Invariants of Lie groups and their applications to differential equations

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Abstract

We consider invariants and differential invariants of Lie group of symmetry of nonlinear Schrödinger equation and Lie group of equivalence transformations of Riccati equations. We use the Tresse theorem in order to construct functional basis of differential invariants for studied Lie groups. By using founded basic invariants we propose some criteria of integrability of Riccati equations and study some group-theoretical properties of nonlinear Schrödinger equation.

1 Basic notions and definitions

We use the notation $x = (x_1, x_2, ..., x_n) \in V \subset \mathbb{R}^n$,
u denotes a function $u(x_1, ..., x_n) : V \to \mathbb{R}$,
u_{x_i} denotes the partial derivative $\dfrac{\partial u}{\partial x_i}$,
$\underset{k}{u}$ denotes the set of all partial derivatives of the order k of a function u,
D_i denotes the operator of the total differentiation over x_i.
G denotes a Lie group of transformations depending on (x, u),
$X = \xi^i \partial_{x_i} + \eta \partial_u$ denotes infinitesimal generator in the Lie algebra of G,
$\underset{m}{X}$ denotes the extension of the m - th order of X to the space
$(x_1, x_2, ...x_n, u, \underset{1}{u}, \underset{2}{u}, ..., \underset{m}{u})$ and define it by the formula

$$\underset{m}{X} = X + \sum_{p=1}^{m} \zeta^{i_1,\ldots,i_p} \partial_{u_{x_{i_1,\ldots,i_p}}},$$

where $\zeta^{i_1,\dots,i_p}$ are the following

$$\zeta^{i_1,\dots,i_p} = D_{i_1,\dots,i_p}(\eta - u_{x_k}\xi^k) + u_{x_{i_1},\dots,x_{i_p},x_{i_k}}\xi^k,$$

Definition 1.1. *Let G be a given Lie group of transformations, $x \in V \subset \mathbb{R}^n$, $u : V \to \mathbb{R}$. A function $F(x,u)$ is called an invariant of the group G, iff*

$$\forall_{\varphi\in G} \quad F(\varphi(x,u)) = F(x,u)$$

Definition 1.2. *A function $F(x,u,\underset{1}{u},...,\underset{m}{u})$ is called a differential invariant (of the m-th order) of G iff*

$$\forall_{\underset{m}{\varphi}\in\underset{m}{G}} \quad F(\underset{m}{\varphi}(x,u,\underset{1}{u},...,\underset{m}{u})) = F(x,u,\underset{1}{u},...,\underset{m}{u}).$$

Definition 1.3. *Let G be a Lie group of transformations depending on (x,u). We denote by $\underset{m}{G}$ the extension of the group G to the space $(x,u,\underset{1}{u},...,\underset{m}{u})$.*
The **general** *(or universal)* **differential invariant** *of the m-th order is the set of all differential invariants from the order zero to the order m inclusive,*

A maximal set of functionally independent invariants of the order $r \leqslant m$ of a Lie group G is called a **functional basis** *of the m–th order differential invariants of G,*

Q is called an ***operator of the invariant differentiation****, if for any differential invariant F of the group G the expression QF is also the differential invariant of the group G.*

Example 1.1. *The group of rotations in $\mathbb{R}^3$:* $\begin{cases} \widetilde{x} = x\cos a - y\sin a \\ \widetilde{y} = x\sin a + y\cos a \\ \widetilde{u} = u \end{cases}$

with infinitesimal generator $X = -y\partial_x + x\partial_y$.
Invariants of the order zero satisfy the equation $X\omega = 0$ and they are

$$\omega_{01} = u, \ \omega_{02} = x^2 + y^2$$

First order invariants are

$$\omega_{11} = u_x^2 + u_y^2, \quad \omega_{12} = xu_x + yu_y$$

The second order basic invariants are

$$\omega_{21} = xu_yu_{yy} - yu_xu_{xx} + (xu_x - yu_y)u_{xy}, \quad \omega_{22} = u_x^2u_{xx} + u_y^2u_{yy} + 2u_xu_yu_{xy},$$

$$\omega_{23} = u_{xx} + u_{yy}$$

Example 1.2. *The Lorentz group in $(x,y,u) \in \mathbb{R}^3$ with the generator $X = y\partial_x + x\partial_y$.*
Invariants of the order zero have the form

$$u, \quad x^2 - y^2$$

First order basic invariants are

$$u_x^2 - u_y^2, \quad xu_x + yu_y$$

2 The Tresse theorem

In the theory of differential invariants of Lie group the most important role plays the following theorem.

