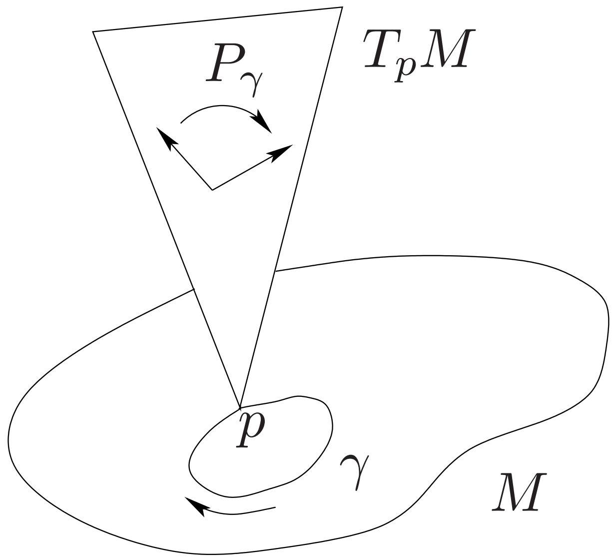
Nonintegrable geometries with parallel characteristic torsion

Thomas Friedrich (Humboldt-Universität zu Berlin)

Braniewo, March 2012

Survey article: I. Agricola, The Srni lectures on non-integrable geometries with torsion, Arch. Math. 42 (2006), 5-84.

Holonomy group of the Levi-Civita connection



- ∇ metric, γ: closed path through p
- $P_\gamma : T_pM \to T_pM$ parallel transport
  (∇ metric $\Rightarrow$ P_γ isometry)
- $C_0(p)$: null-homotopic γ's
  $$\mathrm{Hol}_0(M;\nabla) := \{P_\gamma \mid \gamma \in C_0(p)\} \subset \mathrm{SO}(n)$$

Theorem (Berger / Simons, ≥ 1955). Let M^n be an irreducible, non-symmetric Riemannian manifold. Then the holonomy $\mathrm{Hol}_0(M;\nabla^g)$ group of the LC connection ∇^g is either $\mathrm{SO}(n)$ (generic case) or

$$\mathrm{Sp}(n)\mathrm{Sp}(1)\ \mathrm{qK},\ \ \mathrm{U}(n)\ \mathrm{K}, \underbrace{\mathrm{SU}(n)\ \mathrm{CY},\ \ \mathrm{Sp}(n)\ \mathrm{hK},\ \ G_2,\ \ \mathrm{Spin}(7)}_{\mathrm{Ric}=0}.$$

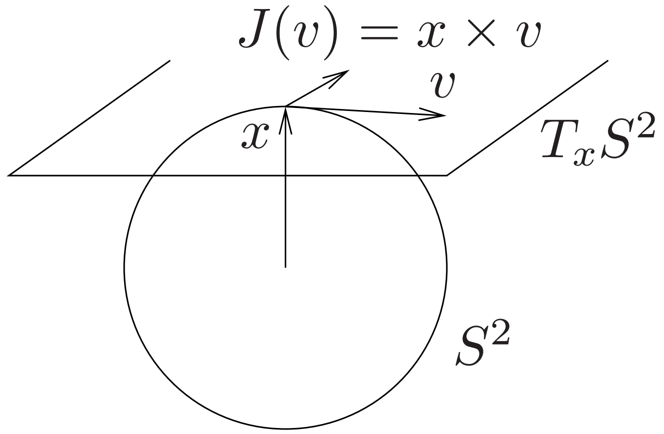
Examples of non-integrable geometries

Example 1:

• (S^6, g_{can}): $S^6 \subset \mathbf{R}^7$ has an almost complex structure J $(J^2 = -\mathrm{id})$ inherited from the "cross product" on $\mathbf{R}^7$.

• J is not integrable, $\nabla^g J \neq 0$

• Problem (Hopf): Does S^6 admit an (integrable) complex structure ?



