group of affine transformations.
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \text{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \implies$ go to ∞ in every orbit Gx).
    - More precisely, the map
$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$
      is a proper map (preimages of compacta are compact).
    - Unlike Riemannian isometries, discreteness does not imply properness.
- Equivalently this structure is a geodesically complete torsionfree affine connection on M.

Complete affine manifolds

- A *complete affine manifold* M^n is a quotient $\mathbb{R}^n/G$ where G is a discrete group of affine transformations.
- For M to be a (Hausdorff) smooth manifold, G must act:
  - Discretely: ($G \subset \text{Homeo}(\mathbb{R}^n)$ discrete);
  - Freely: (No fixed points);
  - Properly: (Go to ∞ in $G \implies$ go to ∞ in every orbit Gx).
    - More precisely, the map

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

      is a proper map (preimages of compacta are compact).
    - Unlike Riemannian isometries, discreteness does not imply properness.
- Equivalently this structure is a geodesically complete torsionfree affine connection on M.

Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - G is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $\mathbb{L}(G) \subset O(2,1)$ is isomorphic to G.
    - M^3 noncompact complete flat Lorentz 3-manifold.
  - Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
  - Closely related to the geometry of M^3 is a *deformation* of the hyperbolic structure on Σ^2.
- Conjecture: *Every Margulis spacetime is diffeomorphic to a solid handlebody.*

Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - G is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $\mathbb{L}(G) \subset \mathrm{O}(2,1)$ is isomorphic to G.
    - M^3 noncompact complete flat Lorentz 3-manifold.
  - Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
  - Closely related to the geometry of M^3 is a *deformation* of the hyperbolic structure on Σ^2.
- Conjecture: *Every Margulis spacetime is diffeomorphic to a solid handlebody.*

Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - G is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $\mathbb{L}(G) \subset O(2,1)$ is isomorphic to G.
    - M^3 noncompact complete flat Lorentz 3-manifold.
  - Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
  - Closely related to the geometry of M^3 is a *deformation* of the hyperbolic structure on Σ^2.
- Conjecture: *Every Margulis spacetime is diffeomorphic to a solid handlebody.*

Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - G is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $\mathbb{L}(G) \subset O(2,1)$ is isomorphic to G.
    - M^3 noncompact complete flat Lorentz 3-manifold.
  - Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
  - Closely related to the geometry of M^3 is a *deformation* of the hyperbolic structure on Σ^2.
- Conjecture: *Every Margulis spacetime is diffeomorphic to a solid handlebody.*

Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - G is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $\mathbb{L}(G) \subset \mathrm{O}(2,1)$ is isomorphic to G.
    - M^3 noncompact complete flat Lorentz 3-manifold.
  - Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
  - Closely related to the geometry of M^3 is a *deformation* of the hyperbolic structure on Σ^2.
- Conjecture: *Every Margulis spacetime is diffeomorphic to a solid handlebody.*

Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - G is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $\mathbb{L}(G) \subset \mathrm{O}(2,1)$ is isomorphic to G.
    - M^3 noncompact complete flat Lorentz 3-manifold.
  - Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
  - Closely related to the geometry of M^3 is a *deformation* of the hyperbolic structure on Σ^2.
- Conjecture: *Every Margulis spacetime is diffeomorphic to a solid handlebody.*

Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - G is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $\mathbb{L}(G) \subset \mathrm{O}(2,1)$ is isomorphic to G.
    - M^3 noncompact complete flat Lorentz 3-manifold.
  - Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
  - Closely related to the geometry of M^3 is a *deformation* of the hyperbolic structure on Σ^2.
- Conjecture: *Every Margulis spacetime is diffeomorphic to a solid handlebody.*

Margulis Spacetimes

- Most interesting examples: Margulis ($\sim$ 1980):
  - G is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $\mathbb{L}(G) \subset \mathrm{O}(2,1)$ is isomorphic to G.
    - M^3 noncompact complete flat Lorentz 3-manifold.
  - Associated to every Margulis spacetime M^3 is a noncompact complete hyperbolic surface Σ^2.
  - Closely related to the geometry of M^3 is a *deformation* of the hyperbolic structure on Σ^2.
- Conjecture: *Every Margulis spacetime is diffeomorphic to a solid handlebody.*

Geometric 3-manifolds

- Unlike the 8 geometries of Thurston's Geometrization, affine structures are *not Riemannian.*
  - No obvious metrics.
  - Usual tools (distance, angle, metric convexity, completeness, volume) NOT available.

Conjecture:

A complete affine 3-manifold $M^3 = \mathbb{R}^3/\Gamma$ is finitely covered by:

- *An iterated fibration by cells and circles; or*
- *An open solid handlebody (Margulis, Drumm examples).*

Geometric 3-manifolds

- Unlike the 8 geometries of Thurston's Geometrization, affine structures are *not Riemannian.*
  - No obvious metrics.
  - Usual tools (distance, angle, metric convexity, completeness, volume) NOT available.

Conjecture:

A complete affine 3-manifold $M^3 = \mathbb{R}^3/\Gamma$ is finitely covered by:

- *An iterated fibration by cells and circles; or*
- *An open solid handlebody (Margulis, Drumm examples).*

Geometric 3-manifolds

- Unlike the 8 geometries of Thurston's Geometrization, affine structures are *not Riemannian.*
  - No obvious metrics.
  - Usual tools (distance, angle, metric convexity, completeness, volume) NOT available.

Conjecture:

A complete affine 3-manifold $M^3 = \mathbb{R}^3/\Gamma$ is finitely covered by:

- *An iterated fibration by cells and circles; or*
- *An open solid handlebody (Margulis, Drumm examples).*

Geometric 3-manifolds

- Unlike the 8 geometries of Thurston's Geometrization, affine structures are *not Riemannian.*
  - No obvious metrics.
  - Usual tools (distance, angle, metric convexity, completeness, volume) NOT available.

Conjecture:

A complete affine 3-manifold $M^3 = \mathbb{R}^3/\Gamma$ is finitely covered by:

- *An iterated fibration by cells and circles; or*
- *An open solid handlebody (Margulis, Drumm examples).*

Geometric 3-manifolds

- Unlike the 8 geometries of Thurston's Geometrization, affine structures are *not Riemannian.*
  - No obvious metrics.
  - Usual tools (distance, angle, metric convexity, completeness, volume) NOT available.

Conjecture:

A complete affine 3-manifold $M^3 = \mathbb{R}^3/\Gamma$ *is finitely covered by:*

- *An iterated fibration by cells and circles; or*
- *An open solid handlebody (Margulis, Drumm examples).*

Geometric 3-manifolds

- Unlike the 8 geometries of Thurston's Geometrization, affine structures are *not Riemannian.*
  - No obvious metrics.
  - Usual tools (distance, angle, metric convexity, completeness, volume) NOT available.

Conjecture:

A complete affine 3-manifold $M^3 = \mathbb{R}^3/\Gamma$ is finitely covered by:

- *An iterated fibration by cells and circles; or*
- *An open solid handlebody (Margulis, Drumm examples).*

Geometric 3-manifolds

- Unlike the 8 geometries of Thurston's Geometrization, affine structures are *not Riemannian.*
  - No obvious metrics.
  - Usual tools (distance, angle, metric convexity, completeness, volume) NOT available.

Conjecture:

A complete affine 3-manifold $M^3 = \mathbb{R}^3/\Gamma$ *is finitely covered by:*

- *An iterated fibration by cells and circles; or*
- *An open solid handlebody (Margulis, Drumm examples).*

Geometric 3-manifolds

- Unlike the 8 geometries of Thurston's Geometrization, affine structures are *not Riemannian.*
  - No obvious metrics.
  - Usual tools (distance, angle, metric convexity, completeness, volume) NOT available.

Conjecture:

A complete affine 3-manifold $M^3 = \mathbb{R}^3/\Gamma$ is finitely covered by:

- *An iterated fibration by cells and circles; or*
- *An open solid handlebody (Margulis, Drumm examples).*

Milnor's Question (1977)

Can a nonabelian free group act properly, freely and discretely by affine transformations on $\mathbb{R}^n$?

- Equivalently (Tits 1971): *"Are there discrete groups other than virtually polycycic groups which act properly, affinely?"*
  - If NO, M^n finitely covered by iterated fibration by cells and circles.
  - Dimension 3: M^3 *compact* $\Longrightarrow$ M^3 finitely covered by T^2-bundle over S^1 (Fried-G 1983),
  - Geometrizable by **Euc**, **Nil** or **Sol**.

Milnor's Question (1977)

Can a nonabelian free group act properly, freely and discretely by affine transformations on $\mathbb{R}^n$?

- Equivalently (Tits 1971): *"Are there discrete groups other than virtually polycycic groups which act properly, affinely?"*
  - If NO, M^n finitely covered by iterated fibration by cells and circles.
  - Dimension 3: M^3 *compact* $\Longrightarrow$ M^3 finitely covered by T^2-bundle over S^1 (Fried-G 1983),
  - Geometrizable by **Euc**, **Nil** or **Sol**.

Milnor's Question (1977)

Can a nonabelian free group act properly, freely and discretely by affine transformations on $\mathbb{R}^n$?

- Equivalently (Tits 1971): *"Are there discrete groups other than virtually polycyclic groups which act properly, affinely?"*
  - If NO, M^n finitely covered by iterated fibration by cells and circles.
  - Dimension 3: M^3 *compact* $\Longrightarrow$ M^3 finitely covered by T^2-bundle over S^1 (Fried-G 1983),
  - Geometrizable by **Euc**, **Nil** or **Sol**.

Milnor's Question (1977)

Can a nonabelian free group act properly, freely and discretely by affine transformations on $\mathbb{R}^n$?

- Equivalently (Tits 1971): *"Are there discrete groups other than virtually polycyclic groups which act properly, affinely?"*
  - If NO, M^n finitely covered by iterated fibration by cells and circles.
  - Dimension 3: M^3 *compact* $\Longrightarrow$ M^3 finitely covered by T^2-bundle over S^1 (Fried-G 1983),
  - Geometrizable by **Euc**, **Nil** or **Sol**.

Milnor's Question (1977)

Can a nonabelian free group act properly, freely and discretely by affine transformations on $\mathbb{R}^n$?

- Equivalently (Tits 1971): *"Are there discrete groups other than virtually polycyclic groups which act properly, affinely?"*
  - If NO, M^n finitely covered by iterated fibration by cells and circles.
  - Dimension 3: M^3 *compact* $\Longrightarrow$ M^3 finitely covered by T^2-bundle over S^1 (Fried-G 1983),
  - Geometrizable by **Euc**, **Nil** or **Sol**.

Milnor's Question (1977)

Can a nonabelian free group act properly, freely and discretely by affine transformations on $\mathbb{R}^n$?

- Equivalently (Tits 1971): *"Are there discrete groups other than virtually polycyclic groups which act properly, affinely?"*
  - If NO, M^n finitely covered by iterated fibration by cells and circles.
  - Dimension 3: M^3 *compact* $\Longrightarrow$ M^3 finitely covered by T^2-bundle over S^1 (Fried-G 1983),
  - Geometrizable by **Euc**, **Nil** or **Sol**.

Evidence?

