Philipps Universität Marburg

Old and new on the exceptional Lie group G_2

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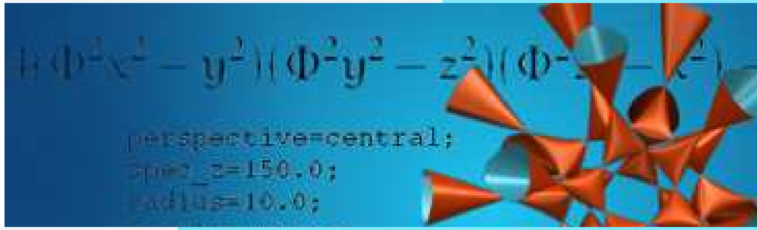

March 2012, Braniewo
Literature: Notices of the AMS 55 (2008), 922-929

A mathematical expedition. . .
to the oeuvre of Killing, Engel, and others 1880-1900

• the mathematical world: was projective, and almost always complex; hence, $\mathfrak{R}_5 = \mathbb{CP}^5$

• linear algebra was brand new, for example: [Alten, 4000 Jahre Algebra]

-1868: Karl Weierstraß publishes theory of 'elementary divisors' for matrices (Elementarteiler)

-1872: Camille Jordan describes the 'Jordan normal form'

-1874: Leopold Kronecker proves the equivalence of normal forms of Weierstraß and Jordan

• no clear difference between *Lie group* and *Lie algebra*

• Lie groups were not objects on their own right, but *transformation groups with transitive actions on interesting spaces:*

> „I find your point of view very interesting that to each transformation group should belong a particular ('besonders beschaffener') space.“
>
> letter nr. 3 from F. Engel to W. Killing, 9.11.1885

• the world being projective, the spaces of interest were mostly *flag manifolds*

Killing's goal (1884): Classification of all 'space forms' $\rightsquigarrow$ classif. of complex simple Lie algebras

In this talk:

- History of the discovery and realisation of G_2
- What was known about G_2 by the 'old masters' ?
- Role (& life) of Killing, Engel, and his Ph. D. student Walter Reichel
- Significance for modern differential geometry

„Moreover, we hereby obtain a direct definition of our 14-dimensional simple group $[G_2]$ which is as elegant as one can wish for.“ Friedrich Engel, 1900.

„Zudem ist hiermit eine direkte Definition unsrer vierzehngliedrigen einfachen Gruppe gegeben, die an Eleganz nichts zu wünschen übrig lässt.“

Friedrich Engel, 1900.

Friedrich Engel in a note to his talk at the Royal Saxonian Academy of Sciences on June 11, 1900.

Wilhelm Killing (1847–1923)

- 1872 thesis in Berlin on 'Flächenbündel 2. Ordnung' (advisor: K. Weierstraß)
- **1882–1892 professor, later rector at the Lyceum Hosianum in Braunsberg (now Braniewo/PL)**
- 1884 *Programmschrift* [Studium der Raumformen über ihre infinitesimalen Bewegungen]
- 1892–1919 professor in Münster (rector 18897-98)
- W. Killing, *Die Zusammensetzung der stetigen endlichen Transformationsgruppen*, Math. Ann. 33 (1889), 1-48.

Lyceum Hosianum Braunsberg (1565–1945)

- founded 1565 as a Jesuit collegium by Stanislaus Hosius, cardinal & prince-bishop of Warmia (Ermland) – one of the biggest in Europe

- consisted of a gymnasium and a seminary for catholic priests; attempts to turn it into a regular university up to the 18th ct.

- 1821: foundation of a theological and a philosophical faculty and awarded same rights as a university

- K. Weierstraß: teacher at the gymnasium 1848-1856, recommended Killing as professor (. . . with extremely low salary)

Wir Wilhelm, von Gottes Gnaden König von Preußen etc.
thun kund und fügen hiermit zu wissen, daß wir allergnädigst geruht haben, den bisherigen Oberlehrer an dem Gymnasium zu Brilon, Dr. Wilhelm Killing zum ordentlichen Professor in der philosophischen Fakultät des Lyceum Hosianum in Braunsberg zu ernennen. Es ist dies in dem Vertrauen geschehen, daß derselbe Uns und Unserem kngl. Hause in unverbrüchlicher Treue ergeben bleiben und die Pflichten des ihm übertragenen Amtes in ihrem ganzen Umfange mit stets regem Eifer erfüllen und insbesondere alle halbe Jahre [. . .] Vorlesungen in seinen Fächern ankündigen werde [. . .].

Gegeben Schloß Babelsberg den 17ten August 1882


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Wir Wilhelm,
von Gottes Gnaden
König von Preußen etc.

Bestallung
für
Dr. Wilhelm Killing
ordentlichen Professor
Braunsberg

1880-1885: simple complex Lie algebras $\mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{sl}(n, \mathbb{C})$ were well-known; Lie and Engel knew about $\mathfrak{sp}(n, \mathbb{C})$, but nothing was published

In 1885, Wilhelm Killing starts a correspondence with Felix Klein, Sophus Lie and, most importantly, Friedrich Engel

April 1886: Killing conjectures that $\mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{sl}(n, \mathbb{C})$ are the *only* simple complex Lie algebras (though Engel had told him that more simple algebras could occur as isotropy groups)

March 1887: Killing discovers the root system of G_2 and claims that it should have a 5-dimensional 'space form'

October 1887: Killing obtains the full classification, prepares a paper after strong encouragements by Engel

Thm (W. Killing, 1887). The only complex simple Lie algebras are $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{sp}(n,\mathbb{C})$, $\mathfrak{sl}(n,\mathbb{C})$ as well as five exceptional Lie algebras,

$$\mathfrak{g}_2 \;:= \mathfrak{g}_2^{14},\ \mathfrak{f}_4^{52},\ \mathfrak{e}_6^{78},\ \mathfrak{e}_7^{133}, \mathfrak{e}_8^{248}.$$

(upper index: dimension, lower index: rank)

Killing's proof contains some gaps and mistakes. In his thesis (1894), Élie Cartan gave a completely revised and polished presentation of the classification, which has therefore become the standard reference for the result.

Notations:

- G_2, $\mathfrak{g}_2$: complex Lie group resp. Lie algebra
- G_2^c, $\mathfrak{g}_2^c$: real *compact* form of G_2, $\mathfrak{g}_2$
- G_2^*, $\mathfrak{g}_2^*$: real *non compact* form of G_2, $\mathfrak{g}_2$

Cartan's thesis

Last section: derives from weight lattice the lowest dimensional irreducible representation of each simple complex Lie algebra

Result. $\mathfrak{g}_2$ admits an irreducible representation on $\mathbb{C}^7$, and it has a $\mathfrak{g}_2$-invariant scalar product

$$\beta := x_0^2 + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Interpreted as a *real* scalar product, it has signature $(4,3)$: Cartan's representation restricts to an irred. $\mathfrak{g}_2^*$ representation inside $\mathfrak{so}(4,3)$.