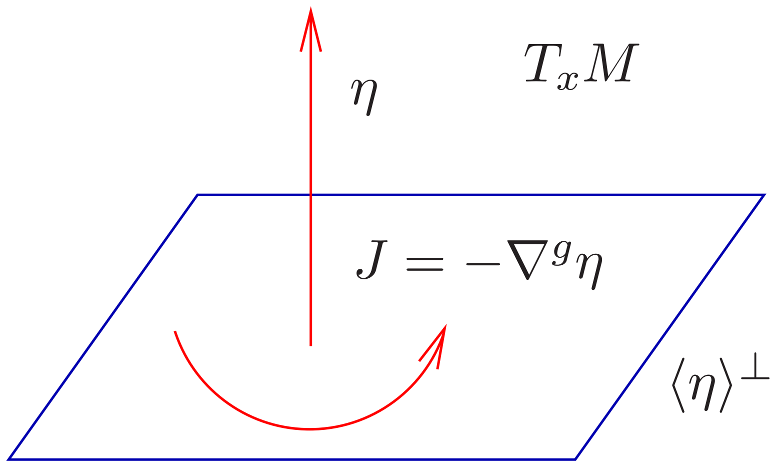
J is an example of a nearly Kähler structure: $\nabla^g_X J(X) = 0$ ($\Rightarrow$ Einstein)

Example 2: (M, J) compact complex mnfd, $b_1(M)$ odd $(S^3 \times S^1 \dots)$

$\Rightarrow$ (M, J) cannot carry a Kähler metric (Hodge theory), but it has many (almost) Hermitian metrics.

Example 3: Contact metric geometries

- (M^{2n+1}, g, η) contact mnfd, η - 1-form ($\cong$ vector field)
- On $\langle\eta\rangle^\perp$ exists an almost complex structure J which is compatible with the metric g



- Contact condition: $\eta \wedge (d\eta)^n \neq 0 \Rightarrow \nabla^g\eta \neq 0$, i. e. contact structures are never integrable !

Example 4: Mnfds with G_2- or $\mathrm{Spin}(7)$-structure $(\mathrm{dim} = 7, 8)$.

Example 5: Homogeneous reductive non-symmetric spaces G/H.

Type II string equations

A. Strominger, 1986: $(M^n, g, \mathrm{T}, \Psi, \Phi)$ a Riemannian manifold,

$$\mathrm{T} \text{ -- a 3-form}\,,\ \ \Psi \text{ -- a spinor field}\,,\ \ \Phi \text{ -- a function}\ .$$

- Bosonic equations: $\delta(e^{-2\Phi}\mathrm{T}) = 0,\ R^g_{ij} - \frac{1}{4}\mathrm{T}_{imn}\mathrm{T}_{jmn} + 2 \cdot \nabla^g_i \partial_j \Phi = 0$
- Fermionic equations: $\big(\nabla^g_X + \frac{1}{4}\, X \lrcorner\, \mathrm{T}\big) \cdot \Psi = 0,\ \big(2 \cdot d\Phi\ - \mathrm{T}\big) \cdot \Psi = 0$

Geometric interpretation

The 3-form T is the torsion form of some metric connection ∇ with totally skew symmetric torsion,

$$\mathrm{T}(X, Y, Z)\ =\ g(\nabla_X Y\ -\ \nabla_Y X\ - [X, Y]\,,\ Z)$$

Then we obtain

$$R^g_{ij} \;-\; \frac{1}{4}\mathrm{T}_{imn}\mathrm{T}_{jmn} \;=\; \mathrm{Ric}^{\nabla}_{ij}$$

and the equations now read as (a, b, μ are constants)

- Fermionic equations:

$$\nabla\Psi \;=\; 0\,, \quad \mathrm{T}\cdot\Psi \;=\; b\cdot d\Phi\cdot\Psi + \mu\cdot\Psi\,.$$

- Conservation law: $\delta(\mathrm{T}) \;=\; a\cdot\big(d\Phi\,\lrcorner\,\mathrm{T}\big)\,.$

Integrable geometric structures $\mathrm{T} = 0$, $\nabla = \nabla^g$

- Calabi-Yau manifolds in dimension 6,
- parallel G_2-structures in dimension 7,
- parallel $\mathrm{Spin}(7)$-structures in dimension 8.

Basic Idea:

Non-integrable G-structures of special geometric type yield solutions of the equations for type II string theory.

Results:

- For contact geometries (in particular $n = 5$).
- For almost complex manifolds (in partic. $n = 6$).
- In dimension $n = 7$ for the subgroup G_2 and
- in dimension $n = 8$ for $\mathrm{Spin}(7)$.