Milnor offers the following results as possible "evidence" for a negative answer to this question.

- *Connected Lie group G admits a proper affine action $\Longleftrightarrow$ G is amenable (compact-by-solvable).*
- *Every virtually polycyclic group admits a proper affine action.*

Evidence?

Milnor offers the following results as possible "evidence" for a negative answer to this question.

- *Connected Lie group G admits a proper affine action $\iff G$ is amenable (compact-by-solvable).*
- *Every virtually polycyclic group admits a proper affine action.*

Evidence?

Milnor offers the following results as possible "evidence" for a negative answer to this question.

- *Connected Lie group G admits a proper affine action $\iff$ G is amenable (compact-by-solvable).*
- *Every virtually polycyclic group admits a proper affine action.*

An idea for a counterexample...

- Clearly a geometric problem: free groups act properly by isometries on H^3 hence by diffeomorphisms of $\mathbb{E}^3$
- These actions are *not* affine.
- Milnor suggests:

  Start with a free discrete subgroup of $O(2,1)$ and add translation components to obtain a group of affine transformations which acts freely. However it seems difficult to decide whether the resulting group action is properly discontinuous."

An idea for a counterexample...

- Clearly a geometric problem: free groups act properly by isometries on H^3 hence by diffeomorphisms of $\mathbb{E}^3$
- These actions are *not* affine.
- Milnor suggests:

> Start with a free discrete subgroup of $O(2,1)$ and add translation components to obtain a group of affine transformations which acts freely.
> However it seems difficult to decide whether the resulting group action is properly discontinuous."

An idea for a counterexample...

- Clearly a geometric problem: free groups act properly by isometries on H^3 hence by diffeomorphisms of $\mathbb{E}^3$
- These actions are *not* affine.
- Milnor suggests:

  Start with a free discrete subgroup of $O(2,1)$ and add translation components to obtain a group of affine transformations which acts freely.
  However it seems difficult to decide whether the resulting group action is properly discontinuous."

An idea for a counterexample...

- Clearly a geometric problem: free groups act properly by isometries on H^3 hence by diffeomorphisms of $\mathbb{E}^3$
- These actions are *not* affine.
- Milnor suggests:

  Start with a free discrete subgroup of $O(2,1)$ and add translation components to obtain a group of affine transformations which acts freely.
  However it seems difficult to decide whether the resulting group action is properly discontinuous."

An idea for a counterexample...

- Clearly a geometric problem: free groups act properly by isometries on H^3 hence by diffeomorphisms of $\mathbb{E}^3$
- These actions are *not* affine.
- Milnor suggests:

  Start with a free discrete subgroup of $O(2,1)$ and add translation components to obtain a group of affine transformations which acts freely.
  However it seems difficult to decide whether the resulting group action is properly discontinuous."

An idea for a counterexample...

- Clearly a geometric problem: free groups act properly by isometries on H^3 hence by diffeomorphisms of $\mathbb{E}^3$
- These actions are *not* affine.
- Milnor suggests:

> Start with a free discrete subgroup of $\mathrm{O}(2,1)$ and add translation components to obtain a group of affine transformations which acts freely. However it seems difficult to decide whether the resulting group action is properly discontinuous."

An idea for a counterexample...

- Clearly a geometric problem: free groups act properly by isometries on H^3 hence by diffeomorphisms of $\mathbb{E}^3$
- These actions are *not* affine.
- Milnor suggests:

> Start with a free discrete subgroup of $O(2,1)$ and add translation components to obtain a group of affine transformations which acts freely.
> However it seems difficult to decide whether the resulting group action is properly discontinuous."

Lorentzian and Hyperbolic Geometry

- $\mathbb{R}^{2,1}$ is the 3-dimensional real vector space with inner product:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} := x_1x_2 + y_1y_2 - z_1z_2$$

  and Minkowski space $\mathbb{E}^{2,1}$ is the corresponding *affine space*, a simply connected geodesically complete Lorentzian manifold.
- The Lorentz metric tensor is $dx^2 + dy^2 - dz^2$.
- $\mathrm{Isom}(\mathbb{E}^{2,1})$ is the semidirect product of $\mathbb{R}^{2,1}$ (the vector group of translations) with the orthogonal group $\mathrm{O}(2,1)$.
- The stabiizer of the origin is the group $\mathrm{O}(2,1)$ which preserves the hyperbolic plane

$$\mathrm{H}^2 := \{v \in \mathbb{R}^{2,1} \mid v \cdot v = -1, z > 0\}.$$

Lorentzian and Hyperbolic Geometry

- $\mathbb{R}^{2,1}$ is the 3-dimensional real vector space with inner product:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} := x_1 x_2 + y_1 y_2 - z_1 z_2$$

and Minkowski space $\mathbb{E}^{2,1}$ is the corresponding *affine space*, a simply connected geodesically complete Lorentzian manifold.

- The Lorentz metric tensor is $dx^2 + dy^2 - dz^2$.
- $\mathrm{Isom}(\mathbb{E}^{2,1})$ is the semidirect product of $\mathbb{R}^{2,1}$ (the vector group of translations) with the orthogonal group $\mathrm{O}(2,1)$.
- The stabiizer of the origin is the group $\mathrm{O}(2,1)$ which preserves the hyperbolic plane

$$\mathrm{H}^2 := \{ v \in \mathbb{R}^{2,1} \mid v \cdot v = -1, z > 0 \}.$$

Lorentzian and Hyperbolic Geometry

- $\mathbb{R}^{2,1}$ is the 3-dimensional real vector space with inner product:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} := x_1 x_2 + y_1 y_2 - z_1 z_2$$

  and Minkowski space $\mathbb{E}^{2,1}$ is the corresponding *affine space*, a simply connected geodesically complete Lorentzian manifold.
- The Lorentz metric tensor is $dx^2 + dy^2 - dz^2$.
- $\mathrm{Isom}(\mathbb{E}^{2,1})$ is the semidirect product of $\mathbb{R}^{2,1}$ (the vector group of translations) with the orthogonal group $O(2,1)$.
- The stabiizer of the origin is the group $O(2,1)$ which preserves the hyperbolic plane

$$H^2 := \{v \in \mathbb{R}^{2,1} \mid v \cdot v = -1, z > 0\}.$$

Lorentzian and Hyperbolic Geometry

- $\mathbb{R}^{2,1}$ is the 3-dimensional real vector space with inner product:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} := x_1 x_2 + y_1 y_2 - z_1 z_2$$

  and Minkowski space $\mathbb{E}^{2,1}$ is the corresponding *affine space*, a simply connected geodesically complete Lorentzian manifold.
- The Lorentz metric tensor is $dx^2 + dy^2 - dz^2$.
- $\mathrm{Isom}(\mathbb{E}^{2,1})$ is the semidirect product of $\mathbb{R}^{2,1}$ (the vector group of translations) with the orthogonal group $O(2,1)$.
- The stabiizer of the origin is the group $O(2,1)$ which preserves the hyperbolic plane

$$H^2 := \{ v \in \mathbb{R}^{2,1} \mid v \cdot v = -1, z > 0 \}.$$

Lorentzian and Hyperbolic Geometry

- $\mathbb{R}^{2,1}$ is the 3-dimensional real vector space with inner product:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} := x_1 x_2 + y_1 y_2 - z_1 z_2$$

  and Minkowski space $\mathbb{E}^{2,1}$ is the corresponding *affine space*, a simply connected geodesically complete Lorentzian manifold.
- The Lorentz metric tensor is $dx^2 + dy^2 - dz^2$.
- $\mathsf{Isom}(\mathbb{E}^{2,1})$ is the semidirect product of $\mathbb{R}^{2,1}$ (the vector group of translations) with the orthogonal group $\mathsf{O}(2,1)$.
- The stabiizer of the origin is the group $\mathsf{O}(2,1)$ which preserves the hyperbolic plane

$$\mathsf{H}^2 := \{ v \in \mathbb{R}^{2,1} \mid v \cdot v = -1, z > 0 \}.$$

Lorentzian and Hyperbolic Geometry

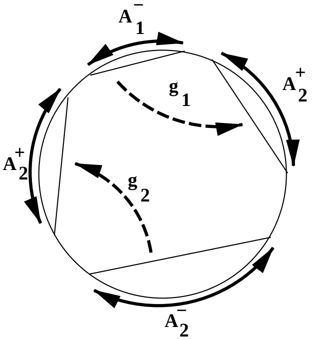
- $\mathbb{R}^{2,1}$ is the 3-dimensional real vector space with inner product:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} := x_1 x_2 + y_1 y_2 - z_1 z_2$$

  and Minkowski space $\mathbb{E}^{2,1}$ is the corresponding *affine space*, a simply connected geodesically complete Lorentzian manifold.
- The Lorentz metric tensor is $dx^2 + dy^2 - dz^2$.
- $\mathsf{Isom}(\mathbb{E}^{2,1})$ is the semidirect product of $\mathbb{R}^{2,1}$ (the vector group of translations) with the orthogonal group $\mathsf{O}(2,1)$.
- The stabiizer of the origin is the group $\mathsf{O}(2,1)$ which preserves the hyperbolic plane

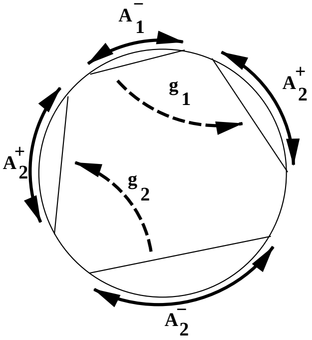
$$\mathsf{H}^2 := \{v \in \mathbb{R}^{2,1} \mid v \cdot v = -1, z > 0\}.$$

A Schottky group



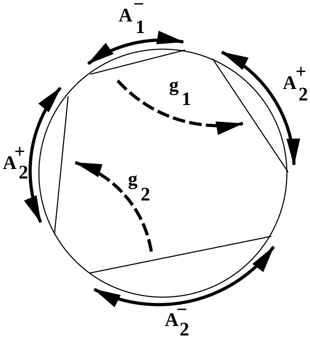
- Generators g_1, g_2 pair half-spaces $A_i^- \longrightarrow \mathrm{H}^2 \setminus A_i^+$.
- g_1, g_2 *freely* generate *discrete* group.
- Action proper with fundamental domain $\mathrm{H}^2 \setminus \bigcup A_i^\pm$.

A Schottky group



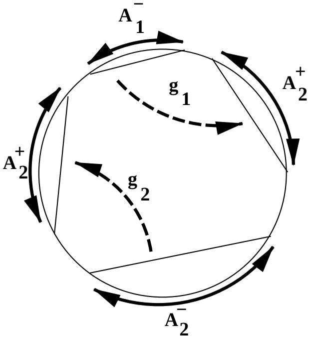
- Generators g_1, g_2 pair half-spaces $A_i^- \longrightarrow \mathrm{H}^2 \setminus A_i^+$.
- g_1, g_2 *freely* generate *discrete* group.
- Action proper with fundamental domain $\mathrm{H}^2 \setminus \bigcup A_i^\pm$.