Problem: direct construction of $\mathfrak{g}_2$ and its real forms $\mathfrak{g}_2^*, \mathfrak{g}_2^c$?

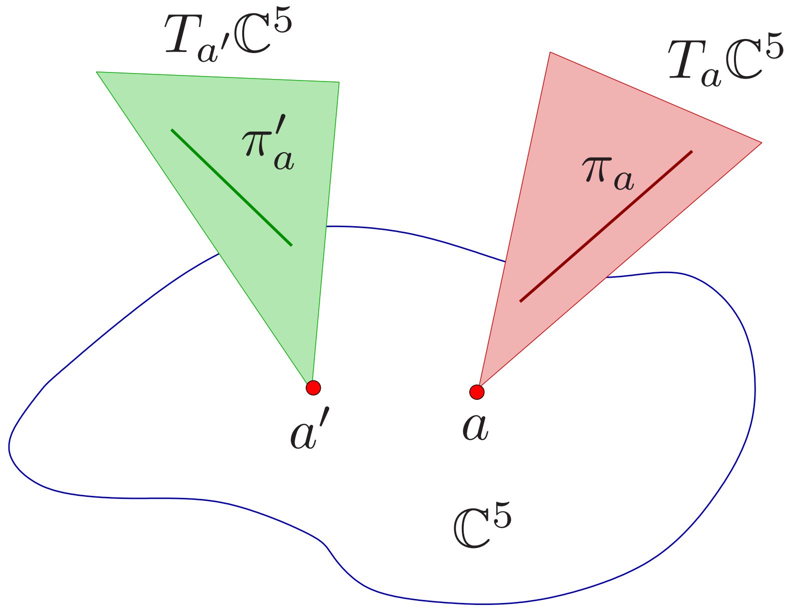
First step: Engel & Cartan, 1893

In 1893, Engel & Cartan publish simultaneously a note in C. R. Acad. Sc. Paris. They give the following construction:

Consider $\mathbb{C}^5$ and the 2-planes $\pi_a \subset T_a\mathbb{C}^5$ defined by

$$dx_3 = x_1\, dx_2 - x_2\, dx_1,$$
$$dx_4 = x_2\, dx_3 - x_3\, dx_2,$$
$$dx_5 = x_3\, dx_1 - x_1\, dx_3.$$

The 14 vector fields whose (local) flows map the planes π_a to each other satisfy the commutator relations of $\mathfrak{g}_2$!



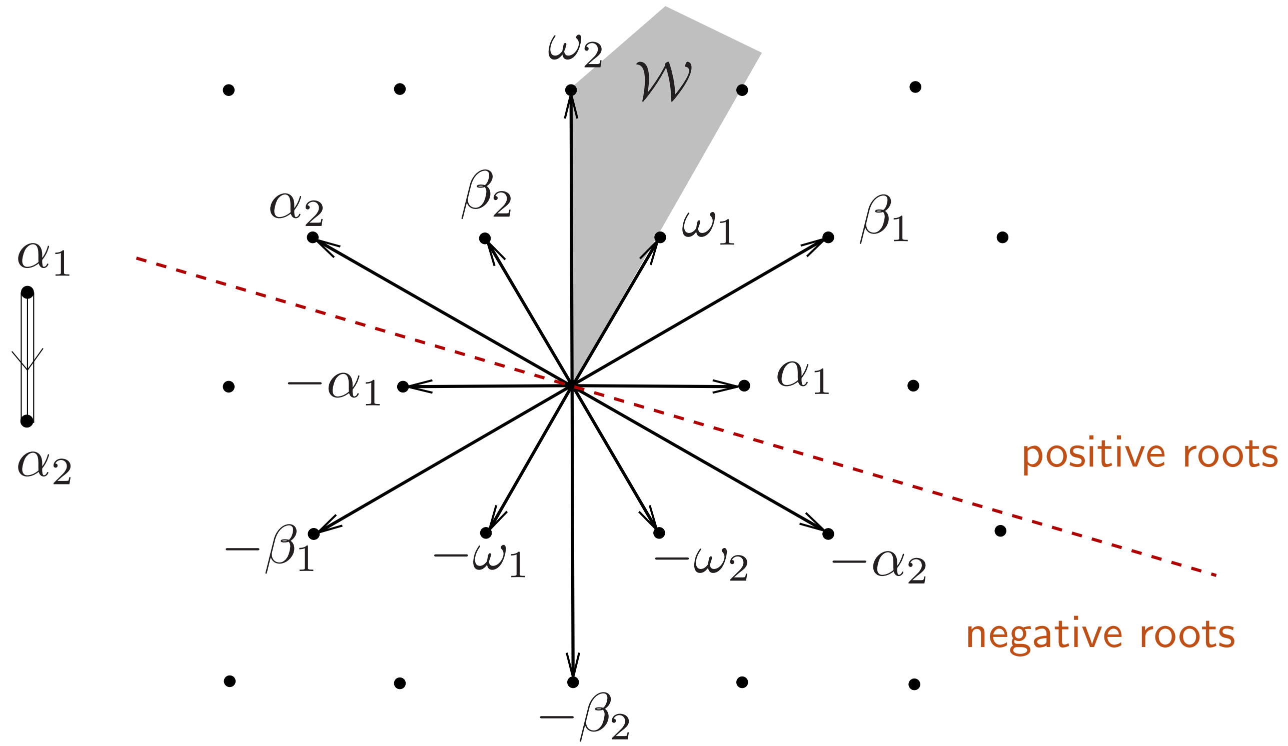
Both give a second, non equivalent realisation of $\mathfrak{g}_2$:

- Engel: through a contact transformation from the first
- Cartan: as symmetries of solution space of the 2nd order pde's ($f = f(x,y)$)

$$f_{xx} = \frac{4}{3}(f_{yy})^3, \;\; f_{xy} = (f_{yy})^2.$$

Root system of $\mathfrak{g}_2$ (II)

For a modern interpretation of the Cartan/Engel result, we need:

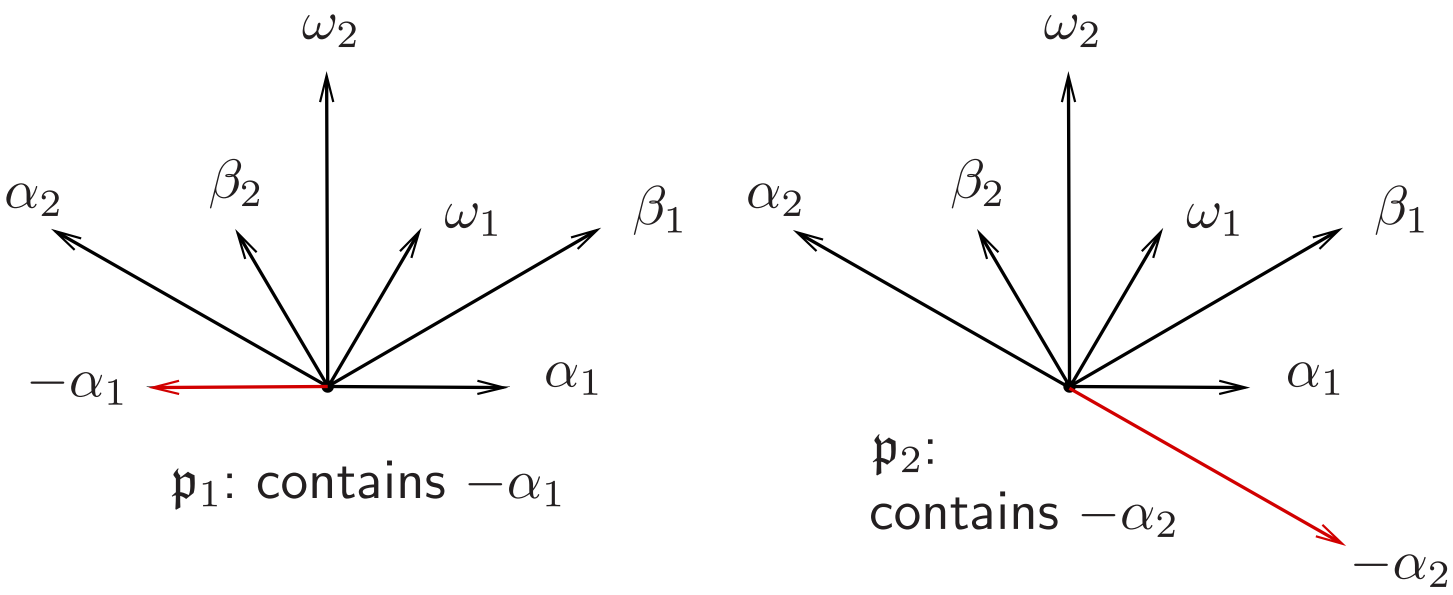


$\alpha_{1,2}$: simple roots

$\omega_{1,2}$: fundamental weights (ω_1: 7-dim. rep., ω_2: adjoint rep.)

$\mathcal{W}$: Weyl chamber = cone spanned by ω_1, ω_2

Parabolic subalgebras of $\mathfrak{g}_2$



Every parabolic subalgebra contains *all positive roots* and (eventually) some negative simple roots:

$\mathfrak{p}_1 = \mathfrak{h} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\beta_2} \oplus \mathfrak{g}_{\omega_2} \oplus \mathfrak{g}_{\omega_1} \oplus \mathfrak{g}_{\beta_1} \oplus \mathfrak{g}_{\alpha_1}$ [9-dimensional]

$\mathfrak{p}_2 = \mathfrak{h} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\beta_2} \oplus \mathfrak{g}_{\omega_2} \oplus \mathfrak{g}_{\omega_1} \oplus \mathfrak{g}_{\beta_1} \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_2}$ [9-dimensional]

$\mathfrak{p}_1 \cap \mathfrak{p}_2 = \mathfrak{h} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\beta_2} \oplus \mathfrak{g}_{\omega_2} \oplus \mathfrak{g}_{\omega_1} \oplus \mathfrak{g}_{\beta_1} \oplus \mathfrak{g}_{\alpha_1}$ [8-dim. Borel alg.]

Modern interpretation

The complex Lie group G_2 has two maximal parabolic subgroups P_1 and P_2 (with Lie algebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$)

$\Rightarrow G_2$ acts on the two 5-dimensional compact homogeneous spaces

- $M_1^5 := G_2/P_1 = \overline{G \cdot [v_{\omega_1}]} \subset \mathbb{P}(\mathbb{C}^7) = \mathbb{CP}^6$: a quadric
- $M_2^5 := G_2/P_2 = \overline{G \cdot [v_{\omega_2}]} \subset \mathbb{P}(\mathbb{C}^{14}) = \mathbb{CP}^{13}$ 'adjoint homogeneous variety'

where v_{ω_1}, v_{ω_2} are h. w. vectors of the reps. with highest weight ω_1, ω_2.

Cartan and Engel described the action of $\mathfrak{g}_2$ on some open subsets of M_i^5.

Real situation: To $P_i \subset G_2$ corresponds a real subgroup $P_i^* \subset G_2^*$, hence the split form G_2^* has two real compact 5-dimensional homogeneous spaces on which it acts.

However, G_2^c has no 9-dim. subgroups! (max. subgroup: 8-dim. $\mathrm{SU}(3) \subset G_2$)

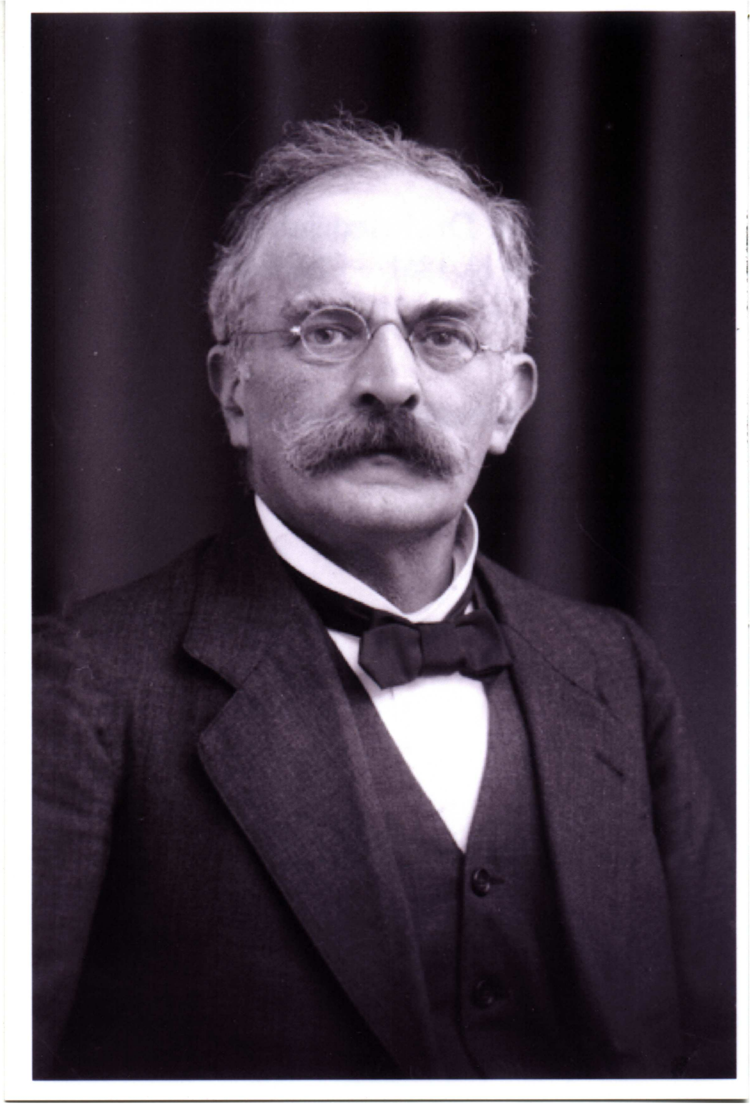
Q: Direct realisation of G_2^c ?