Types of Metric Connections

$(M^n\,,\,g\,,\,\nabla)$ - a metric connection, $\nabla\,g\,=\,0$. Compare ∇ with the Levi-Civita connection

$$\nabla_X Y \;=\; \nabla^g_X Y + A(X,Y)\;.$$

Then $g(A(X,Y)\,,\,Z) \;=\; -\,g(A(X,Z)\,,\,Y),\ A \in \mathbf{R}^n \otimes \Lambda^2(\mathbf{R}^n)\;.$

Decompose under the action of the orthogonal group (E. Cartan 1922 - 1925):

$$\mathbf{R}^n \otimes \Lambda^2(\mathbf{R}^n) \;=\; \mathbf{R}^n \,\oplus\, \Lambda^3(\mathbf{R}^n) \,\oplus\, \mathcal{T}\;.$$

A connection is of type Λ^3 if and only if its torsion is totally skew-symmetric. In this case we have

$$\nabla_X Y \;=\; \nabla^g_X Y + \frac{1}{2}\cdot \mathrm{T}(X,Y,-)\;.$$

The characteristic connection of a geometric structure

- Let $\mathrm{G} \subset \mathrm{SO}(n)$ be a compact subgroup.

- Decompose the Lie algebra $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$.

- Define $\Theta : \Lambda^3(\mathbf{R}^n) \longrightarrow \mathbf{R}^n \otimes \mathfrak{m}$,

$$\Theta(\mathrm{T}^3) \ := \ \sum_{i=1}^{n} e_i \otimes \mathrm{pr}_{\mathfrak{m}}(e_i \lrcorner \mathrm{T}^3) \ .$$

Consider an oriented Riemannian manifold (M^n, g) and denote by $\mathcal{F}(M^n)$ its frame bundle. A geometric structure is G-principal sub-bundle of the frame bundle, $\mathcal{R} \subset \mathcal{F}(M^n)$.

- The Levi-Civita connection $\mathrm{Z} : \mathrm{T}\mathcal{F} \longrightarrow \mathfrak{so}(n)$.

- We split the restriction

$$\mathrm{Z}_{|\mathcal{R}} = \mathrm{Z}^* \oplus \Gamma : \mathrm{T}\mathcal{R} \to \mathfrak{g} \oplus \mathfrak{m} .$$

- Then Γ is a 1-form defined on the manifold M^n with values in $\mathfrak{m}$, $\Gamma \in \mathbf{R}^n \otimes \mathfrak{m}$ (intrinsic torsion).

The types of geometric structures $\mathcal{R} \subset \mathcal{F}(M^n)$ correspond – via Γ – to the irreducible components of the G-representation $\mathbf{R}^n \otimes \mathfrak{m}$.

Proposition: A geometric structure $\mathcal{R} \subset \mathcal{F}(M^n)$ admits a connection with totally skew symmetric torsion T if and only if Γ belongs to the image of

$$\Theta : \Lambda^3(M^n) \to \mathrm{T}^*(M^n) \otimes \mathfrak{m} .$$

Definition: A G-connection with totally skew symmetric torsion of a geometric structure $\mathcal{R} \subset \mathcal{F}(M^n)$ is called a characteristic connection.

Torsion forms and special geometries

- Consider $\mathrm{G}_2 \subset \mathrm{SO}(7)$.
- Decompose $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}^7 = \mathfrak{g}_2 \oplus \mathbf{R}^7$.
- Then $\mathbf{R}^7 \otimes \mathfrak{m}^7 = \mathbf{R}^1 \oplus \mathrm{S}_0(\mathbf{R}^7) \oplus \mathfrak{g}_2 \oplus \mathbf{R}^7$.
- Consequence: Four basic classes of G_2-structures.
- $\Lambda^3(\mathbf{R}^7) = \mathbf{R}^1 \oplus \mathbf{R}^7 \oplus \mathrm{S}_0(\mathbf{R}^7)$.