A Schottky group



- Generators g_1, g_2 pair half-spaces $A_i^- \longrightarrow \mathrm{H}^2 \setminus A_i^+$.
- g_1, g_2 *freely* generate *discrete* group.
- Action proper with fundamental domain $\mathrm{H}^2 \setminus \bigcup A_i^\pm$.

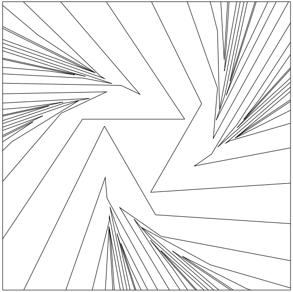
A Schottky group



- Generators g_1, g_2 pair half-spaces $A_i^- \longrightarrow \mathrm{H}^2 \setminus A_i^+$.
- g_1, g_2 *freely* generate *discrete* group.
- Action proper with fundamental domain $\mathrm{H}^2 \setminus \bigcup A_i^{\pm}$.

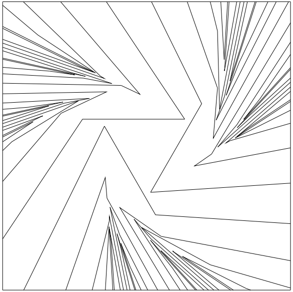
Margulis's examples

Early 1980's: Margulis tried to answer Milnor's question negatively but instead proved that nonabelian free groups can act properly, affinely on $\mathbb{R}^3$.



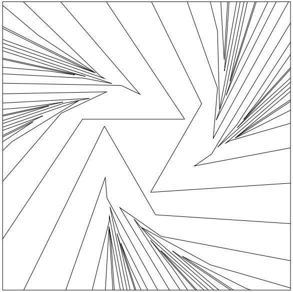
Margulis's examples

Early 1980's: Margulis tried to answer Milnor's question negatively but instead proved that nonabelian free groups can act properly, affinely on $\mathbb{R}^3$.



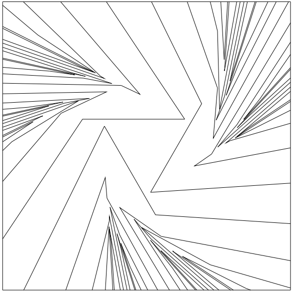
Margulis's examples

Early 1980's: Margulis tried to answer Milnor's question negatively but instead proved that nonabelian free groups can act properly, affinely on $\mathbb{R}^3$.



Margulis's examples

Early 1980's: Margulis tried to answer Milnor's question negatively but instead proved that nonabelian free groups can act properly, affinely on $\mathbb{R}^3$.



Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ acts properly and is not solvable.

- Let $\Gamma \xrightarrow{\mathbb{L}} \mathsf{GL}(3, \mathbb{R})$ be the *linear part.*
  - $\mathbb{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathsf{O}(2, 1)$;
  - $\mathbb{L}$ injective. (Fried-G 1983).
- Homotopy equivalence

$$M^3 := \mathbb{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathbb{L}(\Gamma)$$

  where Σ complete hyperbolic surface.
  - Mess (1990): Σ not compact .
- Γ free;
- Milnor's suggestion is the *only way* to construct examples.

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ acts properly and is not solvable.

- Let $\Gamma \xrightarrow{\mathbb{L}} \mathsf{GL}(3,\mathbb{R})$ be the *linear part.*
  - $\mathbb{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathsf{O}(2,1)$;
  - $\mathbb{L}$ injective. (Fried-G 1983).
- Homotopy equivalence

$$M^3 := \mathbb{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathbb{L}(\Gamma)$$

where Σ complete hyperbolic surface.

  - Mess (1990): Σ not compact .
- Γ free;
- Milnor's suggestion is the *only way* to construct examples.

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ acts properly and is not solvable.

- Let $\Gamma \xrightarrow{\mathbb{L}} \mathsf{GL}(3, \mathbb{R})$ be the *linear part.*
  - $\mathbb{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathsf{O}(2,1)$;
  - $\mathbb{L}$ injective. (Fried-G 1983).
- Homotopy equivalence

$$M^3 := \mathbb{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathbb{L}(\Gamma)$$

where Σ complete hyperbolic surface.

  - Mess (1990): Σ not compact .
- Γ free;
- Milnor's suggestion is the *only way* to construct examples.

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ acts properly and is not solvable.

- Let $\Gamma \xrightarrow{\mathbb{L}} \mathsf{GL}(3,\mathbb{R})$ be the *linear part.*
  - $\mathbb{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathsf{O}(2,1)$;
  - $\mathbb{L}$ injective. (Fried-G 1983).
- Homotopy equivalence

$$M^3 := \mathbb{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathbb{L}(\Gamma)$$

  where Σ complete hyperbolic surface.
  - Mess (1990): Σ not compact .
- Γ free;
- Milnor's suggestion is the *only way* to construct examples.

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathrm{Aff}(\mathbb{R}^3)$ acts properly and is not solvable.

- Let $\Gamma \xrightarrow{\mathbb{L}} \mathrm{GL}(3,\mathbb{R})$ be the *linear part.*
  - $\mathbb{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathrm{O}(2,1)$;
  - $\mathbb{L}$ injective. (Fried-G 1983).
- Homotopy equivalence

$$M^3 := \mathbb{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathrm{H}^2/\mathbb{L}(\Gamma)$$

  where Σ complete hyperbolic surface.
  - Mess (1990): Σ not compact .
- Γ free;
- Milnor's suggestion is the *only way* to construct examples.

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ acts properly and is not solvable.

- Let $\Gamma \xrightarrow{\mathbb{L}} \mathsf{GL}(3,\mathbb{R})$ be the *linear part.*
  - $\mathbb{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathsf{O}(2,1)$;
  - $\mathbb{L}$ injective. (Fried-G 1983).
- Homotopy equivalence

$$M^3 := \mathbb{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathbb{L}(\Gamma)$$

  where Σ complete hyperbolic surface.
  - Mess (1990): Σ not compact .
- Γ free;
- Milnor's suggestion is the *only way* to construct examples.

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ acts properly and is not solvable.

- Let $\Gamma \xrightarrow{\mathbb{L}} \mathsf{GL}(3,\mathbb{R})$ be the *linear part.*
  - $\mathbb{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathsf{O}(2,1)$;
  - $\mathbb{L}$ injective. (Fried-G 1983).
- Homotopy equivalence

$$M^3 := \mathbb{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathbb{L}(\Gamma)$$

  where Σ complete hyperbolic surface.
  - Mess (1990): Σ not compact .
- Γ free;
- Milnor's suggestion is the *only way* to construct examples.

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ acts properly and is not solvable.

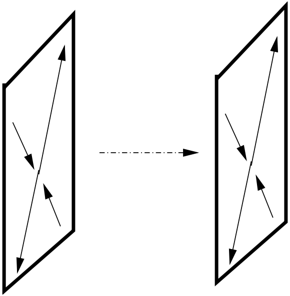
- Let $\Gamma \xrightarrow{\mathbb{L}} \mathsf{GL}(3,\mathbb{R})$ be the *linear part.*
  - $\mathbb{L}(\Gamma)$ (conjugate to) a *discrete* subgroup of $\mathsf{O}(2,1)$;
  - $\mathbb{L}$ injective. (Fried-G 1983).
- Homotopy equivalence

$$M^3 := \mathbb{E}^{2,1}/\Gamma \longrightarrow \Sigma := \mathsf{H}^2/\mathbb{L}(\Gamma)$$

  where Σ complete hyperbolic surface.
  - Mess (1990): Σ not compact .
- Γ free;
- Milnor's suggestion is the *only way* to construct examples.

Cyclic groups

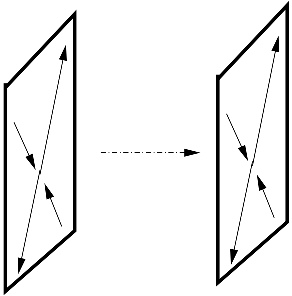
- Most elements $\gamma \in \Gamma$ are *boosts*, affine deformations of hyperbolic elements of $O(2,1)$. A fundamental domain is the *slab* bounded by two parallel planes.



A boost identifying two parallel planes.

Cyclic groups

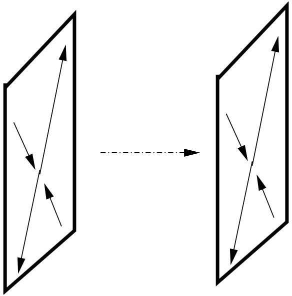
- Most elements $\gamma \in \Gamma$ are *boosts*, affine deformations of hyperbolic elements of $O(2,1)$. A fundamental domain is the *slab* bounded by two parallel planes.



A boost identifying two parallel planes

Cyclic groups

- Most elements $\gamma \in \Gamma$ are *boosts*, affine deformations of hyperbolic elements of $O(2,1)$. A fundamental domain is the *slab* bounded by two parallel planes.



A boost identifying two parallel planes

Closed geodesics and holonomy

- Each such element leaves invariant a unique (spacelike) line, whose image in $\mathbb{E}^{2,1}/\Gamma$ is a *closed geodesic.* Just as for hyperbolic surfaces, most loops are freely homotopic to closed geodesics.

$$\gamma = \begin{bmatrix} e^{\ell(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\ell(\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha(\gamma) \\ 0 \end{bmatrix}$$

- $\ell(\gamma) \in \mathbb{R}^+$: *geodesic length* of γ
- $\alpha(\gamma) \in \mathbb{R}$: (signed) *Lorentzian length* of γ.
- The unique γ-invariant geodesic C_γ inherits a natural orientation and metric and γ translates along C_γ by $\alpha(\gamma)$.

Closed geodesics and holonomy

- Each such element leaves invariant a unique (spacelike) line, whose image in $\mathbb{E}^{2,1}/\Gamma$ is a *closed geodesic.* Just as for hyperbolic surfaces, most loops are freely homotopic to closed geodesics.

$$\gamma = \begin{bmatrix} e^{\ell(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\ell(\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha(\gamma) \\ 0 \end{bmatrix}$$

- $\ell(\gamma) \in \mathbb{R}^+$: *geodesic length* of γ
- $\alpha(\gamma) \in \mathbb{R}$: (signed) *Lorentzian length* of γ.
- The unique γ-invariant geodesic C_γ inherits a natural orientation and metric and γ translates along C_γ by $\alpha(\gamma)$.

Closed geodesics and holonomy

- Each such element leaves invariant a unique (spacelike) line, whose image in $\mathbb{E}^{2,1}/\Gamma$ is a *closed geodesic.* Just as for hyperbolic surfaces, most loops are freely homotopic to closed geodesics.

$$\gamma = \begin{bmatrix} e^{\ell(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\ell(\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha(\gamma) \\ 0 \end{bmatrix}$$

- $\ell(\gamma) \in \mathbb{R}^+$: *geodesic length* of γ
- $\alpha(\gamma) \in \mathbb{R}$: (signed) *Lorentzian length* of γ.
- The unique γ-invariant geodesic C_γ inherits a natural orientation and metric and γ translates along C_γ by $\alpha(\gamma)$.