Theorem 2.1. *(Tresse, 1894)*
For a given r–dimensional Lie group of transformations G $(r < +\infty)$, depending on variables (x, u), $x \in V \subset \mathbb{R}^n$, $u : V \to \mathbb{R}$ there exists a finite basis of functionally independent invariants and exist operators of the invariant differentiation Q such that an arbitrary fixed order invariant of G can be obtained in a finite number of invariant differentiations and functional operations on invariants from the basis.

Lemma 2.1. *Let $\mathcal{A}$ be the Lie algebra of the Lie group G from the Tresse theorem, operators $X_\nu = \xi^i_\nu(x,u)\partial_{x_i} + \eta_\nu(x,u)\partial_u$ for $\nu = 1, ..., r$ be the generators of $\mathcal{A}$ and $\xi^i(x,u) = [\xi^i_1, ..., \xi^i_r]^T$, $\eta(x,y) = [\eta_1, ..., \eta_r]^T$.*
Then the properties of the functional basis from the Tresse theorem are the following
1) This finite basis of invariants is included in the general differential invariant of the minimal order $s \geqslant 1$ such that

$$r = rank\left[\xi^i(x,u), \eta(x,u), \zeta^{i_1}(x,u,\underset{1}{u}), ..., \zeta^{i_1,...,i_{s-1}}(x,u,...,\underset{s-1}{u})\right].$$

2) Operators of the invariant differentiation are defined by

$$Q = \lambda^k(x,u,\underset{1}{u},...,\underset{s}{u})D_{x_k},$$

where λ satisfies the condition

$$\underset{s}{X}_\nu \lambda = \lambda^k D_{x_k}(\xi_\nu).$$

3) If a group G acts in the space of n independent and k dependent variables, then the number of elements in a functional basis of the m–th order is given by the formula

$$R(m) = n + k \cdot \binom{n+m}{n} - r_m,$$

where r_m is a rank of the matrix of coefficients of the m–th extension of operators X_ν.
4) If all invariants of the order s can be obtained from invariants of the order $s-1$ by a finite number of invariant differentiation and functional operations then the basis of invariants from the Tresse theorem is included in the general invariant of the order $s-1$. Analogical property holds for the invariants of lower orders.
5) The basis of differential invariants from the Tresse theorem of a Lie group G with the operators of invariant differentiation uniquely define G.

Example 2.1. *Consider the group of rotations in $\mathbb{R}^3$:* $\begin{cases} \widetilde{x} = x\cos a - y\sin a \\ \widetilde{y} = x\sin a + y\cos a \\ \widetilde{u} = u \end{cases}$,

with infinitesimal generator $X = -y\partial_x + x\partial_y$.
Operators of invariant differentiation have the form

$$Q_1 = u_x D_x + u_y D_y, \quad Q_2 = -u_y D_x + u_x D_y.$$

Invariants of the order zero satisfy the equation $X\omega = 0$ *and they are*

$$\omega_{01} = u, \ \omega_{02} = x^2 + y^2$$

$$\underset{1}{X} = -y\partial_x + x\partial_y - u_y\partial_{u_x} + u_x\partial_{u_y}$$

The basis of a general invariant of the first order consists of four elements

$$u, \ x^2 + y^2, \ u_x^2 + u_y^2 = Q_1(\omega_{01}), \ xu_x + yu_y = \frac{1}{2}Q_1(\omega_{02})$$

3 Differential invariants connected with the nonlinear Schrödinger equation

Consider the NSE of the form

$$i\psi_t + \psi_{xx} + W(|\psi|) \cdot \psi = 0,$$

where $x \in \mathbb{R}^1$ and W is an arbitrary smooth function.
It admits the four–dimensional, solvable Lie algebra of symmetry with generators of the form