Theorem: A 7-dimensional Riemannian manifold (M^7, g, ω) with a fixed G_2-structure admits a characteristic connection if and only if $\delta^g(\omega) = -(\beta \lrcorner \omega)$. In this case, the connection is unique and its torsion form is given by

$$\mathrm{T} = - * d\omega - \frac{1}{6} \cdot (d\omega, *\omega) \cdot \omega + *(\beta \wedge \omega) .$$

- Consider $\mathrm{Spin}(7) \subset \mathrm{SO}(8) = \mathrm{SO}(\Delta_7)$.
- Decompose $\mathfrak{so}(8) = \mathfrak{spin}(7) \oplus \mathfrak{m}^7 = \mathfrak{spin}(7) \oplus \mathbf{R}^7$.
- Then $\mathbf{R}^8 \otimes \mathfrak{m}^7 = \Delta_7 \otimes \mathbf{R}^7$ splits into 2 irreducible components (Clifford multiplication).
- Consequence: Two basic classes of $\mathrm{Spin}(7)$-structures.
- $\Lambda^3(\mathbf{R}^8) = \mathbf{R}^8 \otimes \mathfrak{m}^7$.

Theorem: Any 8-dimensional Riemannian manifold equipped with a $\mathrm{Spin}(7)$-structure admits a unique characteristic connection.

A formula for the characteristic torsion is known.

Theorem: An almost metric contact manifold $(M^{2k+1}, g, \xi, \eta, \phi)$ admits a connection ∇ with skew-symmetric torsion and preserving the structure if and only if ξ is a Killing vector field and the tensor $N(X,Y,Z) := g(N(X,Y),Z)$ is totally skew-symmetric. In this case, the connection is unique, and its torsion form is given by the formula

$$\mathrm{T} \;=\; \eta \wedge d\eta + d^{\phi}F + N - \eta \wedge \xi \lrcorner N \,.$$

Theorem: An almost complex manifold $(M^{2k}, g, \mathcal{J})$ admits a connection with skew-symmetric torsion if and only if the Nijenhuis tensor $N(X,Y,Z) \;:=\; g\big(N(X,Y),\, Z\big)$ is skew-symmetric. In this case, the connection is unique, and its torsion form is given by the formula

$$\mathrm{T}(X,Y,Z) \;=\; -\,d\Omega(\mathcal{J}X, \mathcal{J}Y, \mathcal{J}Z) + N(X,Y,Z)\,.$$

Folklore: Any reductive Riemannian manifold G/H admits a 1-parameter family of invariant connections with skew-symmetric torsion.

Geometric structures with parallel characteristic torsion

- Naturally reductive space $(\mathrm{K}/\mathrm{G}, \nabla^c, \mathrm{T}^c)$:

$$\nabla^c \mathrm{T}^c = 0 \, , \; \nabla^c \mathrm{R}^c = 0 \, .$$

A larger category:

$(M^n, g, \mathcal{R}, \nabla^c)$ – Riemannian manifolds with a geometric structure admitting a characteristic connection such that $\nabla^c \mathrm{T}^c = 0$.

- The condition $\nabla^c \mathrm{T}^c = 0$ implies the conservation law of string theory, $\delta(\mathrm{T}^c) = 0$.

First example:

$(M^{2k+1}, g, \eta, \xi, \varphi)$ - Sasakian manifold. It admits a characteristic connection and

$$\mathrm{T}^c = \eta \wedge d\eta \, , \quad \nabla^c \mathrm{T}^c = 0 \, .$$

Second example:

Any nearly parallel G_2-manifold (M^7, g, ω^3) satisfies this condition.

Theorem: (Matsumoto/Takamatsu/ Gray/Kirichenko, 1970 - 1978)

Any nearly Kähler manifold admits a characteristic connection with $\nabla^c \mathrm{T}^c = 0$.

- In dimension $n = 6$ this result implies:

1. Any nearly Kähler M^6 is Einstein.

2. Any nearly Kähler M^6 is spin.

3. The first Chern class $c_1(M^6) = 0$ vanishes.

Problem: Describe all almost hermitian manifolds manifolds $(n = 6)$ or G_2-manifolds $(n = 7)$ admitting a characteristic connection ∇^c such that $\nabla^c \mathrm{T}^c = 0$.

Metric connections with parallel torsion

- T – a 3-form on a Riemannian manifold (M^n, g).