Closed geodesics and holonomy

- Each such element leaves invariant a unique (spacelike) line, whose image in $\mathbb{E}^{2,1}/\Gamma$ is a *closed geodesic.* Just as for hyperbolic surfaces, most loops are freely homotopic to closed geodesics.

$$\gamma = \begin{bmatrix} e^{\ell(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\ell(\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha(\gamma) \\ 0 \end{bmatrix}$$

- $\ell(\gamma) \in \mathbb{R}^+$: *geodesic length* of γ
- $\alpha(\gamma) \in \mathbb{R}$: (signed) *Lorentzian length* of γ.
- The unique γ-invariant geodesic C_γ inherits a natural orientation and metric and γ translates along C_γ by $\alpha(\gamma)$.

Closed geodesics and holonomy

- Each such element leaves invariant a unique (spacelike) line, whose image in $\mathbb{E}^{2,1}/\Gamma$ is a *closed geodesic.* Just as for hyperbolic surfaces, most loops are freely homotopic to closed geodesics.

$$\gamma = \begin{bmatrix} e^{\ell(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\ell(\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha(\gamma) \\ 0 \end{bmatrix}$$

- $\ell(\gamma) \in \mathbb{R}^+$: *geodesic length* of γ
- $\alpha(\gamma) \in \mathbb{R}$: (signed) *Lorentzian length* of γ.
- The unique γ-invariant geodesic C_γ inherits a natural orientation and metric and γ translates along C_γ by $\alpha(\gamma)$.

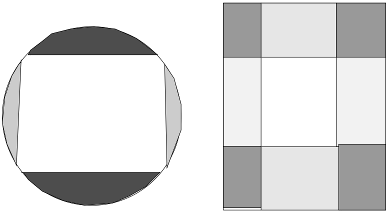
Closed geodesics and holonomy

- Each such element leaves invariant a unique (spacelike) line, whose image in $\mathbb{E}^{2,1}/\Gamma$ is a *closed geodesic.* Just as for hyperbolic surfaces, most loops are freely homotopic to closed geodesics.

$$\gamma = \begin{bmatrix} e^{\ell(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\ell(\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha(\gamma) \\ 0 \end{bmatrix}$$

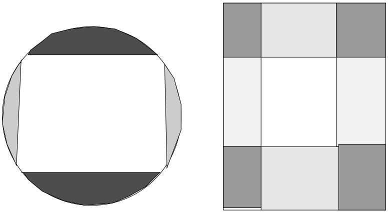
- $\ell(\gamma) \in \mathbb{R}^+$: *geodesic length* of γ
- $\alpha(\gamma) \in \mathbb{R}$: (signed) *Lorentzian length* of γ.
- The unique γ-invariant geodesic C_γ inherits a natural orientation and metric and γ translates along C_γ by $\alpha(\gamma)$.

Slabs don't work!



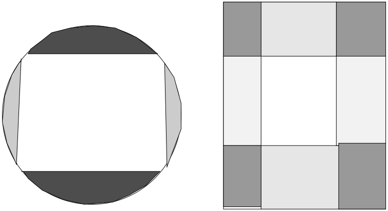
- In H^2, the half-spaces $A_i^{\pm}$ are disjoint;
- Their complement is a fundamental domain.
- In affine space, half-spaces disjoint $\Rightarrow$ parallel!
- Complements of slabs always intersect,
- Unsuitable for building Schottky groups!

Slabs don't work!



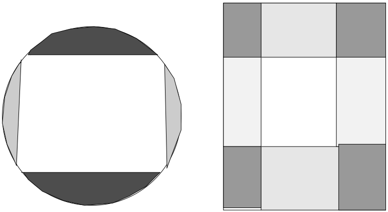
- In H^2, the half-spaces $A_i^{\pm}$ are disjoint;
- Their complement is a fundamental domain.
- In affine space, half-spaces disjoint $\Rightarrow$ parallel!
- Complements of slabs always intersect,
- Unsuitable for building Schottky groups!

Slabs don't work!



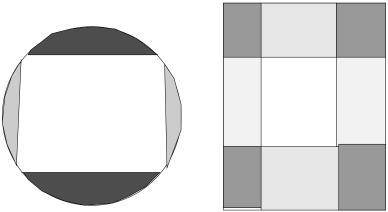
- In H^2, the half-spaces $A_i^{\pm}$ are disjoint;
- Their complement is a fundamental domain.
- In affine space, half-spaces disjoint $\Rightarrow$ parallel!
- Complements of slabs always intersect,
- Unsuitable for building Schottky groups!

Slabs don't work!



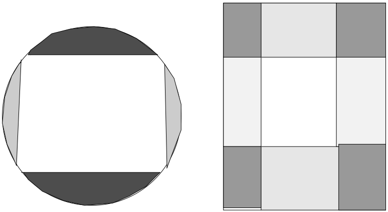
- In H^2, the half-spaces $A_i^{\pm}$ are disjoint;
- Their complement is a fundamental domain.
- In affine space, half-spaces disjoint $\Rightarrow$ parallel!
- Complements of slabs always intersect,
- Unsuitable for building Schottky groups!

Slabs don't work!



- In H^2, the half-spaces $A_i^{\pm}$ are disjoint;
- Their complement is a fundamental domain.
- In affine space, half-spaces disjoint $\Rightarrow$ parallel!
- Complements of slabs always intersect,
- Unsuitable for building Schottky groups!

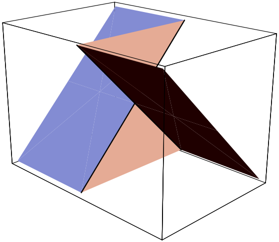
Slabs don't work!



- In H^2, the half-spaces $A_i^{\pm}$ are disjoint;
- Their complement is a fundamental domain.
- In affine space, half-spaces disjoint $\Rightarrow$ parallel!
- Complements of slabs always intersect,
- Unsuitable for building Schottky groups!

Crooked Planes (Drumm 1990)

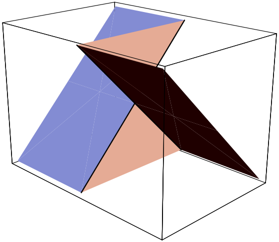
- *Crooked Planes:* Flexible polyhedral surfaces bound fundamental polyhedra for free affine groups.



- Two null half-planes connected by lines inside light-cone.

Crooked Planes (Drumm 1990)

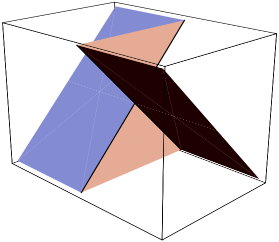
- *Crooked Planes:* Flexible polyhedral surfaces bound fundamental polyhedra for free affine groups.



- Two null half-planes connected by lines inside light-cone.

Crooked Planes (Drumm 1990)

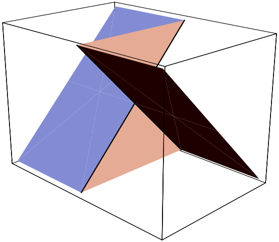
- *Crooked Planes:* Flexible polyhedral surfaces bound fundamental polyhedra for free affine groups.



- Two null half-planes connected by lines inside light-cone.

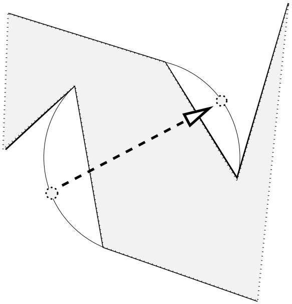
Crooked Planes (Drumm 1990)

- *Crooked Planes:* Flexible polyhedral surfaces bound fundamental polyhedra for free affine groups.



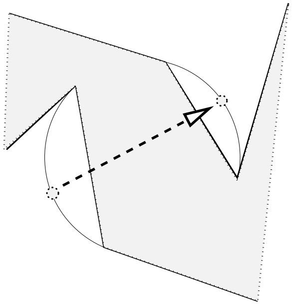
- Two null half-planes connected by lines inside light-cone.

Crooked polyhedron for a boost



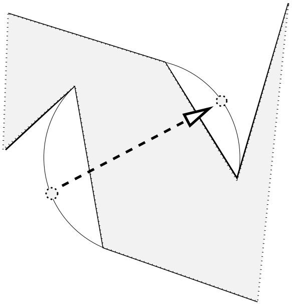
- Start with a *hyperbolic slab* in H^2.
- Extend into light cone in $\mathbb{E}^{2,1}$;
- Extend outside light cone in $\mathbb{E}^{2,1}$;
- Action proper except at the origin and two null half-planes.

Crooked polyhedron for a boost



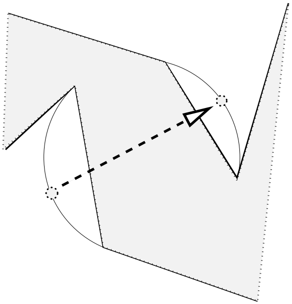
- Start with a *hyperbolic slab* in H^2.
- Extend into light cone in $\mathbb{E}^{2,1}$;
- Extend outside light cone in $\mathbb{E}^{2,1}$;
- Action proper except at the origin and two null half-planes.

Crooked polyhedron for a boost



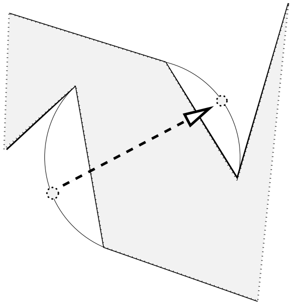
- Start with a *hyperbolic slab* in H^2.
- Extend into light cone in $\mathbb{E}^{2,1}$;
- Extend outside light cone in $\mathbb{E}^{2,1}$;
- Action proper except at the origin and two null half-planes.

Crooked polyhedron for a boost



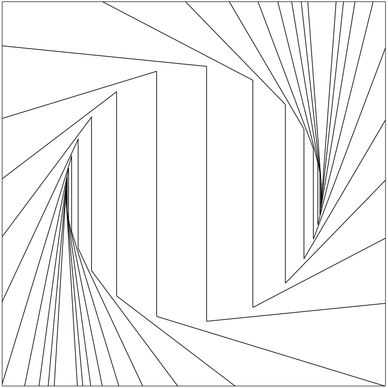
- Start with a *hyperbolic slab* in H^2.
- Extend into light cone in $\mathbb{E}^{2,1}$;
- Extend outside light cone in $\mathbb{E}^{2,1}$;
- Action proper except at the origin and two null half-planes.

Crooked polyhedron for a boost



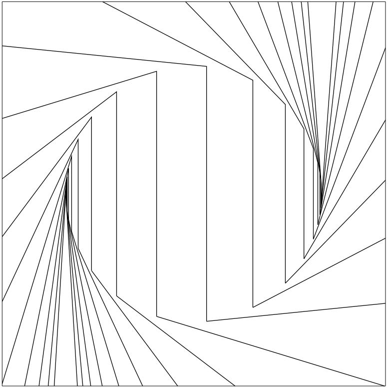
- Start with a *hyperbolic slab* in H^2.
- Extend into light cone in $\mathbb{E}^{2,1}$;
- Extend outside light cone in $\mathbb{E}^{2,1}$;
- Action proper except at the origin and two null half-planes.