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = \psi\partial_\psi - \psi^*\partial_{\psi^*}, \quad X_4 = t\partial_x + \frac{i}{2}x(\psi\partial_\psi - \psi^*\partial_{\psi^*})$$

The commutation relations are as follows

$$[X_1, X_2] = 0, \quad [X_1, X_3] = 0, \quad [X_1, X_4] = X_2, \quad [X_2, X_3] = 0,$$

$$[X_2, X_4] = \frac{i}{2}X_3, \quad [X_3, X_4] = 0.$$

The invariant of the order zero is $\omega_0 = \psi\psi^* = |\psi|^2$.
The first order invariants are

$$\omega_1 = \frac{\psi_x}{\psi} + \frac{\psi_x^*}{\psi^*}, \quad \omega_2 = \frac{\psi_t}{\psi} - i \cdot \left(\frac{\psi_x}{\psi}\right)^2, \quad \omega_3 = \frac{\psi_t^*}{\psi^*} + i \cdot \left(\frac{\psi_x^*}{\psi^*}\right)^2.$$

Note that $\omega_i, \ i = 0, 1, 2, 3$ are functionally independent (over $\mathbb{R}$) and form the maximal system of the first order invariants of this algebra.

One can easily seen that general second order differential invariant of this Lie algebra have 10 generators and among these we have ω_i for $i = 0, 1, 2, 3$.
We find the invariant differentiation operators Q_1, Q_2 and show that $\omega_i, \ i = 0, 1, 2, 3$ and Q_1, Q_2 suffice to express all second order invariants.

According to Tresse theorem from equation

$$\underset{s}{X}_\nu \lambda = \lambda^k D_{x_k}(\xi_\nu)$$

for X_1, X_2 we obtain that λ not depend on t, x. For X_3, X_4 we have

$$\psi \cdot \begin{bmatrix} \lambda^1_\psi \\ \lambda^2_\psi \end{bmatrix} - \psi^* \cdot \begin{bmatrix} \lambda^1_{\psi^*} \\ \lambda^2_{\psi^*} \end{bmatrix} + \psi_t \cdot \begin{bmatrix} \lambda^1_{\psi_t} \\ \lambda^2_{\psi_t} \end{bmatrix} + \psi_x \cdot \begin{bmatrix} \lambda^1_{\psi_x} \\ \lambda^2_{\psi_x} \end{bmatrix} - \psi_t^* \cdot \begin{bmatrix} \lambda^1_{\psi_t^*} \\ \lambda^2_{\psi_t^*} \end{bmatrix} - \psi_x^* \cdot \begin{bmatrix} \lambda^1_{\psi_x^*} \\ \lambda^2_{\psi_x^*} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\frac{i}{2}x\psi \cdot \begin{bmatrix}\lambda^1_{\psi}\\ \lambda^2_{\psi}\end{bmatrix} - \frac{i}{2}x\psi^* \cdot \begin{bmatrix}\lambda^1_{\psi^*}\\ \lambda^2_{\psi^*}\end{bmatrix} + \left(\frac{i}{2}x\psi_t - \psi_x\right) \cdot \begin{bmatrix}\lambda^1_{\psi_t}\\ \lambda^2_{\psi_t}\end{bmatrix} + \left(\frac{i}{2}\psi + \frac{i}{2}x\psi_x\right) \cdot \begin{bmatrix}\lambda^1_{\psi_x}\\ \lambda^2_{\psi_x}\end{bmatrix} -$$

$$-\left(\frac{i}{2}x\psi^*_t + \psi^*_x\right) \cdot \begin{bmatrix}\lambda^1_{\psi^*_t}\\ \lambda^2_{\psi^*_t}\end{bmatrix} - \left(\frac{i}{2}\psi^* + \frac{i}{2}x\psi^*_x\right) \cdot \begin{bmatrix}\lambda^1_{\psi^*_x}\\ \lambda^2_{\psi^*_x}\end{bmatrix} = \begin{bmatrix}0\\ \lambda^1\end{bmatrix}.$$