- The connection ∇ :

$$\nabla_X Y \ := \ \nabla^g_X Y + \frac{1}{2}\mathrm{T}(X, Y, *)$$

- $\nabla \mathrm{T} = 0$ implies $\delta(\mathrm{T}) = 0$ and

$$d\mathrm{T} \ = \ \sum_{i=1}^{n} (e_i \lrcorner \mathrm{T}) \wedge (e_i \lrcorner \mathrm{T}) \ .$$

- If Ψ is a ∇-parallel spinor field, then

$$2\,\mathrm{Ric}^{\nabla}(X) \cdot \Psi \ = \ \big(X \lrcorner d\mathrm{T}\big) \cdot \Psi \, .$$

$$\nabla \mathrm{Ric}^{\nabla} \;=\; 0\,,\;\; \mathrm{div}\big(\mathrm{Ric}^{\nabla}\big) \;=\; 0\,.$$

$$\mathrm{T}^2 \cdot \Psi \;=\; \frac{1}{4}(2\,\mathrm{Scal}^g \;+\; ||\mathrm{T}||^2) \cdot \Psi\;.$$

- Solutions of the equations for the common sector of type II superstring theory,

$$\nabla \Psi \;=\; 0\,,\;\; \mathrm{T}\cdot\Psi \;=\; a\cdot\Psi\,,\quad \delta(\mathrm{T}) \;=\; 0\;,\;\; \nabla \mathrm{Ric}^{\nabla} \;=\; 0$$

- $$(D^{1/3}_{\mathrm{T}})^2 \;=\; \Delta_{\mathrm{T}} + \frac{1}{4}\mathrm{Scal}^g + \frac{1}{8}||\mathrm{T}||^2 - \frac{1}{4}\mathrm{T}^2\;.$$

  In particular, the endomorphism T commutes with the square of the Dirac operator $(D^{1/3}_{\mathrm{T}})^2$.

- Eigenvalue estimates for $(D^{1/3}_{\mathrm{T}})^2$ – see I. Agricola at el 2008-2012.

Results and Examples:

- $n = 6$: B. Alexandrov, Th. Friedrich, N. Schoemann, J. Geom. Phys. 53 (2005), 1-30 and J. Geom. Phys. 57 (2007), 2187-2212.

- $n = 7$: Th. Friedrich, Diff. Geom. Appl. 25 (2007), 632-648.

- $n = 8$: C. Puhle, Comm. Math. Phys. 291 (2009), 303-320.

Cocalibrated G_2-manifolds

Definition: A G_2-manifold (M^7, g, φ) is called cocalibrated if the 3-form φ satisfies the differential equation

$$d * \varphi \;=\; 0.$$

- There exists a unique connection ∇^{c} preserving the G_2-structure with totally skew-symmetric torsion, the characteristic connection (Friedrich/Ivanov 2002),

$$\mathrm{T}^{\mathrm{c}} \;=\; \frac{1}{6}(d\varphi\,,\, *\varphi)\cdot\varphi \;-\; *\,d\varphi.$$

- There exists at least one ∇^{c}-parallel spinor field Ψ.

- If $\nabla^{\mathrm{c}}\mathrm{T}^{\mathrm{c}} = 0$, $\mathrm{T}^{\mathrm{c}} \neq 0$ and if (M^7, g, φ) is not nearly parallel ($d\varphi = *\varphi$), then the holonomy algebra $\mathfrak{hol}(\nabla^{\mathrm{c}}) \subset \mathfrak{g}_2$ is a proper subalgebra.

Result: Classification of cocalibrated G_2-manifolds with parallel characteristic torsion and non-abelian holonomy $\mathfrak{hol}(\nabla^c) \neq \mathfrak{g}_2$.

Method: There are 8 non-abelian subalgebra of $\mathfrak{g}_2$ (Dynkin 1952). We compute explicitely the family of admissible torsion forms for any of these algebras. Then we study the corresponding geometry using the formulas for the torsion forms.

The non-abelian subalgebras of $\mathfrak{g}_2$

- $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ is the subalgebra preserving one spinor.
- $\mathfrak{su}(3) \subset \mathfrak{g}_2$ is the subalgebra preserving two spinors.
- $\mathfrak{u}(2) \subset \mathfrak{su}(3) \subset \mathfrak{g}_2$. Two spinors are preserved.
- $\mathfrak{su}(2) \subset \mathfrak{g}_2$ is the subalgebra preserving four spinors.
- $\mathfrak{su}(2)_c \subset \mathfrak{g}_2$ - the centralizer of the subalgebra $\mathfrak{su}(2) \subset \mathfrak{g}_2$.
- $\mathbb{R}^1 \oplus \mathfrak{su}(2)_c \subset \mathfrak{g}_2$. One spinor is preserved.
- $\mathfrak{su}(2) \oplus \mathfrak{su}(2)_c \subset \mathfrak{g}_2$. One spinor is preserved.
- $\mathfrak{so}(3) \subset \mathfrak{su}(3) \subset \mathfrak{g}_2$. Two spinors are preserved.