Images of crooked planes under a linear cyclic group



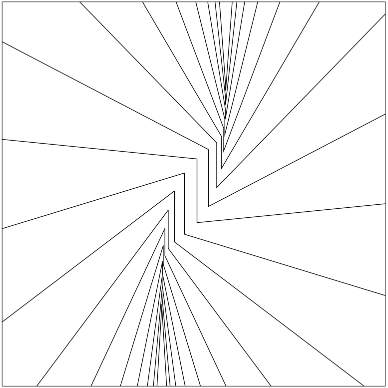
The resulting tesselation for a linear boost.

Images of crooked planes under a linear cyclic group



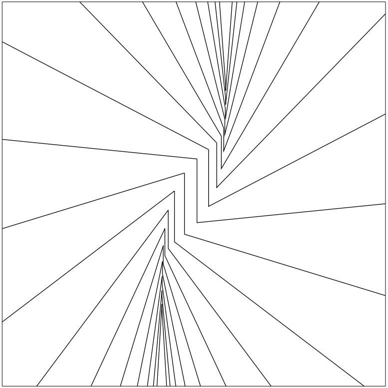
The resulting tesselation for a linear boost.

Images of crooked planes under an affine deformation



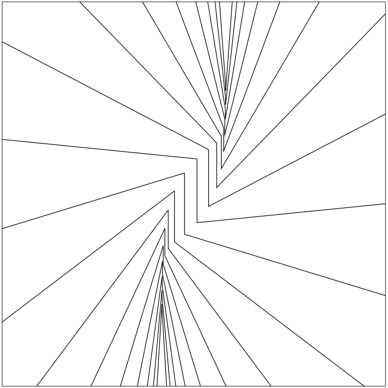
- Adding translations frees up the action
- — which is now proper on *all* of $\mathbb{E}^{2,1}$.

Images of crooked planes under an affine deformation



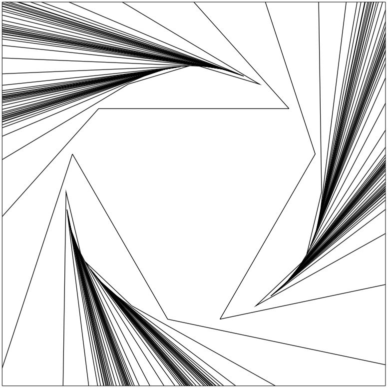
- Adding translations frees up the action
- — which is now proper on *all* of $\mathbb{E}^{2,1}$.

Images of crooked planes under an affine deformation



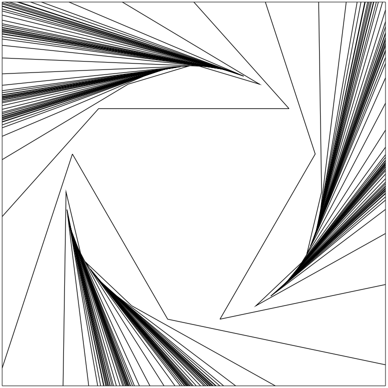
- Adding translations frees up the action
- — which is now proper on *all* of $\mathbb{E}^{2,1}$.

Linear action of Schottky group



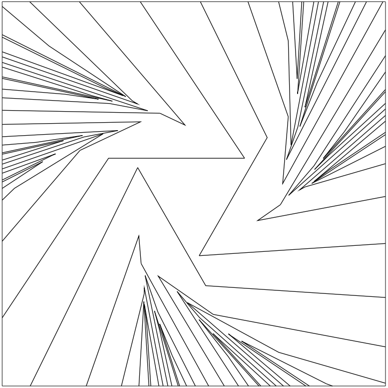
Crooked polyhedra tile H^2 for subgroup of $O(2,1)$.

Linear action of Schottky group



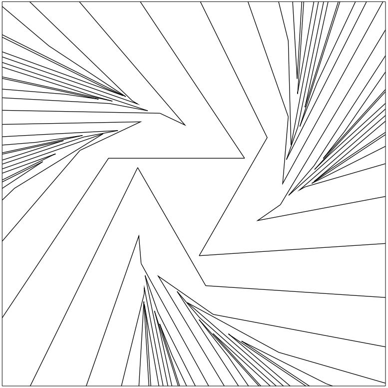
Crooked polyhedra tile H^2 for subgroup of $O(2,1)$.

Affine action of Schottky group



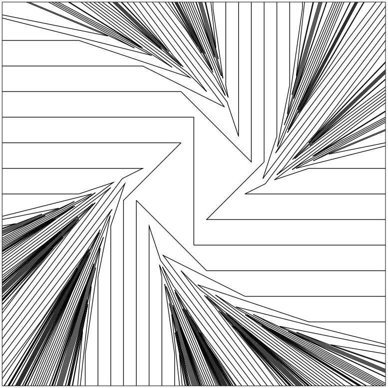
Carefully chosen affine deformation acts properly on $\mathbb{E}^{2,1}$.

Affine action of Schottky group



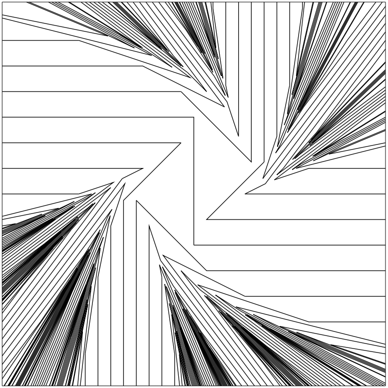
Carefully chosen affine deformation acts properly on $\mathbb{E}^{2,1}$.

Affine action of level 2 congruence subgroup of $\mathrm{GL}(2, \mathbb{Z})$



Proper affine deformations exist even for *lattices* (Drumm).

Affine action of level 2 congruence subgroup of $GL(2, \mathbb{Z})$



Proper affine deformations exist even for *lattices* (Drumm).

The linear part

- Mess's theorem (Σ noncompact) is the only obstruction for the existence of a proper affine deformation:
- (Drumm 1990) Let Σ be a *noncompact* complete hyperbolic surface. Then its holonomy group admits a proper affine deformation and M^3 is a solid handlebody.
- BASIC PROBLEM:

Classify, both geometrically and topologically, all proper affine deformations of a non-cocompact Fuchsian group.

The linear part

- Mess's theorem (Σ noncompact) is the only obstruction for the existence of a proper affine deformation:
- (Drumm 1990) Let Σ be a *noncompact* complete hyperbolic surface. Then its holonomy group admits a proper affine deformation and M^3 is a solid handlebody.
- BASIC PROBLEM:

Classify, both geometrically and topologically, all proper affine deformations of a non-cocompact Fuchsian group.

The linear part

- Mess's theorem (Σ noncompact) is the only obstruction for the existence of a proper affine deformation:
- (Drumm 1990) Let Σ be a *noncompact* complete hyperbolic surface. Then its holonomy group admits a proper affine deformation and M^3 is a solid handlebody.
- BASIC PROBLEM:

Classify, both geometrically and topologically, all proper affine deformations of a non-cocompact Fuchsian group.

The linear part

- Mess's theorem (Σ noncompact) is the only obstruction for the existence of a proper affine deformation:
- (Drumm 1990) Let Σ be a *noncompact* complete hyperbolic surface. Then its holonomy group admits a proper affine deformation and M^3 is a solid handlebody.
- BASIC PROBLEM:

  Classify, both geometrically and topologically, all proper affine deformations of a non-cocompact Fuchsian group.

The linear part

- Mess's theorem (Σ noncompact) is the only obstruction for the existence of a proper affine deformation:
- (Drumm 1990) Let Σ be a *noncompact* complete hyperbolic surface. Then its holonomy group admits a proper affine deformation and M^3 is a solid handlebody.
- BASIC PROBLEM:

  Classify, both geometrically and topologically, all proper affine deformations of a non-cocompact Fuchsian group.

The linear part

- Mess's theorem (Σ noncompact) is the only obstruction for the existence of a proper affine deformation:
- (Drumm 1990) Let Σ be a *noncompact* complete hyperbolic surface. Then its holonomy group admits a proper affine deformation and M^3 is a solid handlebody.
- BASIC PROBLEM:

  Classify, both geometrically and topologically, all proper affine deformations of a non-cocompact Fuchsian group.

Marked Signed Lorentzian Length Spectrum

- $\forall$ affine deformation $\Gamma \xrightarrow{\rho=(\mathbb{L},u)} \mathsf{Isom}(\mathbb{E}^{2,1})^0$, define $\alpha_u(\gamma) \in \mathbb{R}$ as the (signed) displacement of γ along the unique γ-invariant geodesic C_γ, when $\mathbb{L}(\gamma)$ is hyperbolic.
- α_u is a class function on Γ;
- When ρ acts properly, $|\alpha_u(\gamma)|$ is the *Lorentzian length* of the closed geodesic in M^3 corresponding to γ;
- (Margulis 1983) If ρ acts properly, either
  - $\alpha_u(\gamma) > 0 \ \forall \gamma \neq 1$, or
  - $\alpha_u(\gamma) < 0 \ \forall \gamma \neq 1$.
- The Margulis invariant $\Gamma \xrightarrow{\alpha} \mathbb{R}$ determines Γ up to conjugacy (Charette-Drumm 2004).

Marked Signed Lorentzian Length Spectrum

- $\forall$ affine deformation $\Gamma \xrightarrow{\rho=(\mathbb{L},u)} \mathrm{Isom}(\mathbb{E}^{2,1})^0$, define $\alpha_u(\gamma) \in \mathbb{R}$ as the (signed) displacement of γ along the unique γ-invariant geodesic C_γ, when $\mathbb{L}(\gamma)$ is hyperbolic.
- α_u is a class function on Γ;
- When ρ acts properly, $|\alpha_u(\gamma)|$ is the *Lorentzian length* of the closed geodesic in M^3 corresponding to γ;
- (Margulis 1983) If ρ acts properly, either
  - $\alpha_u(\gamma) > 0 \ \forall \gamma \neq 1$, or
  - $\alpha_u(\gamma) < 0 \ \forall \gamma \neq 1$.
- The Margulis invariant $\Gamma \xrightarrow{\alpha} \mathbb{R}$ determines Γ up to conjugacy (Charette-Drumm 2004).

Marked Signed Lorentzian Length Spectrum

- $\forall$ affine deformation $\Gamma \xrightarrow{\rho=(\mathbb{L},u)} \mathsf{Isom}(\mathbb{E}^{2,1})^0$, define $\alpha_u(\gamma) \in \mathbb{R}$ as the (signed) displacement of γ along the unique γ-invariant geodesic C_γ, when $\mathbb{L}(\gamma)$ is hyperbolic.
- α_u is a class function on Γ;
- When ρ acts properly, $|\alpha_u(\gamma)|$ is the *Lorentzian length* of the closed geodesic in M^3 corresponding to γ;
- (Margulis 1983) If ρ acts properly, either
  - $\alpha_u(\gamma) > 0 \ \forall \gamma \neq 1$, or
  - $\alpha_u(\gamma) < 0 \ \forall \gamma \neq 1$.
- The Margulis invariant $\Gamma \xrightarrow{\alpha} \mathbb{R}$ determines Γ up to conjugacy (Charette-Drumm 2004).