Solving this system we obtain

$$\begin{bmatrix}\lambda^1\\ \lambda^2\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix} \qquad \vee \qquad \begin{bmatrix}\lambda^1\\ \lambda^2\end{bmatrix} = \begin{bmatrix}\psi\psi^*\\ i(\psi\psi^*_x - \psi_x\psi^*)\end{bmatrix}.$$

Hence the invariant differentiation operators are in the form

$$Q_1 = D_x, \qquad Q_2 = \psi\psi^* D_t + i(\psi\psi^*_x - \psi_x\psi^*)D_x.$$

Note that the coefficients of the invariant differentiation operators are real and all basic second order differential invariants one can obtain from the first order ones by invariant differentiation.

$$\omega_{ij} = Q_i(\omega_j), \;\; i = 1,2; \;\; j = 1,2,3.$$

3.1 Construction of NSE using differential invariants

Consider a new invariants ω_4, Ω

$$\omega_4 = \frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2}, \qquad \Omega = i \cdot \omega_2 + \omega_4 = i\frac{\psi_t}{\psi} + \frac{\psi_{xx}}{\psi}.$$

Now we take invariant equation

$$\Omega = F(\omega_0),$$

where F is an arbitrary function. Hence

$$i\frac{\psi_t}{\psi} + \frac{\psi_{xx}}{\psi} = F(|\psi|^2).$$

Now multiplying by ψ and putting $F(|\psi|^2) = -W(|\psi|)$ we obtain the studied nonlinear Schrödinger equation $i\psi_t + \psi_{xx} + W(|\psi|) \cdot \psi = 0$.

3.2 Nonlinear Schrödinger equation - special case

Consider the NSE of the form

$$i\psi_t + \psi_{xx} + |\psi|^2\psi = 0.$$

It admits the five–dimensional, solvable Lie algebra of point symmetry

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = \psi\partial_\psi - \psi^*\partial_{\psi^*},$$

$$X_4 = t\partial_x + \frac{i}{2}x(\psi\partial_\psi - \psi^*\partial_{\psi^*}), \quad X_5 = 2t\partial_t + x\partial_x - \psi\partial_\psi - \psi^*\partial_{\psi^*}$$

where ψ^* is the complex conjugation of ψ and $\psi\psi^* = |\psi|^2$.
Further we calculate cardinal number of functional basis and differential invariants for this Lie algebra.

$$R(0) = 2 + 2\cdot\binom{2+0}{2} - 4 = 0, \quad R(1) = 2 + 2\cdot\binom{2+1}{2} - 5 = 3, \quad R(2) = 9$$

$$\omega_1 = \frac{\psi_x}{|\psi|\psi} + \frac{\psi_x^*}{|\psi|\psi^*}, \quad \omega_2 = \frac{\psi_t}{|\psi|^2\psi} - i\left(\frac{\psi_x}{|\psi|\psi}\right)^2, \quad \omega_3 = \frac{\psi_t^*}{|\psi|^2\psi^*} + i\left(\frac{\psi_x^*}{|\psi|\psi^*}\right)^2$$

$$\omega_4 = \frac{1}{|\psi|^2\psi^2}(\psi_{xx}\psi - \psi_x^2), \qquad \omega_4^* = \overline{\omega_4},$$

$$\omega_5 = \frac{\psi_{tx}\psi - \psi_x\psi_t}{|\psi|^3\psi^2} + \frac{i(\psi\psi_x^* - \psi_x\psi^*)}{|\psi^5|}\left(\frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2}\right), \quad \omega_5^* = \overline{\omega_5},$$

$$\omega_6 = \frac{1}{|\psi|^6}[\psi\psi^* D_t(\Omega) + i(\psi\psi_x^* - \psi_x\psi^*)D_x(\Omega)], \quad \omega_6^* = \overline{\omega_6}$$

where $\quad \Omega = \dfrac{\psi_t}{\psi} - i\left(\dfrac{\psi_x}{\psi}\right)^2.$