- $\mathfrak{so}(3)_{ir} \subset \mathfrak{g}_2$, the irreducible 7-dimensional representation of $\mathfrak{so}(3)$. One spinor is preserved.

G_2-manifolds with parallel torsion and $\mathfrak{hol}(\nabla^c) = \mathfrak{so}_{ir}(3)$

Theorem: A complete, simply-connected and cocalibrated G_2-manifold with parallel characteristic torsion and $\mathfrak{hol}(\nabla^c) = \mathfrak{so}_{ir}(3)$ is isometric to the nearly parallel G_2-manifold $\mathrm{SO}(5)/\mathrm{SO}_{ir}(3)$.

G_2-manifolds with parallel torsion and $\mathfrak{hol}(\nabla^c) = \mathfrak{su}_c(2)$

Theorem: There exists a unique simply-connected, complete, cocalibrated G_2-manifold with

$$\nabla^c \mathrm{T}^c = 0, \quad \mathfrak{hol}(\nabla^c) = \mathfrak{su}_c(2).$$

The manifold is homogeneous naturally reductive.

Remark:
$M^7 = \mathrm{G}/\mathrm{SU}_c(2)$ is a homogeneous space with an 10-dimensional automorphism group G. Its Lie algebra $\mathfrak{g}$ contains a 7-dimensional nilpotent radical $\mathfrak{r}$ and $\mathfrak{g}/\mathfrak{r} = \mathfrak{su}_c(2)$ is isomorphic to the holonomy algebra.

Theorem: All simply-connected, complete, cocalibrated G_2-manifolds with parallel characteristic torsion and holonomy $\mathbb{R}^1 \oplus \mathfrak{su}_c(2)$ are naturally reductive. Up to a scaling, the family depends on one parameter.

G_2-manifolds with parallel torsion and $\mathfrak{hol}(\nabla^c) = \mathfrak{so}(3)$

- A cocalibrated G_2-manifold with characteristic holonomy $\mathfrak{hol}(\nabla^c) = \mathfrak{so}(3)$ admits two ∇^c-parallel spinor fields Ψ_1, Ψ_2. The torsion form T^c may act on it by the same eigenvalue or by opposite eigenvalues. Consequently, we have to discuss two cases.

Theorem: A simply-connected, complete, cocalibrated G_2-manifold with characteristic holonomy $\mathfrak{hol}(\nabla^c) = \mathfrak{so}(3)$ such that T^c acts with the same eigenvalue on the parallel spinors is isometric to the Stiefel manifold $SO(5)/SO(3)$. The metric is a Riemannian submersion over the Grassmanian manifold $G_{5,2}$.

Theorem: A simply-connected, complete, cocalibrated G_2-manifold with characteristic holonomy $\mathfrak{hol}(\nabla^c) = \mathfrak{so}(3)$ such that T^c acts with opposite eigenvalues on the parallel spinors splits into the Riemannian product $Y^6 \times \mathbf{R}^1$, where Y^6 is an almost Hermitian manifold of Gray-Hervella-type $\mathcal{W}_1 \oplus \mathcal{W}_3$ with characteristic holonomy $\mathfrak{so}(3) \subset \mathfrak{su}(3)$.

G_2-manifolds with parallel torsion and $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(3)$

Theorem: Any cocalibrated G_2-manifold such that the characteristic torsion acts on both ∇^c-parallel spinors by the same eigenvalue and

$$\nabla^c \mathrm{T}^c \, = \, 0\,, \quad \mathrm{T}^c \, \neq \, 0\,, \quad \mathfrak{hol}(\nabla^c) \, = \, \mathfrak{su}(3)$$

holds is homothetic to an η-Einstein Sasakian manifold. Its Ricci tensor is given by the formula

$$\mathrm{Ric}^g = \, 10 \cdot g \, - \, 4 \cdot e_7 \otimes e_7 \, .$$

Conversely, a simply-connected η-Einstein Sasakian manifold with Ricci tensor $\mathrm{Ric}^g = 10 \cdot g \, - \, 4 \cdot e_7 \otimes e_7$ admits a cocalibrated G_2-structure with parallel characteristic torsion and characteristic holonomy contained in $\mathfrak{su}(3)$.