Marked Signed Lorentzian Length Spectrum

- $\forall$ affine deformation $\Gamma \xrightarrow{\rho=(\mathbb{L},u)} \mathsf{Isom}(\mathbb{E}^{2,1})^0$, define $\alpha_u(\gamma) \in \mathbb{R}$ as the (signed) displacement of γ along the unique γ-invariant geodesic C_γ, when $\mathbb{L}(\gamma)$ is hyperbolic.
- α_u is a class function on Γ;
- When ρ acts properly, $|\alpha_u(\gamma)|$ is the *Lorentzian length* of the closed geodesic in M^3 corresponding to γ;
- (Margulis 1983) If ρ acts properly, either
  - $\alpha_u(\gamma) > 0\ \forall \gamma \neq 1$, or
  - $\alpha_u(\gamma) < 0\ \forall \gamma \neq 1$.
- The Margulis invariant $\Gamma \xrightarrow{\alpha} \mathbb{R}$ determines Γ up to conjugacy (Charette-Drumm 2004).

Marked Signed Lorentzian Length Spectrum

- $\forall$ affine deformation $\Gamma \xrightarrow{\rho=(\mathbb{L},u)} \mathsf{Isom}(\mathbb{E}^{2,1})^0$, define $\alpha_u(\gamma) \in \mathbb{R}$ as the (signed) displacement of γ along the unique γ-invariant geodesic C_γ, when $\mathbb{L}(\gamma)$ is hyperbolic.
- α_u is a class function on Γ;
- When ρ acts properly, $|\alpha_u(\gamma)|$ is the *Lorentzian length* of the closed geodesic in M^3 corresponding to γ;
- (Margulis 1983) If ρ acts properly, either
  - $\alpha_u(\gamma) > 0 \ \forall \gamma \neq 1$, or
  - $\alpha_u(\gamma) < 0 \ \forall \gamma \neq 1$.
- The Margulis invariant $\Gamma \xrightarrow{\alpha} \mathbb{R}$ determines Γ up to conjugacy (Charette-Drumm 2004).

Marked Signed Lorentzian Length Spectrum

- $\forall$ affine deformation $\Gamma \xrightarrow{\rho=(\mathbb{L},u)} \mathsf{Isom}(\mathbb{E}^{2,1})^0$, define $\alpha_u(\gamma) \in \mathbb{R}$ as the (signed) displacement of γ along the unique γ-invariant geodesic C_γ, when $\mathbb{L}(\gamma)$ is hyperbolic.
- α_u is a class function on Γ;
- When ρ acts properly, $|\alpha_u(\gamma)|$ is the *Lorentzian length* of the closed geodesic in M^3 corresponding to γ;
- (Margulis 1983) If ρ acts properly, either
  - $\alpha_u(\gamma) > 0 \ \forall \gamma \neq 1$, or
  - $\alpha_u(\gamma) < 0 \ \forall \gamma \neq 1$.
- The Margulis invariant $\Gamma \xrightarrow{\alpha} \mathbb{R}$ determines Γ up to conjugacy (Charette-Drumm 2004).

Affine deformations

- Start with a Fuchsian group $\Gamma_0 \subset \mathrm{O}(2,1)$. An *affine deformation* is a representation $\rho = \rho_u$ with image $\Gamma = \Gamma_u$

$$\begin{array}{ccc} & & \mathrm{Isom}(\mathbb{R}^{2,1}) \\ & \overset{\rho}{\nearrow} & \downarrow \mathbb{L} \\ \Gamma_0 & \hookrightarrow & \mathrm{O}(2,1) \end{array}$$

  determined by its translational part

$$u \in Z^1(\Gamma_0, \mathbb{R}^{2,1}).$$

- Conjugating ρ by a translation $\Longleftrightarrow$ adding a coboundary to u.
- *Translational conjugacy classes* of affine deformations of Γ_0 form the vector space $H^1(\Gamma_0, \mathbb{R}^{2,1})$.

Affine deformations

- Start with a Fuchsian group $\Gamma_0 \subset \mathrm{O}(2,1)$. An *affine deformation* is a representation $\rho = \rho_u$ with image $\Gamma = \Gamma_u$

$$\begin{array}{ccc} & & \mathrm{Isom}(\mathbb{R}^{2,1}) \\ & \overset{\rho}{\nearrow} & \downarrow \mathbb{L} \\ \Gamma_0 & \hookrightarrow & \mathrm{O}(2,1) \end{array}$$

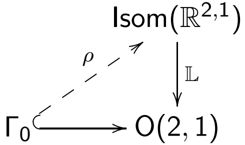
determined by its translational part

$$u \in Z^1(\Gamma_0, \mathbb{R}^{2,1}).$$

- Conjugating ρ by a translation $\Longleftrightarrow$ adding a coboundary to u.
- *Translational conjugacy classes* of affine deformations of Γ_0 form the vector space $H^1(\Gamma_0, \mathbb{R}^{2,1})$.

Affine deformations

- Start with a Fuchsian group $\Gamma_0 \subset \mathrm{O}(2,1)$. An *affine deformation* is a representation $\rho = \rho_u$ with image $\Gamma = \Gamma_u$



  determined by its translational part

$$u \in Z^1(\Gamma_0, \mathbb{R}^{2,1}).$$

- Conjugating ρ by a translation $\Longleftrightarrow$ adding a coboundary to u.
- *Translational conjugacy classes* of affine deformations of Γ_0 form the vector space $H^1(\Gamma_0, \mathbb{R}^{2,1})$.

Affine deformations

- Start with a Fuchsian group $\Gamma_0 \subset \mathrm{O}(2,1)$. An *affine deformation* is a representation $\rho = \rho_u$ with image $\Gamma = \Gamma_u$

$$\begin{array}{ccc} & & \mathrm{Isom}(\mathbb{R}^{2,1}) \\ & \overset{\rho}{\nearrow} & \downarrow \mathbb{L} \\ \Gamma_0 & \hookrightarrow & \mathrm{O}(2,1) \end{array}$$

  determined by its translational part

$$u \in Z^1(\Gamma_0, \mathbb{R}^{2,1}).$$

- Conjugating ρ by a translation $\iff$ adding a coboundary to u.
- *Translational conjugacy classes* of affine deformations of Γ_0 form the vector space $H^1(\Gamma_0, \mathbb{R}^{2,1})$.

Deformations of hyperbolic structures

- Translational conjugacy classes of affine deformations of Γ_0 $\longleftrightarrow$ infinitesimal deformations of the hyperbolic surface Σ.
  - The Lorentzian vector space $\mathbb{R}^{2,1}$ corresponds to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ with the Killing form, and the action of $\mathsf{O}(2,1)$ is the adjoint representation.
  - This Lie algebra comprises the *Killing vector fields,* infinitesimal isometries, of H^2.
- Infinitesimal deformations of the hyperbolic structure on Σ comprise $H^1(\Sigma, \mathfrak{sl}(2,\mathbb{R})) \cong H^1(\Gamma_0, \mathbb{R}^{2,1})$.

Deformations of hyperbolic structures

- Translational conjugacy classes of affine deformations of Γ_0 $\longleftrightarrow$ infinitesimal deformations of the hyperbolic surface Σ.
  - The Lorentzian vector space $\mathbb{R}^{2,1}$ corresponds to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ with the Killing form, and the action of $\mathrm{O}(2,1)$ is the adjoint representation.
  - This Lie algebra comprises the *Killing vector fields,* infinitesimal isometries, of H^2.
- Infinitesimal deformations of the hyperbolic structure on Σ comprise $H^1(\Sigma,\mathfrak{sl}(2,\mathbb{R}))\cong H^1(\Gamma_0,\mathbb{R}^{2,1})$.

Deformations of hyperbolic structures

- Translational conjugacy classes of affine deformations of Γ_0 $\longleftrightarrow$ infinitesimal deformations of the hyperbolic surface Σ.
  - The Lorentzian vector space $\mathbb{R}^{2,1}$ corresponds to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ with the Killing form, and the action of $\mathrm{O}(2,1)$ is the adjoint representation.
  - This Lie algebra comprises the *Killing vector fields,* infinitesimal isometries, of H^2.
- Infinitesimal deformations of the hyperbolic structure on Σ comprise $H^1(\Sigma,\mathfrak{sl}(2,\mathbb{R})) \cong H^1(\Gamma_0,\mathbb{R}^{2,1})$.

Deformations of hyperbolic structures

- Translational conjugacy classes of affine deformations of Γ_0 $\longleftrightarrow$ infinitesimal deformations of the hyperbolic surface Σ.
  - The Lorentzian vector space $\mathbb{R}^{2,1}$ corresponds to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ with the Killing form, and the action of $\mathsf{O}(2,1)$ is the adjoint representation.
  - This Lie algebra comprises the *Killing vector fields,* infinitesimal isometries, of H^2.
- Infinitesimal deformations of the hyperbolic structure on Σ comprise $H^1(\Sigma, \mathfrak{sl}(2,\mathbb{R})) \cong H^1(\Gamma_0, \mathbb{R}^{2,1})$.

Deformations of hyperbolic structures

- Translational conjugacy classes of affine deformations of Γ_0 $\longleftrightarrow$ infinitesimal deformations of the hyperbolic surface Σ.
  - The Lorentzian vector space $\mathbb{R}^{2,1}$ corresponds to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ with the Killing form, and the action of $\mathrm{O}(2,1)$ is the adjoint representation.
  - This Lie algebra comprises the *Killing vector fields*, infinitesimal isometries, of H^2.
- Infinitesimal deformations of the hyperbolic structure on Σ comprise $H^1(\Sigma,\mathfrak{sl}(2,\mathbb{R})) \cong H^1(\Gamma_0,\mathbb{R}^{2,1})$.

Deformation-theoretic interpretation of Margulis invariant

- Suppose $u \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$ defines an *infinitesimal deformation* tangent to a smooth deformation Σ_t of Σ.
  - The marked length spectrum ℓ_t of Σ_t varies smoothly with t.
  - Margulis's invariant $\alpha_u(\gamma)$ represents the derivative

$$\frac{d}{dt}\bigg|_{t=0} \ell_t(\gamma)$$

  (G-Margulis 2000).
- Γ_u is proper $\Longrightarrow$ all closed geodesics lengthen (or shorten) under the deformation Σ_t.
- When Σ is homeomorphic to a three-holed sphere, the converse holds. (Jones 2004, Charette-Drumm-G 2009).

Deformation-theoretic interpretation of Margulis invariant

- Suppose $u \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$ defines an *infinitesimal deformation* tangent to a smooth deformation Σ_t of Σ.
  - The marked length spectrum ℓ_t of Σ_t varies smoothly with t.
  - Margulis's invariant $\alpha_u(\gamma)$ represents the derivative

$$\frac{d}{dt}\bigg|_{t=0} \ell_t(\gamma)$$

  (G-Margulis 2000).
- Γ_u is proper $\implies$ all closed geodesics lengthen (or shorten) under the deformation Σ_t.
- When Σ is homeomorphic to a three-holed sphere, the converse holds. (Jones 2004, Charette-Drumm-G 2009).