Operators of invariant differentiation are

$$Q_1 = \frac{1}{|\psi|}D_x, \quad Q_2 = \frac{1}{|\psi|^2}D_t + \frac{i(\psi\psi_x^* - \psi_x\psi^*)}{|\psi|^4}D_x$$

and it appears that all second order differential invariants can be obtained from the first order ones.
We have the invariant form of the studied NSE

$$i\omega_2 + \omega_4 + 1 = 0$$

$$i\frac{\psi_t}{|\psi|^2\psi} + \left(\frac{\psi_x}{|\psi|\psi}\right)^2 + \frac{1}{|\psi|^2\psi^2}(\psi_{xx}\psi - \psi_x^2) + 1 = 0$$

4 Differential invariants of equivalence transformations of Riccati equations and their applications

Let us consider the family of general Riccati equations, described by three arbitrary functions $a(x), b(x), c(x)$ in the form

$$y' = a(x)y^2 + b(x)y + c(x)$$

Equivalence transformation of the family of general Riccati equations is a nonsingular change of variables

$$\widetilde{x} = \alpha(x,y), \ \widetilde{y} = \beta(x,y),$$

preserving the set

$$\Omega_R = \{y' = a(x)y^2 + b(x)y + c(x) : \ y, a, b, c : \mathbb{R} \to \mathbb{R}\},$$

i.e. carrying every equation $y' = a(x)y^2 + b(x)y + c(x)$ from Ω_R to equation

$$\widetilde{y}\,' = \widetilde{a}(\widetilde{x})\widetilde{y}\,^2 + \widetilde{b}(\widetilde{x})\widetilde{y} + \widetilde{c}(\widetilde{x}),$$

where functions $\widetilde{a}, \widetilde{b}, \widetilde{c}$ may be different from a, b, c.

We use the infinitesimal technique of finding the Lie algebra of Lie group G_R of equivalence transformations of Riccati equations. Treating a, b, c as new dependent variables we are searching for the symmetry operators

$$X = \xi(x,y)\partial_x + \eta(x,y)\partial_y + \mu^1(x,y,a,b,c)\partial_a + \mu^2(x,y,a,b,c)\partial_b +$$

$$+\mu^3(x,y,a,b,c)\partial_c$$

of the system of equations

$$\begin{cases} y' = ay^2 + by + c \\ \dfrac{\partial a}{\partial y} = 0, \quad \dfrac{\partial b}{\partial y} = 0, \quad \dfrac{\partial c}{\partial y} = 0. \end{cases}$$

Theorem 4.1. *The Lie algebra $\mathcal{A}$ of Lie group G_R is generated by the operators*

$$\begin{aligned} X = A(x)\partial_x + \big(B(x)y^2 - C(x)y + D(x)\big)\partial_y + \\ +\big((C(x) - A'(x))a + B(x)b + B'(x)\big)\partial_a + \\ +\big(2B(x)c - 2D(x)a - A'(x)b - C'(x)\big)\partial_b + \\ +\big(D'(x) - D(x)b - (A'(x) + C(x))c\big)\partial_c, \end{aligned}$$

where $A(x), B(x), C(x), D(x)$ are arbitrary smooth functions.

By integrating Lie equations for each operator we obtain the group transformations for $A(x)$

$$X_A = A\partial_x - A'a\,\partial_a - A'b\,\partial_b - A'c\,\partial_c$$

Equivalence group transformations are in the form

$$\begin{cases} \widetilde{x} = \widetilde{A}^{-1}(\varepsilon + \widetilde{A}(x)) = \alpha(x) \\ \widetilde{y} = y, \end{cases}$$

where $\widetilde{A}(s) = \displaystyle\int \frac{ds}{A(s)}$.

for $B(x)$

$$X_B = By^2\partial_y + (Bb + B')\partial_a + 2Bc\,\partial_b$$

$$\begin{cases} \widetilde{x} = x \\ \widetilde{y} = \dfrac{y}{1 - \varepsilon B(x)y} \end{cases}$$

for $C(x)$

$$X_C = -Cy\partial_y + Ca\,\partial_a - C'\partial_b - Cc\,\partial_c$$

$$\begin{cases} \widetilde{x} = x \\ \widetilde{y} = y \cdot e^{-\varepsilon \cdot C(x)} \end{cases}$$

for $D(x)$

$$X_D = D\partial_y - 2Da\,\partial_b + (D' - Db)\partial_c$$

$$\begin{cases} \widetilde{x} = x \\ \widetilde{y} = y + \varepsilon \cdot D(x) \end{cases}$$

4.1 Basic invariants of equivalence transformations of Riccati equations

Invariants of whole group G_R do not exist, therefore we find basic invariants for some subgroups of G_R.
In particular for subgroups generated by operators X_A, X_B, X_C, X_D.