Theorem: A complete, simply-connected cocalibrated G_2-manifold such that the characteristic torsion acts on ∇^c-parallel spinors by opposite eigenvalues and

$$\nabla^c \mathrm{T}^c = 0, \quad \mathrm{T}^c \neq 0, \quad \mathfrak{hol}(\nabla^c) = \mathfrak{su}(3)$$

holds is isometric the the product of a nearly Kähler 6-manifold by $\mathbf{R}$. Conversely, any such product admits a cocalibrated G_2-structure with parallel torsion and holonomy contained in $\mathfrak{su}(3)$.

G_2-manifolds with parallel torsion and $\mathfrak{hol}(\nabla^c) = \mathfrak{u}(2)$

Theorem: Let (M^7, g, φ) be a complete, cocalibrated G_2-manifold such that

$$\nabla^c \mathrm{T}^c \;=\; 0, \quad \mathfrak{hol}(\nabla^c) = \mathfrak{u}(2)$$

and suppose that T^c acts with opposite eigenvalues $\pm 7\,c \neq 0$ on the ∇^c-parallel spinors $\Psi_1\,,\,\Psi_2$. Moreover, suppose that M^7 is regular. Then M^7 is a principal S^1-bundle and a Riemannian submersion over the projective space $\mathbb{CP}^3$ or the flag manifold $\mathbb{F}(1,2)$ equipped with their standard nearly Kähler structure coming from the twistor construction. The Chern class of the fibration $\pi : M^7 \longrightarrow \mathbb{CP}^3\,, \mathbb{F}(1,2)$ is proportional to the Kähler form. Conversely, any of these fibrations admits a G_2-structure with parallel characteristic torsion and characteristic holonomy contained in $\mathfrak{u}(2)$.

Theorem: Let (M^7, g, φ) be a complete, cocalibrated G_2-manifold such that

$$\nabla^c \mathrm{T}^c \ = \ 0, \quad \mathfrak{hol}(\nabla^c) = \mathfrak{u}(2)$$

and suppose that T^c acts with eigenvalue $-\,7\,c \neq 0$ on the ∇^c-parallel spinors Ψ_1, Ψ_2. Moreover, suppose that M^7 is regular. Then M^7 is a principal S^1-bundle and a Riemannian submersion over a Kähler manifold $\tilde{X}^6$. This manifold has the following properties:

1. The universal covering of $\tilde{X}^6$ splits into a 4-dimensional Kähler-Einstein manifold and a 2-dimensional surface with constant curvature.

2. The scalar curvature $\tilde{S} = \tilde{S}_1 + \tilde{S}_2 > 0$ is positive.

3. The Kähler forms $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are globally defined on $\tilde{X}^6$.

The bundle $\pi : M^7 \longrightarrow \tilde{X}^6$ is defined by a connection form. Its curvature is proportional to the Ricci form of $\tilde{X}^6$. Finally, the flat bundle $\Lambda^3_2(\tilde{X}^6) \otimes G_k$ admits a parallel section. Conversely, any S^1-bundle resulting from this construction admits a cocalibrated G_2-structure such that the characteristic torsion is parallel and the characteristic holonomy is contained in $\mathfrak{u}(2)$.

Example: Let $\tilde{Y}_1$ be a simply-connected Kähler-Einstein manifold with negative scalar curvature $\tilde{S}_1 = -1$, for example a hypersurface of degree $d \geq 5$ in $\mathbb{CP}^3$. For the second factor we choose the round sphere normalized by the condition $\tilde{S}_2 = +2$. Then the product $\tilde{X}^6 = \tilde{Y}_1 \times \tilde{Y}_2$ is simply-connected and the S^1-bundle defined by the Ricci form admits a cocalibrated G_2-structure with parallel torsion. Since the product $\tilde{X}^6 = \tilde{Y}_1 \times \tilde{Y}_2$ is simply-connected, the flat bundle $\Lambda^3_2(\tilde{X}^6) \otimes G_1$ admits a parallel section σ.