Élie Cartan (1869–1951)

- 1894 thesis at ENS (Paris), *Sur la structure des groupes de transformations finis et continus*.
- 1894–1912 maître de conférences in Montpellier, Nancy, Lyon, Paris
- 1912-1940 Professor in Paris
- É. Cartan, *Sur la structure des groupes simples finis et continus*, C. R. Acad. Sc. 116 (1893), 784-786.
- É. Cartan, *Nombres complexes*, Encyclop. Sc. Math. 15, 1908, 329-468.
- É. Cartan, *Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. Éc. Norm. 27 (1910), 109-192.

Friedrich Engel (1861–1941)

- 1883 thesis in Leipzig on contact transformations
- 1885–1904 Privatdozent in Leipzig
- 1904–1913 Professor in Greifswald, since 1913 in Gießen
- F. Engel, *Sur un groupe simple à quatorze paramètres*, C. R. Acad. Sc. 116 (1893), 786-788.
- F. Engel, *Ein neues, dem linearen Complexe analoges Gebilde*, Leipz. Ber. 52 (1900), 63-76, 220-239.
- editor of the complete works of S. Lie and H. Grassmann



G_2 and 3-forms in 7 variables

Non-degenerate 2-forms are at the base of symplectic geometry and define the Lie groups $\mathrm{Sp}(n, \mathbb{C})$.

Q: Is there a geometry based on 3-forms ?

• Generic 3-forms (i. e. with dense $\mathrm{GL}(n, \mathbb{C})$ orbit inside $\Lambda^3\mathbb{C}^n$) exist only for $n \leq 8$.

• To do geometry, we need existence of a compatible inner product, i. e. we want for generic $\omega \in \Lambda^3\mathbb{C}^n$

$$G_\omega := \{g \in \mathrm{GL}(n, \mathbb{C}) \mid \omega = g^*\omega\} \subset \mathrm{SO}(n, \mathbb{C}).$$

This implies (dimension count!) $n = 7, 8$.

And indeed: for $n = 7$: $G_\omega = G_2$, for $n = 8$: $G_\omega = \mathrm{SL}(3, \mathbb{C})$.

In fact, Engel had had this idea already in 1886. From a letter to Killing (8.4.1886):

> "There seem to be relatively few simple groups. Thus first of all, the two types mentioned by you [$\mathrm{SO}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$]. If I am not mistaken, the group of a linear complex in space of $2n-1$ dimensions ($n>1$) with $(2n+1)2n/2$ parameters [$\mathrm{Sp}(n,\mathbb{C})$] is distinct from these. In 3-fold space [$\mathbb{CP}^3$] this group [$\mathrm{Sp}(4,\mathbb{C})$] is isomorphic to that [$\mathrm{SO}(5,\mathbb{C})$] of a surface of second degree in 4-fold space. I do not know whether a similar proposition holds in 5-fold space. The projective group of 4-fold space [$\mathbb{CP}^4$] that leaves invariant a trilinear expression of the form
>
> $$\sum_{ijk}^{1\ldots5} a_{ijk} \begin{vmatrix} x_i & y_i & z_i \\ x_k & y_k & z_k \\ x_j & y_j & z_j \end{vmatrix} = 0$$
>
> will probably also be simple. This group has 15 parameters, the corresponding group in 5-fold space has 16, in 6-fold space [$\mathbb{CP}^6$] has 14, in 7-fold space [$\mathbb{CP}^7$] has 8 parameters. In 8-fold space there is no such group. These numbers are already interesting. Are the corresponding groups simple? Probably this is worth investigating. But also Lie, who long ago thought about similar things, has not yet done so."

Thm (Engel, 1900). A generic complex 3-form has precisely one $\mathrm{GL}(7,\mathbb{C})$ orbit. One such 3-form is

$$\omega_0 := (e_1e_4 + e_2e_5 + e_3e_6)e_7 - 2e_1e_2e_3 + 2e_4e_5e_6.$$

Every generic complex 3-form $\omega \in \Lambda^3(\mathbb{C}^7)^*$ satisfies:

1) The isotropy group G_ω is isomorphic to the simple group G_2;

2) ω defines a non degenerate symmetric BLF β_ω, which is cubic in the coefficients of ω and the quadric M_1^5 is its isotropic cone in $\mathbb{CP}^6$. In particular, G_ω is contained in some $\mathrm{SO}(7,\mathbb{C})$.

3) There exists a G_2-invariant polynomial $\lambda_\omega \neq 0$, which is of degree 7 in the coefficients of ω.

„Zudem ist hiermit eine direkte Definition unsrer vierzehngliedrigen einfachen Gruppe gegeben, die an Eleganz nichts zu wünschen übrig lässt.“
F. Engel, 1900

In modern notation: Set $V = \mathbb{C}^7$. Then

$$\beta_\omega : V \times V \to \Lambda^7 V^*, \quad \beta_\omega(X,Y) := (X \lrcorner \omega) \wedge (Y \lrcorner \omega) \wedge \omega$$

is a symmetric BLF with values in the 1-dim. space $\Lambda^7(\mathbb{C}^7)^*$ [R. Bryant, 1987]

Hence β_ω defines a map $K_\omega : V \to V^* \otimes \Lambda^7 V^*$, and

$$\det K_\omega \in (\Lambda^7 V)^* \otimes \Lambda^7(V^* \otimes \Lambda^7 V^*) = \Lambda^9(\Lambda^7 V^*).$$

Assume V is oriented $\Rightarrow$ fix an element $(\det K_\omega)^{1/9} \in \Lambda^7 V^*$ and set

$g_\omega := \frac{\beta_\omega}{(\det K_\omega)^{1/9}}$: this is a true scalar product, and $g_\omega = g_{-\omega}$.