Basis of subalgebra	Basic differential invariants		Invariant different.
	with y	with a, b, c	
X_A	y	$\frac{b}{a},\ \frac{c}{a}$	$\frac{1}{a}D_x$
X_B	$\frac{b}{2c} + \frac{1}{y}$	$c,\ 2ac - \frac{1}{2}b^2 - b' + b\frac{c'}{c}$	D_x
X_C	ay	$ac,\ \frac{a'}{a} + b$	D_x
X_D	$\frac{b}{a} + 2y$	$a,\ 2ac - \frac{1}{2}b^2 + b' - b\frac{a'}{a}$	D_x
(X_A, X_B)	$\frac{b}{2c} + \frac{1}{y}$	$\frac{2a}{c} + \frac{b^2}{2c^2} - \frac{b'c - bc'}{c^3}$	$\frac{1}{c}D_x$
(X_A, X_C)	$\frac{ay^2}{c}$	$\frac{1}{\sqrt{\lvert ac\rvert}}\left(\frac{a'}{a} - \frac{c'}{c} + 2b\right)$	$\frac{1}{\sqrt{\lvert ac\rvert}}D_x$
(X_A, X_D)	$\frac{b}{a} + 2y$	$2\frac{c}{a} - \frac{b^2}{2a^2} + \frac{b'a - ba'}{a^3}$	$\frac{1}{a}D_x$
(X_B, X_C)	$\frac{2c}{y} + b - \frac{c'}{c}$	$b^2 - 4ac + 2b' - 2\frac{c''}{c} + 3\frac{c'^2}{c^2} - 2\frac{bc'}{c}$	D_x
(X_C, X_D)	$2ay + b + \frac{a'}{a}$	$b^2 - 4ac - 2b' - 2\frac{a''}{a} + 3\frac{a'^2}{a^2} + 2\frac{ba'}{a}$	D_x

4.2 Applications of differential invariants of subgroups of G_R

We construct criteria of equivalence (necessary condition).

Theorem 4.2. *(I.Tsyfra) Let* $\omega_1, \omega_2, ..., \omega_n,$ *where* $\omega_i = \omega_i(x, a, b, a', b', ..., a^{(k)}, b^{(k)})$ *for* $k \in \mathbb{N}$, *are differential invariants of Lie group generated by operator* X.
Assume that Riccati equation with $a = a_1(x), b = b_1(x), c = c_1(x)$ *can be transformed by some change of variables from* G_R *in Riccati equation with* $a = a_2(x), b = b_2(x), c = c_2(x)$ *and let*

$$H(\omega_1, \omega_2, ..., \omega_n)\Big|_{\substack{a=a_1(x)\\ b=b_1(x)\\ c=c_1(x)}} = \lambda = const.$$

for some smooth function H.

Then

$$H(\omega_1, \omega_2, ..., \omega_n)\Big|_{\substack{a=a_2(x)\\ b=b_2(x)\\ c=c_2(x)}} = \lambda$$

Example 4.1. *We investigate the equivalence of the following equations* $y' = y^2 - 2x^2y + x^4 + 2x + 4, \quad z' = 3t^2z^2 - 6t^8z + 3t^{14} + 6t^5 + 12t^2$

We test invariants of successive subgroups of equivalence transformations, starting from transformations with one arbitrary function. Because a, c, ac *change, then it is not transformation, generated by* X_B, X_C, X_D. *Further we study invariants of* X_A *and corresponding function* H *for the first equation.*

$$\omega_{11} = \frac{b_1}{a_1} = -2x^2, \quad \omega_{12} = \frac{c_1}{a_1} = x^4 + 2x + 4.$$