Deformation-theoretic interpretation of Margulis invariant

- Suppose $u \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$ defines an *infinitesimal deformation* tangent to a smooth deformation Σ_t of Σ.
  - The marked length spectrum ℓ_t of Σ_t varies smoothly with t.
  - Margulis's invariant $\alpha_u(\gamma)$ represents the derivative

$$\frac{d}{dt}\bigg|_{t=0} \ell_t(\gamma)$$

    (G-Margulis 2000).
- Γ_u is proper $\implies$ all closed geodesics lengthen (or shorten) under the deformation Σ_t.
- When Σ is homeomorphic to a three-holed sphere, the converse holds. (Jones 2004, Charette-Drumm-G 2009).

Deformation-theoretic interpretation of Margulis invariant

- Suppose $u \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$ defines an *infinitesimal deformation* tangent to a smooth deformation Σ_t of Σ.
  - The marked length spectrum ℓ_t of Σ_t varies smoothly with t.
  - Margulis's invariant $\alpha_u(\gamma)$ represents the derivative

$$\frac{d}{dt}\bigg|_{t=0} \ell_t(\gamma)$$

  (G-Margulis 2000).
- Γ_u is proper $\Longrightarrow$ all closed geodesics lengthen (or shorten) under the deformation Σ_t.
- When Σ is homeomorphic to a three-holed sphere, the converse holds. (Jones 2004, Charette-Drumm-G 2009).

Deformation-theoretic interpretation of Margulis invariant

- Suppose $u \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$ defines an *infinitesimal deformation* tangent to a smooth deformation Σ_t of Σ.
  - The marked length spectrum ℓ_t of Σ_t varies smoothly with t.
  - Margulis's invariant $\alpha_u(\gamma)$ represents the derivative

$$\frac{d}{dt}\bigg|_{t=0} \ell_t(\gamma)$$

(G-Margulis 2000).

- Γ_u is proper $\Longrightarrow$ all closed geodesics lengthen (or shorten) under the deformation Σ_t.
- When Σ is homeomorphic to a three-holed sphere, the converse holds. (Jones 2004, Charette-Drumm-G 2009).

Deformation-theoretic interpretation of Margulis invariant

- Suppose $u \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$ defines an *infinitesimal deformation* tangent to a smooth deformation Σ_t of Σ.
  - The marked length spectrum ℓ_t of Σ_t varies smoothly with t.
  - Margulis's invariant $\alpha_u(\gamma)$ represents the derivative

$$\left.\frac{d}{dt}\right|_{t=0} \ell_t(\gamma)$$

  (G-Margulis 2000).
- Γ_u is proper $\Longrightarrow$ all closed geodesics lengthen (or shorten) under the deformation Σ_t.
- When Σ is homeomorphic to a three-holed sphere, the converse holds. (Jones 2004, Charette-Drumm-G 2009).

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathbb{L}(\gamma)$. (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - The Margulis invariant extends to a continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2009, G-Labourie 2010)
- When $\mathbb{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\Longleftrightarrow$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
- Since $\mathcal{C}(\Sigma)$ is connected, either the $\Psi_u(\mu)$ are all positive or all negative.

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathbb{L}(\gamma)$. (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - The Margulis invariant extends to a continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2009, G-Labourie 2010)
- When $\mathbb{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\Longleftrightarrow$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
- Since $\mathcal{C}(\Sigma)$ is connected, either the $\Psi_u(\mu)$ are all positive or all negative.

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathbb{L}(\gamma)$. (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - The Margulis invariant extends to a continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2009, G-Labourie 2010)
- When $\mathbb{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\Longleftrightarrow$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
- Since $\mathcal{C}(\Sigma)$ is connected, either the $\Psi_u(\mu)$ are all positive or all negative.

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathbb{L}(\gamma)$. (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - The Margulis invariant extends to a continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2009, G-Labourie 2010)
- When $\mathbb{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\Longleftrightarrow$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
- Since $\mathcal{C}(\Sigma)$ is connected, either the $\Psi_u(\mu)$ are all positive or all negative.

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathbb{L}(\gamma)$. (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - The Margulis invariant extends to a continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2009, G-Labourie 2010)
- When $\mathbb{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\Longleftrightarrow$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
- Since $\mathcal{C}(\Sigma)$ is connected, either the $\Psi_u(\mu)$ are all positive or all negative.

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathbb{L}(\gamma)$. (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - The Margulis invariant extends to a continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2009, G-Labourie 2010)
- When $\mathbb{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\Longleftrightarrow$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
- Since $\mathcal{C}(\Sigma)$ is connected, either the $\Psi_u(\mu)$ are all positive or all negative.

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathbb{L}(\gamma)$. (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - The Margulis invariant extends to a continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2009, G-Labourie 2010)
- When $\mathbb{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\iff$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
- Since $\mathcal{C}(\Sigma)$ is connected, either the $\Psi_u(\mu)$ are all positive or all negative.

Extensions of the Margulis invariant

- α_u extends to parabolic $\mathbb{L}(\gamma)$. (Charette-Drumm 2005).
- (Margulis 1983) $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$.
  - Therefore $\alpha_u(\gamma)/\ell(\gamma)$ is *constant* on cyclic (hyperbolic) subgroups of Γ.
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow Φ of $U\Sigma$.
  - The Margulis invariant extends to a continuous functional $\Psi_u(\mu)$ on the space $\mathcal{C}(\Sigma)$ of Φ-invariant probability measures μ on $U\Sigma$. (G-Labourie-Margulis 2009, G-Labourie 2010)
- When $\mathbb{L}(\Gamma)$ is convex cocompact, Γ_u acts properly $\Longleftrightarrow$ $\Psi_u(\mu) \neq 0$ for all invariant probability measures μ.
- Since $\mathcal{C}(\Sigma)$ is connected, either the $\Psi_u(\mu)$ are all positive or all negative.

The Deformation Space

- The deformation space of marked Margulis space-times arising from a topological surface S is a bundle over the Fricke space $\mathfrak{F}(S)$ of marked hyperbolic structures $S \longrightarrow \Sigma$ on S.
  - The fiber is the subspace of $H^1(\Sigma, \mathbb{R}^{2,1})$ (equivalence classes of *all* affine deformations) consisting of *proper* deformations.
  - Consists of equivalence classes of proper affine deformations of a fixed hyperbolic surface Σ and is nonempty (Drumm 1990).
  - (G-Labourie-Margulis 2010) Convex domain in $H^1(\Sigma, \mathbb{R}^{2,1})$ defined by the generalized CDM-invariants of measured geodesic laminations on Σ.
- Thus the deformation space is a cell with some boundary components corresponding to the ends of S.

The Deformation Space

- The deformation space of marked Margulis space-times arising from a topological surface S is a bundle over the Fricke space $\mathfrak{F}(S)$ of marked hyperbolic structures $S \longrightarrow \Sigma$ on S.
  - The fiber is the subspace of $H^1(\Sigma, \mathbb{R}^{2,1})$ (equivalence classes of *all* affine deformations) consisting of *proper* deformations.
  - Consists of equivalence classes of proper affine deformations of a fixed hyperbolic surface Σ and is nonempty (Drumm 1990).
  - (G-Labourie-Margulis 2010) Convex domain in $H^1(\Sigma, \mathbb{R}^{2,1})$ defined by the generalized CDM-invariants of measured geodesic laminations on Σ.
- Thus the deformation space is a cell with some boundary components corresponding to the ends of S.

The Deformation Space

- The deformation space of marked Margulis space-times arising from a topological surface S is a bundle over the Fricke space $\mathfrak{F}(S)$ of marked hyperbolic structures $S \longrightarrow \Sigma$ on S.
  - The fiber is the subspace of $H^1(\Sigma, \mathbb{R}^{2,1})$ (equivalence classes of *all* affine deformations) consisting of *proper* deformations.
  - Consists of equivalence classes of proper affine deformations of a fixed hyperbolic surface Σ and is nonempty (Drumm 1990).
  - (G-Labourie-Margulis 2010) Convex domain in $H^1(\Sigma, \mathbb{R}^{2,1})$ defined by the generalized CDM-invariants of measured geodesic laminations on Σ.
- Thus the deformation space is a cell with some boundary components corresponding to the ends of S.

The Deformation Space

- The deformation space of marked Margulis space-times arising from a topological surface S is a bundle over the Fricke space $\mathfrak{F}(S)$ of marked hyperbolic structures $S \longrightarrow \Sigma$ on S.
  - The fiber is the subspace of $H^1(\Sigma, \mathbb{R}^{2,1})$ (equivalence classes of *all* affine deformations) consisting of *proper* deformations.
  - Consists of equivalence classes of proper affine deformations of a fixed hyperbolic surface Σ and is nonempty (Drumm 1990).
  - (G-Labourie-Margulis 2010) Convex domain in $H^1(\Sigma, \mathbb{R}^{2,1})$ defined by the generalized CDM-invariants of measured geodesic laminations on Σ.
- Thus the deformation space is a cell with some boundary components corresponding to the ends of S.

The Deformation Space

- The deformation space of marked Margulis space-times arising from a topological surface S is a bundle over the Fricke space $\mathfrak{F}(S)$ of marked hyperbolic structures $S \longrightarrow \Sigma$ on S.
  - The fiber is the subspace of $H^1(\Sigma, \mathbb{R}^{2,1})$ (equivalence classes of *all* affine deformations) consisting of *proper* deformations.
  - Consists of equivalence classes of proper affine deformations of a fixed hyperbolic surface Σ and is nonempty (Drumm 1990).
  - (G-Labourie-Margulis 2010) Convex domain in $H^1(\Sigma, \mathbb{R}^{2,1})$ defined by the generalized CDM-invariants of measured geodesic laminations on Σ.
- Thus the deformation space is a cell with some boundary components corresponding to the ends of S.

The Deformation Space

- The deformation space of marked Margulis space-times arising from a topological surface S is a bundle over the Fricke space $\mathfrak{F}(S)$ of marked hyperbolic structures $S \longrightarrow \Sigma$ on S.
  - The fiber is the subspace of $H^1(\Sigma, \mathbb{R}^{2,1})$ (equivalence classes of *all* affine deformations) consisting of *proper* deformations.
  - Consists of equivalence classes of proper affine deformations of a fixed hyperbolic surface Σ and is nonempty (Drumm 1990).
  - (G-Labourie-Margulis 2010) Convex domain in $H^1(\Sigma, \mathbb{R}^{2,1})$ defined by the generalized CDM-invariants of measured geodesic laminations on Σ.
- Thus the deformation space is a cell with some boundary components corresponding to the ends of S.

The three-holed sphere (Charette-Drumm-G 2009)

- Suppose Σ is a three-holed sphere with boundary $\partial_1, \partial_2, \partial_3$.
- CDM-invariants of $\partial\Sigma$ identify the deformation space $H^1(\Gamma_0, \mathbb{R}^{2,1})$ of equivalence classes of all affine deformations with $\mathbb{R}^3$.
- If $\alpha(\partial_i) > 0$ for $i = 1, 2, 3.$ then Γ admits a crooked fundamental polyhedron:
  - Γ acts properly;
  - M^3 is a solid handlebody of genus two.
- Corollary (in hyperbolic geometry): If each component of $\partial\Sigma$ lengthens, then *every curve* lengthens under a deformation of the hyperbolic surface Σ.