$\det g_\omega := \lambda_\omega^3$ for an element of 'order' 7 in ω

$$\boxed{\lambda_\omega \neq 0 \Leftrightarrow \omega \text{ is generic} \Leftrightarrow g_\omega \text{ is nondegenerate}}$$

This allows a more concise description of the 2nd homogeneous space G_2/P_2:

Consider

$G_0^7(2,7) = \{\pi^2 \subset \mathbb{C}^7 \; : \; \beta_\omega\big|_{\pi^2} = 0\} \subset G^{10}(2,7) \subset \mathbb{P}(\Lambda^2\mathbb{C}^7)$ (Plücker emb.)

Then $G_2/P_2 = \{\pi^2 \subset G_0^7(2,7) \; : \; \pi^2 \lrcorner\, \omega = 0\}$

On the other hand, we know that

$G_2/P_2 = \overline{G \cdot [v_{\omega_2}]} \subset \mathbb{P}(\mathfrak{g}^2) \subset \mathbb{P}(\Lambda^2 V)$ (because $\Lambda^2 V = \mathfrak{g}_2 \oplus V$)

$\longrightarrow$ turns out: $G_2/P_2 = G^{10}(2,7) \cap \mathbb{P}(\mathfrak{g}^2)$ inside $\mathbb{P}(\Lambda^2 V)$

[Landsberg-Manivel, 2002/04]

Facts:

- G_2/P_2 has degree 18

- a smooth complete intersection of G_2/P_2 with 3 hyperplanes is a K3 surface of genus 10. [Borcea, Mukai]

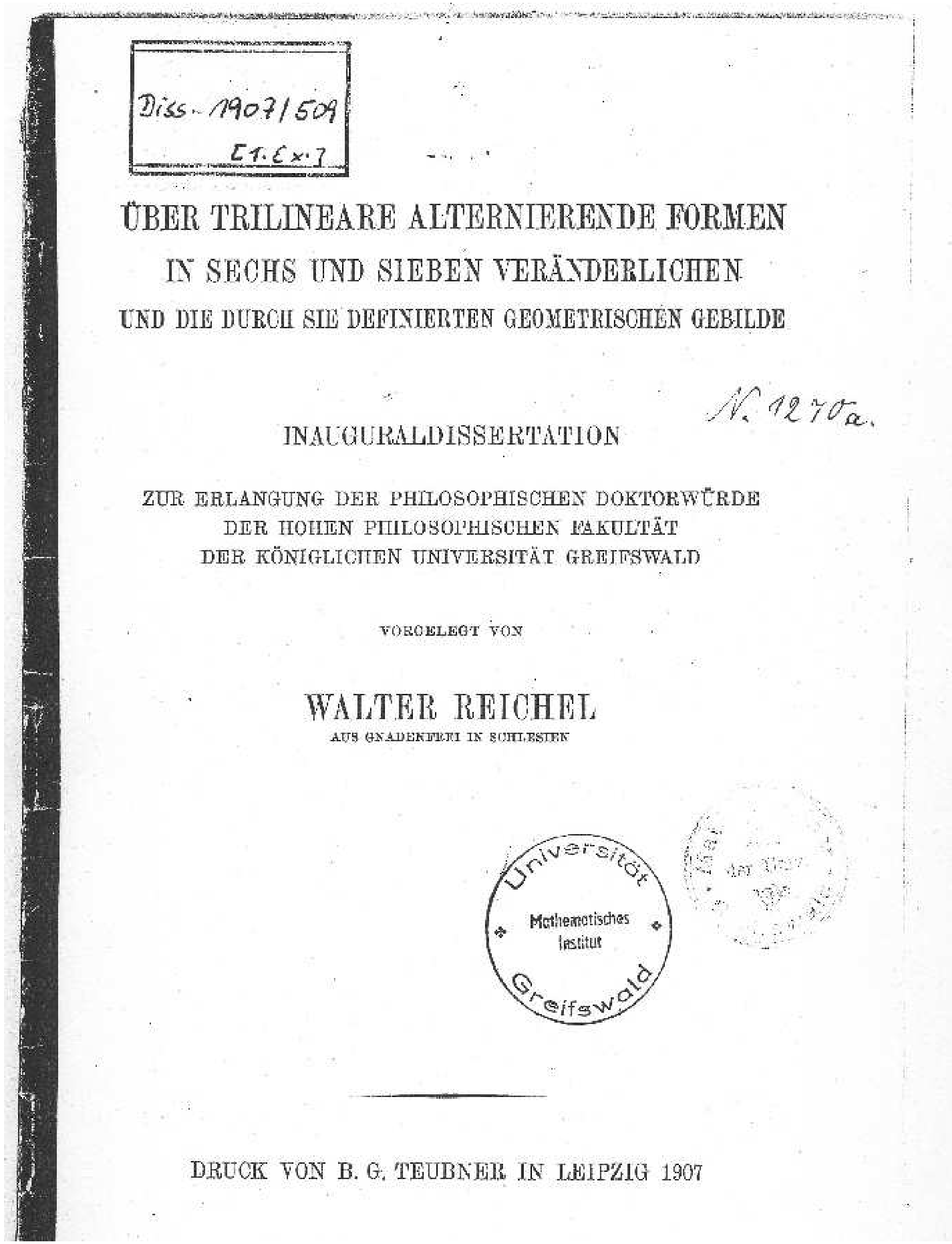
Walter Reichel's thesis (Greifswald, 1907)

- complete system of invariants for complex 3-forms in 6 und 7 variables through Study's symbolic method

- normal forms for 3-forms under $\mathrm{GL}(6,\mathbb{C}), \mathrm{GL}(7,\mathbb{C})$.

$n = 7$: vanishing of λ_ω for non generic 3-forms and rank of β_ω play a decisice role

- Lie-Algebra $\mathfrak{g}_\omega$ for any 3-form ω expressed in terms of its coefficients



ÜBER TRILINEARE ALTERNIERENDE FORMEN

IN SECHS UND SIEBEN VERÄNDERLICHEN

UND DIE DURCH SIE DEFINIERTEN GEOMETRISCHEN GEBILDE

INAUGURALDISSERTATION

ZUR ERLANGUNG DER PHILOSOPHISCHEN DOKTORWÜRDE
DER HOHEN PHILOSOPHISCHEN FAKULTÄT
DER KÖNIGLICHEN UNIVERSITÄT GREIFSWALD

VORGELEGT VON

WALTER REICHEL

AUS GNADENFREI IN SCHLESIEN

DRUCK VON B. G. TEUBNER IN LEIPZIG 1907

Over $\mathbb{R}$, there are two $\mathrm{GL}(7,\mathbb{R})$ orbits of generic 3-forms!