They depend on x, *hence we calculate function* H. *We have*

$$x = \sqrt{\left|\frac{\omega_{11}}{-2}\right|} \quad and \quad \omega_{12} = \frac{\omega_{11}^2}{4} + 2\sqrt{\left|\frac{\omega_{11}}{-2}\right|} + 4.$$

The corresponding function H *is*

$$H(\omega_{11}, \omega_{12}) = \omega_{12} - \frac{\omega_{11}^2}{4} - 2\sqrt{\left|\frac{\omega_{11}}{-2}\right|} = 4.$$

Now the second equation yields

$$\omega_{21} = \frac{b_2}{a_2} = -2t^6, \quad \omega_{22} = \frac{c_2}{a_2} = t^{12} + 2t^3 + 4.$$

We calculate

$$H(\omega_{21}, \omega_{22}) = \omega_{22} - \frac{\omega_{21}^2}{4} - 2\sqrt{\left|\frac{\omega_{21}}{-2}\right|} = t^{12} + 2t^3 + 4 - t^{12} - 2\sqrt{t^6} = 4.$$

We obtain equality for the basic invariants, then using Lemma p.5 we state that these equations are equivalent.
Moreover we can find the change of variables that carries one equation to another using invariant depending on x.

$$x = \sqrt{\left|\frac{\omega_{11}}{-2}\right|} = \sqrt{\left|\frac{\omega_{21}}{-2}\right|} = t^3.$$

Example 4.2. *Let consider two following equations*

$$y' = -y^2 + x^2 + 1, \quad z' = -\frac{z^2}{t\sqrt{\ln t}} + \frac{z}{2t\ln t} + \frac{\sqrt{\ln t}}{t}(1 + \ln^2 t)$$

As above, we check all basic invariants and corresponding function H. We obtain that for (X_A, X_C) they are the same. Indeed

$$\omega_{11} = \frac{1}{\sqrt{|a_1 c_1|}}\left(\frac{a_1'}{a_1} - \frac{c_1'}{c_1} + 2b_1\right) = \frac{-2x}{(x^2+1)^{3/2}},$$

$$\omega_{12} = Q(\omega_{11}) = \frac{1}{\sqrt{|a_1 c_1|}} D_x(\omega_{11}) = 2\frac{2x^2-1}{(x^2+1)^3},$$

$$H(\omega_{11}, \omega_{12}) = \frac{2\omega_{11}^2}{\omega_{11}^2 - \omega_{12}} - 4\sqrt[3]{\frac{2}{\omega_{11}^2 - \omega_{12}}} = -4,$$

$$\omega_{21} = \frac{1}{\sqrt{|a_2 c_2|}}\left(\frac{a_2'}{a_2} - \frac{c_2'}{c_2} + 2b_2\right) = \frac{-2\ln t}{(1+\ln^2 t)^{3/2}},$$

$$\omega_{22} = Q(\omega_{21}) = \frac{1}{\sqrt{|a_2 c_2|}} D_x(\omega_{21}) = \frac{4\ln^2 t - 2}{(1+\ln^2 t)^3},$$

$$H(\omega_{21}, \omega_{22}) = \frac{2\omega_{21}^2}{\omega_{21}^2 - \omega_{22}} - 4\sqrt[3]{\frac{2}{\omega_{21}^2 - \omega_{22}}} = -4.$$

Further we find the equivalence transformations, using invariants with x, y

$$x^2 = \sqrt[3]{\frac{2}{\omega_{11}^2 - \omega_{12}}} - 1 = \sqrt[3]{\frac{2}{\omega_{21}^2 - \omega_{22}}} = \ln^2 t,$$

$$\frac{a_1 y^2}{c_1} = \frac{a_2 z^2}{c_2} \quad \Longrightarrow \quad y = \frac{z}{\sqrt{\ln t}}.$$

The change of variables $\begin{cases} x = \ln t \\ y = \dfrac{z}{\sqrt{\ln t}} \end{cases}$ *is the sought equivalence transformation.*

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