The three-holed sphere (Charette-Drumm-G 2009)

- Suppose Σ is a three-holed sphere with boundary $\partial_1, \partial_2, \partial_3$.
- CDM-invariants of $\partial\Sigma$ identify the deformation space $H^1(\Gamma_0, \mathbb{R}^{2,1})$ of equivalence classes of all affine deformations with $\mathbb{R}^3$.
- If $\alpha(\partial_i) > 0$ for $i = 1, 2, 3$. then Γ admits a crooked fundamental polyhedron:
  - Γ acts properly;
  - M^3 is a solid handlebody of genus two.
- Corollary (in hyperbolic geometry): If each component of $\partial\Sigma$ lengthens, then *every curve* lengthens under a deformation of the hyperbolic surface Σ.

The three-holed sphere (Charette-Drumm-G 2009)

- Suppose Σ is a three-holed sphere with boundary $\partial_1, \partial_2, \partial_3$.
- CDM-invariants of $\partial\Sigma$ identify the deformation space $H^1(\Gamma_0, \mathbb{R}^{2,1})$ of equivalence classes of all affine deformations with $\mathbb{R}^3$.
- If $\alpha(\partial_i) > 0$ for $i = 1, 2, 3$. then Γ admits a crooked fundamental polyhedron:
  - Γ acts properly;
  - M^3 is a solid handlebody of genus two.
- Corollary (in hyperbolic geometry): If each component of $\partial\Sigma$ lengthens, then *every curve* lengthens under a deformation of the hyperbolic surface Σ.

The three-holed sphere (Charette-Drumm-G 2009)

- Suppose Σ is a three-holed sphere with boundary $\partial_1, \partial_2, \partial_3$.
- CDM-invariants of $\partial\Sigma$ identify the deformation space $H^1(\Gamma_0, \mathbb{R}^{2,1})$ of equivalence classes of all affine deformations with $\mathbb{R}^3$.
- If $\alpha(\partial_i) > 0$ for $i = 1, 2, 3.$ then Γ admits a crooked fundamental polyhedron:
  - Γ acts properly;
  - M^3 is a solid handlebody of genus two.
- Corollary (in hyperbolic geometry): If each component of $\partial\Sigma$ lengthens, then *every curve* lengthens under a deformation of the hyperbolic surface Σ.

The three-holed sphere (Charette-Drumm-G 2009)

- Suppose Σ is a three-holed sphere with boundary $\partial_1, \partial_2, \partial_3$.
- CDM-invariants of $\partial\Sigma$ identify the deformation space $H^1(\Gamma_0, \mathbb{R}^{2,1})$ of equivalence classes of all affine deformations with $\mathbb{R}^3$.
- If $\alpha(\partial_i) > 0$ for $i = 1, 2, 3.$ then Γ admits a crooked fundamental polyhedron:
  - Γ acts properly;
  - M^3 is a solid handlebody of genus two.
- Corollary (in hyperbolic geometry): If each component of $\partial\Sigma$ lengthens, then *every curve* lengthens under a deformation of the hyperbolic surface Σ.

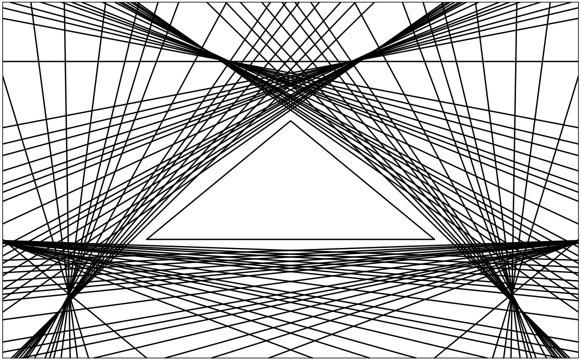
The three-holed sphere (Charette-Drumm-G 2009)

- Suppose Σ is a three-holed sphere with boundary $\partial_1, \partial_2, \partial_3$.
- CDM-invariants of $\partial\Sigma$ identify the deformation space $H^1(\Gamma_0, \mathbb{R}^{2,1})$ of equivalence classes of all affine deformations with $\mathbb{R}^3$.
- If $\alpha(\partial_i) > 0$ for $i = 1, 2, 3.$ then Γ admits a crooked fundamental polyhedron:
  - Γ acts properly;
  - M^3 is a solid handlebody of genus two.
- Corollary (in hyperbolic geometry): If each component of $\partial\Sigma$ lengthens, then *every curve* lengthens under a deformation of the hyperbolic surface Σ.

The three-holed sphere (Charette-Drumm-G 2009)

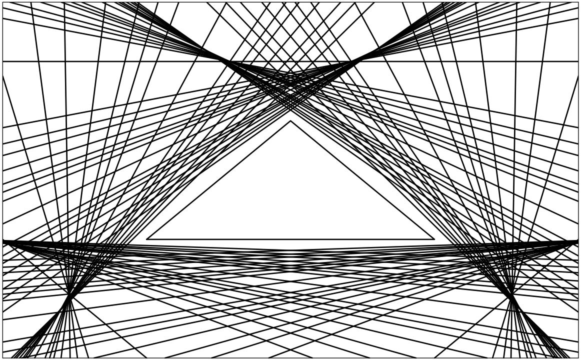
- Suppose Σ is a three-holed sphere with boundary $\partial_1, \partial_2, \partial_3$.
- CDM-invariants of $\partial\Sigma$ identify the deformation space $H^1(\Gamma_0, \mathbb{R}^{2,1})$ of equivalence classes of all affine deformations with $\mathbb{R}^3$.
- If $\alpha(\partial_i) > 0$ for $i = 1, 2, 3$. then Γ admits a crooked fundamental polyhedron:
  - Γ acts properly;
  - M^3 is a solid handlebody of genus two.
- Corollary (in hyperbolic geometry): If each component of $\partial\Sigma$ lengthens, then *every curve* lengthens under a deformation of the hyperbolic surface Σ.

Linear functionals $\alpha(\gamma)$ when Σ is a three-holed sphere



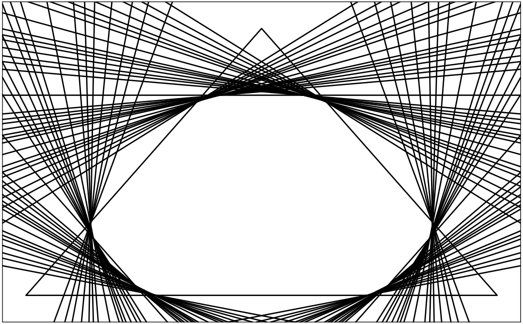
The triangle is bounded by the lines corresponding to $\gamma \subset \partial\Sigma$.
Its interior parametrizes proper affine deformations.

Linear functionals $\alpha(\gamma)$ when Σ is a three-holed sphere



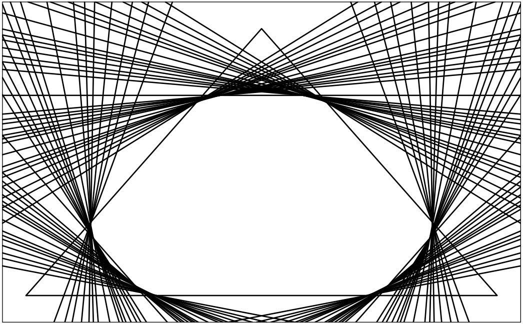
The triangle is bounded by the lines corresponding to $\gamma \subset \partial\Sigma$.
Its interior parametrizes proper affine deformations.

Linear functionals $\alpha(\gamma)$ when Σ is a one-holed torus



Properness region bounded by infinitely many intervals, each corresponding to a simple loop on Σ. Boundary points lie on intervals or are points of strict convexity (irrational laminations) (G-Margulis-Minsky).

Linear functionals $\alpha(\gamma)$ when Σ is a one-holed torus



Properness region bounded by infinitely many intervals, each corresponding to a simple loop on Σ. Boundary points lie on intervals or are points of strict convexity (irrational laminations) (G-Margulis-Minsky).

Questions for the future

- Does every proper affine deformation admit a crooked fundamental polyhedron?
- Is every nonsolvable complete affine 3-manifold M^3 a solid handlebody?
- Which $\mu \in \mathcal{C}(\Sigma)$ maximize (minimize) the generalized Margulis invariant?
- Can other hyperbolic groups (closed surface, 3-manifold groups) act properly and affinely?
- *Lorentzian Kleinian groups:* How do affine Schottky groups deform as discrete groups of conformal transformations of Einstein $2+1$-space?

Questions for the future

- Does every proper affine deformation admit a crooked fundamental polyhedron?
- Is every nonsolvable complete affine 3-manifold M^3 a solid handlebody?
- Which $\mu \in \mathcal{C}(\Sigma)$ maximize (minimize) the generalized Margulis invariant?
- Can other hyperbolic groups (closed surface, 3-manifold groups) act properly and affinely?
- *Lorentzian Kleinian groups:* How do affine Schottky groups deform as discrete groups of conformal transformations of Einstein $2+1$-space?

Questions for the future

- Does every proper affine deformation admit a crooked fundamental polyhedron?
- Is every nonsolvable complete affine 3-manifold M^3 a solid handlebody?
- Which $\mu \in \mathcal{C}(\Sigma)$ maximize (minimize) the generalized Margulis invariant?
- Can other hyperbolic groups (closed surface, 3-manifold groups) act properly and affinely?
- *Lorentzian Kleinian groups:* How do affine Schottky groups deform as discrete groups of conformal transformations of Einstein $2+1$-space?

Questions for the future

- Does every proper affine deformation admit a crooked fundamental polyhedron?
- Is every nonsolvable complete affine 3-manifold M^3 a solid handlebody?
- Which $\mu \in \mathcal{C}(\Sigma)$ maximize (minimize) the generalized Margulis invariant?
- Can other hyperbolic groups (closed surface, 3-manifold groups) act properly and affinely?
- *Lorentzian Kleinian groups:* How do affine Schottky groups deform as discrete groups of conformal transformations of Einstein $2+1$-space?

Questions for the future

- Does every proper affine deformation admit a crooked fundamental polyhedron?
- Is every nonsolvable complete affine 3-manifold M^3 a solid handlebody?
- Which $\mu \in \mathcal{C}(\Sigma)$ maximize (minimize) the generalized Margulis invariant?
- Can other hyperbolic groups (closed surface, 3-manifold groups) act properly and affinely?
- *Lorentzian Kleinian groups:* How do affine Schottky groups deform as discrete groups of conformal transformations of Einstein $2+1$-space?

Questions for the future

- Does every proper affine deformation admit a crooked fundamental polyhedron?
- Is every nonsolvable complete affine 3-manifold M^3 a solid handlebody?
- Which $\mu \in \mathcal{C}(\Sigma)$ maximize (minimize) the generalized Margulis invariant?
- Can other hyperbolic groups (closed surface, 3-manifold groups) act properly and affinely?
- *Lorentzian Kleinian groups:* How do affine Schottky groups deform as discrete groups of conformal transformations of Einstein $2+1$-space?