$\Rightarrow$ Reichel's formulas allow to compute the isotropy Lie group on both orbits, and indeed:

- one isotropy group is G_2^*, and the scalar product β_ω has signature $(4,3)$
- the other isotropy group is G_2^c, and the scalar product β_ω is positive definite.

Hence, Walter Reichel's thesis establishes for the first time a geometric realisation of G_2^c – in fact, the one which explains its importance in modern geometry.

N.B. G_2^c and the octonians:

- 1908 and 1914, É. Cartan: observes that $G_2^c \cong \mathrm{Aut}\mathbb{O}$
- this approach becomes popular by the work of H. Freudenthal (after 1951)

In fact, the 3-form approach and the the octonian picture are equivalent (a third equivalent description is through 'vector cross products')

[see J. Baez, 2002, for a modern account]

G_2 and spinors

The projective point of view lead Engel to further deep insights:

$l = [x : y]_6 := [x_0 : \ldots : x_3 : y_1 : \ldots : y_3]$ homogeneous coords. of $\mathbb{CP}^6$,

$M_1^5 = \{l \in \mathbb{CP}^6 | \, \beta_\omega(l, l) = 0\} = \{[x : y]_6 \in \mathbb{CP}^6 \,|\, x_0^2 + x_1 y_1 + x_2 y_2 + x_3 y_3 = 0\}$

Prop. A smooth quadric of dim.$2n+1$ contains an irred. $\frac{(n+1)(n+2)}{2}$-dim. family of n-planes. [Griffiths-Harris, Ch.6]

$\Rightarrow$ M_1^5 contains a 6-dim. family of 2-planes, check:
$\forall \, [a : b]_6 \in M_1^5$, the 8 eqs. ($i, j, k$ cyclic perm. of $1, 2, 3$)

$$b_i y_k - b_k y_i + a_0 x_j - a_j x_0 = 0, \quad a_0 x_0 + a_1 y_1 + a_2 y_2 + a_3 y_3 = 0$$

$$a_i x_k - a_k x_i + a_0 y_j - b_j x_0 = 0, \quad a_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 = 0$$

define a 2-plane $\pi_{a,b} \subset M_1^5$ *invariant under G_2*.

Now introduce a further coord. y_0 and homog. coords. on $\mathbb{CP}^7$

$$[x:y]_7 := [x_0 : \ldots : x_3 : y_0 : y_1 : \ldots : y_3]$$

Every plane $\pi_{a,b}$ defines a unique point $[a:b]_7 \in \mathbb{CP}^7$ by

$a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$ and the 8 eqs.

$$b_i y_k - b_k y_i + a_0 x_j - a_j x_0 = 0, \quad a_0 x_0 + a_1 y_1 + a_2 y_2 + a_3 y_3 = 0$$
$$a_i x_k - a_k x_i + b_0 y_j - b_j x_0 = 0, \quad b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 = 0$$

To conclude: $g: \ \{\pi_{a,b} \,|\, [a:b]_6 \in M_1^5\} \longrightarrow \mathbb{CP}^7, \quad \pi_{a,b} \longmapsto [a:b]_7 \in \mathbb{CP}^7$, and $\operatorname{im} g$ *is a smooth quadric* $Q^6 \subset \mathbb{CP}^7$.

Thm. (Engel, 1900)

1) The map g is $\mathfrak{so}(7)$-equivariant, and extends to an irreducible $\mathfrak{so}(7)$-representation on $\mathbb{CP}^7$;

2) The stabilizer of $[a_0 : 0 : 0 : 0 : -a_0 : 0 : 0 : 0]$ (not on Q^6) is G_2.

Modern description

Consider spin representation $\kappa^{\mathbb{C}}: \ \mathrm{Spin}(7) \to \mathrm{End}(\Delta_7^{\mathbb{C}}), \quad \Delta_7^{\mathbb{C}} \cong \mathbb{C}^8$.

In dim.7, this turns out to be complexification of 8-dim. real rep.,

$$\kappa: \ \mathrm{Spin}(7) \to \mathrm{End}(\Delta_7), \quad \Delta_7 \cong \mathbb{R}^8.$$

and for a generic spinor $\psi \in \Delta_7$: $\boxed{G_2^c = \{A \in \mathrm{Spin}(7) \mid A\psi = \psi\}}$

$\Rightarrow$ explains physicists' interest in G_2^c.

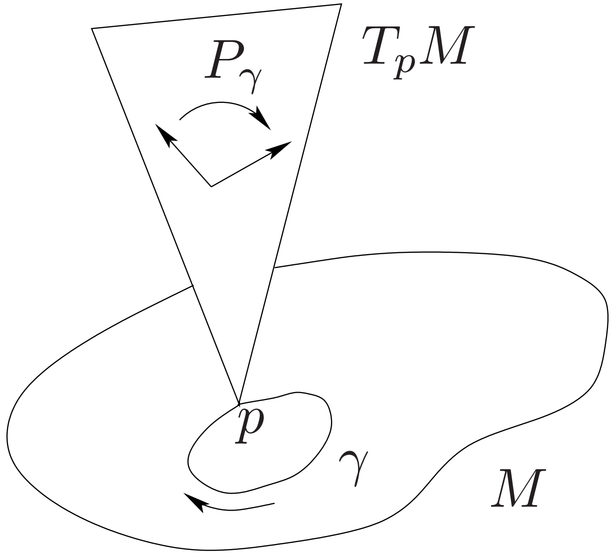
The work of Engel, Killing, and Reichel on G_2 and the description of G_2^c

- as a stabilizer of a generic 3-form $\in \Lambda^3(\mathbb{R}^7)$
- as a stabilizer of a generic spinor in Δ_7

was forgotten for more than 60 years.

Work in progress: Engel sketches how the algebraic geometry of $Q^6 \subset \mathbb{CP}^7$ gives an analogon of Plücker's line complexes ($\mathbb{CP}^3 \supset \pi^1 \longmapsto l \in Q \subset \mathbb{CP}^5$, Q: Klein quadric, 2-forms $\cong$ Plücker coords.)