Theorem: Let (M^7, g, φ) be a complete G_2-manifold of pure type $\mathcal{W}_3$ such that

$$\nabla^c \mathrm{T}^c \;=\; 0, \quad \mathrm{T}^c \;\neq 0, \quad \mathfrak{hol}(\nabla^c) = \mathfrak{u}(2).$$

Moreover, suppose that M^7 is regular. Then M^7 is a principal S^1-bundle and a Riemannian submersion over a Ricci-flat Kähler manifold $\tilde{X}^6$. This manifold has the following properties:

1. The universal covering of $\tilde{X}^6$ splits into a 4-dimensional Ricci-flat Kähler manifold and the 2-dimensional flat space $\mathbf{R}^2$.

2. The Kähler forms $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are globally defined on $\tilde{X}^6$.

3. There exists a parallel form $\Sigma \in \Lambda^3_2(\tilde{X}^6)$.

The bundle $\pi : M^7 \longrightarrow \tilde{X}^6$ is defined by a connection form. Its curvature is proportional to the form

$$\tilde{\Omega}_1 \;-\; 2\,\tilde{\Omega}_2.$$

Conversely, any S^1-bundle resulting from this construction admits a G_2-structure of pure type $\mathcal{W}_3$ such that the characteristic torsion is parallel and the characteristic holonomy is contained in $\mathfrak{u}(2)$.

Example: Consider a $K3$-surface and denote by $\tilde{\Omega}_1$ its Kähler form. Then there exist two parallel forms η_1, η_2 in $\Lambda^2_+(K3)$ being orthogonal to $\tilde{\Omega}_1$. Let e_5 and e_6 be a parallel frame on the torus T^2. The product $\tilde{X}^6 = K3 \times T^2$ satisfies the conditions of the latter Theorem. Indeed, we can construct the following parallel form

$$\Sigma \;=\; \eta_1 \wedge e_5 + \eta_2 \wedge e_6 .$$

Moreover, the cohomology class of $\tilde{\Omega}_1 - 2\tilde{\Omega}_2$ has to be proportional to an integral class. This implies the condition that $\tilde{\Omega}_1/\mathrm{vol}(T^2) \in \mathrm{H}^2(K3;\mathbb{Q})$ is a rational cohomology class.

G_2-manifolds with parallel torsion and $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$

Example: Starting with a 3-Sasakian manifold and rescaling its metric along the three-dimensional bundle spanned by e_5, e_6, e_7, one obtains a family (M^7, g_s, φ_s) of cocalibrated G_2-manifold such that

$$d *_s \varphi_s \;=\; 0,\; \mathrm{T}^c_s \;=\; \left[\frac{2}{s} - 10\right] e^*_{567} + 2s\varphi_s,\;\; \nabla^c \mathrm{T}^c_s \;=\; 0.$$

The characteristic connection preserves the splitting of the tangent bundle and, consequently, its holonomy is $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$. If $s = 1/\sqrt{5}$, the structure is nearly parallel (type $\mathcal{W}_1$). Since $(\mathrm{T}^c_s, \varphi_s) = 4s + 2/s > 0$, these structures are never of pure type $\mathcal{W}_3$.

Remark: A G_2-structure with characteristic holonomy $\mathfrak{su}(2)\oplus\mathfrak{su}_c(2)$ has not to be naturally reductive. However, the naturally reductive structures are classified.

Theorem: Up to scaling there exists a one-parameter family of naturally reductive homogeneous G_2-manifolds with $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$.

The general case:

- We do not know the complete classification.
- Some necessary conditions can be derived.

Theorem: M^7 admits a 3-dimensional foliation. The leaves are totally geoedesic and have constant, non-negative sectional curvature. If the space of leaves is smooth, then it is an Einstein space.

Final Remark: Any nearly parallel G_2-manifold different from $\mathrm{SO}(5)/\mathrm{SO}_{ir}(3)$ and $N(1,1) = (\mathrm{SU}(3) \times \mathrm{SU}(2))/(\mathrm{S}^1 \times \mathrm{SU}(2))$ has characteristic holonomy $\mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$ or $\mathfrak{g}_2$.

Even the classification of all nearly parallel G_2-manifolds with characteristic holonomy $\mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$ seems to be not known.