Holonomy group of a connection ∇



Thm (Berger [& Simons], ≥ 1955).
The reduced holonomy $\mathrm{Hol}_0(M;\nabla^g)$ of the LC connection ∇^g is either that of a symmetric space or
$\mathrm{Sp}(n)\mathrm{Sp}(1)$ [qK], $\mathrm{U}(n)$ [K], $\mathrm{SU}(n)$ [CY], $\mathrm{Sp}(n)$ [hK], G_2^c, $\mathrm{Spin}(7)$.

However, Berger missed that [Bonan, 1966]

- manifolds with holonomy G_2^c have a ∇^g-parallel 3-form,
- manifolds with holonomy $\mathrm{Spin}(7)$ have a ∇^g-parallel 4-form,

and, in consequence, both have to be Ricci-flat.

This is—up to our knowledge—the first reappearance of the 3-form defining a G_2^c structure after Engel and Reichel.

Weak holonomy (A. Gray, 1971):

Idea: Enlarge the successful holonomy concept to wider classes of manifolds (contact manifolds, almost Hermitian manifolds etc.)

Dfn. ***'nearly parallel G_2^c-manifold':*** has structure group G_2^c, but 3-form ω is not parallel, but rather satisfies

$$d\omega = \lambda * \omega \text{ for some } \lambda \neq 0.$$

Fernandez-Gray, 1982: Show that there are 4 basic classes of manifolds with G_2^c-structure and construct first examples:

$S^7 = \mathrm{Spin}(7)/G_2^c$, $\mathrm{SU}(3)/S^1$ (Aloff-Wallach sp.), ext. of Heisenberg groups. . .

Progress in the parallel G_2^c case:

- 1987-89, R. Bryant and S. Salamon: local complete metrics with Riemannian holonomy G_2^c
- 1996, D. Joyce: existence of compact Riemannian 7-dimensional manifolds with Riemannian holonomy G_2^c

Today's general philosophy:

Given a mnfd M^n with G-structure ($G \subset \mathrm{SO}(n)$), replace ∇^g by a *metric connection ∇ with torsion that preserves the geometric structure!*

$$\text{torsion: } T(X,Y,Z) \;:=\; g(\nabla_X Y - \nabla_Y X - [X,Y], Z)$$

Special case: require $T \in \Lambda^3(M^n)$ ($\Leftrightarrow$ same geodesics as ∇^g)

$$\Rightarrow \quad g(\nabla_X Y, Z) \;=\; g(\nabla^g_X Y, Z) + \tfrac{1}{2} T(X,Y,Z)$$

- representation theory yields

- a clear answer *which* G-structures admit such a connection; if existent, it's unique and called the *'characteristic connection'*

- a *classification scheme* for G-structures with characteristic connection: $T_x \in \Lambda^3(T_x M) \stackrel{G}{=} V_1 \oplus \ldots \oplus V_p$

- study Dirac operator $\not{D}$ of the metric connection with torsion $T/3$: *'characteristic Dirac operator'* (generalizes the Dolbeault operator, Kostant's cubic Dirac operator) [IA-TF et.al. since 2003]

G_2^c-manifolds

Dfn: mnfds with reduction of the frame bundle to G_2; reduction induces 3-form and hence a metric, automatically spin

- $\exists$ char. connection $\nabla \Leftrightarrow \exists$ VF β s. t. $\delta\omega = -\beta \lrcorner \omega$, torsion:

$$T = - * d\omega - \frac{1}{6}(d\omega, *\omega)\omega + *(\beta \wedge \omega)$$ [TF-Ivanov, 2002]

- $\nabla\omega^3 = 0$, at least on spinor field with $\nabla\psi = 0$ and $\mathrm{Hol}_0(\nabla) \subset G_2 \subset \mathrm{SO}(7)$

Superstring theory:

torsion $\cong$ field, ∇-parallel spinor $\cong$ supersymmetry transformation.

Duality:

$T = 0$: 'vacuum solutions' of superstring theory $\longrightarrow$ algebraic geometry (K3 surfaces, Calabi-Yau manifolds. . .)

$T \neq 0$: 'non vacuum solutions' of superstring theory $\longrightarrow$ differential geometry, connections with torsion

Exceptional G_2^c-manifolds—the flat case

Suppose (M, g) Riemannian, ∇ metric with torsion $T \in \Lambda^3(M)$

Q: What are the manifolds with a flat metric connection with skew torsion?

$\Rightarrow$ (M, g) is parallelisable (and therefore spin): take any frame in $p \in M$ and transport it to all other points

Example 1: Lie groups

Let $M = G$ be a connected Lie group, g a biinvariant metric

Ansatz: T proportional to $[,]$, i. e. $\nabla_X Y := \lambda[X, Y]$

- torsion: $T^\nabla(X, Y) = (2\lambda - 1)[X, Y]$ ($T \in \Lambda^3(G) \Leftrightarrow g$ biinv.), $\nabla T = 0$
- curvature:

$$\mathcal{R}^\nabla(X, Y)Z = \lambda(1 - \lambda)[Z, [X, Y]] = \begin{cases} \frac{1}{4}[Z, [X, Y]] & \text{for LC conn.}(\lambda = \frac{1}{2}) \\ 0 & \text{for } \lambda = 0, 1 \end{cases}$$

[$\pm$-connection, Cartan-Schouten, 1926]

Example 2: S^7

- only parallelisable sphere that is not a Lie group (but almost. . .)

Consider spin representation $\kappa: \ \mathrm{Spin}(7) \to \mathrm{End}(\Delta_7), \quad \Delta_7 \cong \mathbb{R}^8$

κ is in fact a repr. of the Clifford algebra over $\mathbb{R}^7$ ($\mathrm{Spin}(7) \subset \mathrm{Cl}(\mathbb{R}^7)$!),

$$\kappa: \ \mathbb{R}^7 \subset \mathrm{Cl}(\mathbb{R}^7) \to \mathrm{End}(\Delta_7).$$

Choose $e_1, \ldots, e_7$ an ON basis of $\mathbb{R}^7$, and set $\kappa_i = \kappa(e_i)$.

- Embed $S^7 \subset \Delta_7$ as spinors of length 1,
- the VFs $V_i(x) = \kappa_i \cdot x$ for all $x \in S^7 \subset \Delta^7$ realize ON trivialization of S^7
- the connection ∇ defined by $\nabla V_i = 0$ is metric, flat, and with torsion

$$T(V_i, V_j, V_k)(x) \ = \ -\langle [V_i, V_j], V_k \rangle \ = \ 2\langle \kappa_i \kappa_j \kappa_k x, x \rangle \in \Lambda^3(S^7)$$

- $\nabla T \neq 0$ (check that T does not have constant coefficients)
- ∇ is a G_2 connection of Fernandez-Gray type $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$

Classification

Goal: Show that any irreducible, complete, and simply connected M with a flat, metric connection with antisymmetric torsion $T \in \Lambda^3(M)$ is one of these examples.

- 1926: Cartan-Schouten "On manifolds with absolute parallelism" – wrong proof.

- 1968: d'Atri-Nickerson "On the existence of special orthonormal frames" – when does (M, g) admit an ONF of Killing vectors?

  This is mainly an equivalent problem:

  $$V \text{ is Killing VF} \Leftrightarrow g(\nabla^g_X V, Y) + g(X, \nabla_Y V) = 0 \qquad (*)$$

  If V is parallel for ∇ with torsion T, then $\nabla^g_X V = -\frac{1}{2}T(X, V)$, hence

  $$(*) \Leftrightarrow g(T(X,V),Y) + g(X, T(Y,V)) = 0 \Leftrightarrow T \in \Lambda^3(M)$$

- 1972: J. Wolf "On the geometry and classification of absolute parallelisms" – 2 long papers in J. Diff.Geom.

Q: Both proofs rely on classification of symmetric spaces. Direct proof?

Sketch of proof

(1) General identities: [common to all authors]

- $\mathrm{Ric}^g(X,Y) = \frac{1}{4}\sum_i \langle T(X,e_i), T(Y,e_i)\rangle$, ($\Rightarrow$ $\mathrm{Ric}^g(X,X) \geq 0$)
- $K^g(X,Y) = \frac{\|T(X,Y)\|^2}{4[\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2]} \geq 0$ (sectional curvature)
- $\delta T = 0$ ($=$ antisymmetric part of Ric^∇)

(2) General tools: $\sigma_T = \frac{1}{2}\sum_i (e_i \lrcorner T) \wedge (e_i \lrcorner T) \in \Lambda^4(M)$ satisfies

- $T^2 = -2\sigma_T + \|T\|^2$ (as endomorphisms on Δ_7)
- $\nabla T = 0$ implies $dT = 2\sigma_T$ [recall: true for G, wrong for S^7]
- All spinors with constant coeff. are parallel $\Rightarrow$ $3dT = 2\sigma_T$ (SL formula)
- Bianchi I:

$$\overset{X,Y,Z}{\mathfrak{S}}\ \mathcal{R}(X,Y,Z,V) = dT(X,Y,Z,V) - \sigma^T(X,Y,Z,V) + (\nabla_V T)(X,Y,Z)$$

(3) Rescaling of connection:

[implicit in Cartan]

Consider the rescaled connection $\nabla^{1/3}$,

$$\nabla^{1/3}{}_X Y \;=\; \nabla^g_X Y + \frac{1}{6} T(X,Y)$$

– $\nabla^{1/3}$ plays a prominent role for Dirac operators with torsion

Thm.

- $\nabla^{1/3} T = 0 \quad (\Leftrightarrow \nabla_V T = -\frac{1}{3} V \lrcorner\, \sigma_T \Leftrightarrow \nabla^g_V T = \frac{1}{6} V \lrcorner\, \sigma_T)$

In particular, $\|T\|$ and the scalar curvature are constant, and for any tensor field $\mathcal{T}$ polynomial in T:

$$\nabla \mathcal{T} = -2\nabla^g \mathcal{T};\ \text{in particular: } \nabla \mathcal{T} = 0 \Leftrightarrow \nabla^g \mathcal{T} = 0$$

- $\nabla^{1/3} \mathcal{R}^g = 0$

By the Ambrose-Singer Thm, M is a naturally reductive space (in particular, homogeneous).

(4) Splitting principle:

Thm. Let $M = M_1 \times M_2$ be a mnfd with a flat metric connection ∇ with torsion $T \in \Lambda^3(M)$. Then $T = T_1 + T_2$ with $T_i \in \Lambda^3(M_i)$.

(5) Type of M:

Thm. Let $e_1, \ldots, e_n$ be a ONF of ∇-parallel VFs. Then:

- $\mathcal{R}^g(e_i, e_j)e_k = -\frac{1}{4}[[e_i, e_j], e_k]$ [$\Rightarrow$ M is Einstein]
- $e_m\langle[e_i, e_j], e_k\rangle = -(\nabla_{e_m} T)(e_i, e_j, e_k) = -\frac{1}{3}\sigma_T(e_i, e_j, e_k, e_m)$ $\quad (*)$

Cor. $e_i(R_{jklm}) = 0$, hence $\nabla^g \mathcal{R}^g = 0$ and, by (2), $\nabla \mathcal{R}^g = 0$ and

$$(\nabla_X - \nabla^g_X)\mathcal{R}^g = [X \lrcorner T, \mathcal{R}^g] = 0 \qquad (**)$$

Cor. (M, g) is a compact symmetric Einstein space.

1st case: $\sigma_T = 0$. $(*)$ $\Rightarrow$ all $\langle[e_i, e_j], e_k\rangle$ = const $\Rightarrow$ M is Lie group

2nd case: $\sigma_T \neq 0$ $(n > 4)$. Consider the Lie algebra

$$\mathfrak{g}_T(p) \;:=\; \mathrm{Lie}\langle X \lrcorner T | X \in T_pM\rangle \subset \Lambda^2 T_pM \;\cong\; \mathfrak{so}(T_pM).$$

By the splitting principle, may assume: $\mathfrak{g}_T(p)$ acts irreducibly on T_pM.

Idea: Let $G_T(p)$ be a Lie group with Lie algebra $\mathfrak{g}_T(p)$ and consider its action on unit sphere $S \subset T_pM$.

Thm (Skew holonomy theorem). There are only two possible cases:

(1) $G_T(p)$ does not act not transitively on S:

$T(X,Y) =: [X,Y]$ defines a Lie bracket and M is a Lie group,

(2) or $G_T(p)$ acts transitively on S:

$\mathfrak{g}_T(p) = \mathfrak{so}(T_pM)$.

dfn of $\mathfrak{g}_T(p)$: AF. (1): Olmos-Reggiani;

(2): AF indirectly if one uses the classification of transitive sphere actions, except for the qK case (OR); or OR for a more systematic proof.

Cor. If M is not a Lie group, $\mathfrak{g}_T(p) = \mathfrak{so}(T_pM)$ and

$$(**) \Rightarrow \mathcal{R}^g = c \cdot \mathrm{Id} \Rightarrow K^g(X, Y) = c \cdot \mathrm{Id}$$

$\Rightarrow$ M is a sphere

$\Rightarrow$ formula for $K^g(X, Y)$ states that T defines a vector cross product

$$\Rightarrow \qquad \boxed{M = S^7}$$