On the Selberg class of L-functions

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Automorphic *L*-functions?

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2. (*Analytic continuation*) There exists an integer $m \geq 0$ such that $(s-1)^m F(s)$ is entire of finite order.
3. (*Functional equation*)

$$\Phi(s) = \omega \overline{\Phi(1-\overline{s})},$$

where

$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s) = \gamma(s) F(s)$, and $r \geq 0$, $Q > 0$, $\lambda_j > 0$, $\Re \mu_j \geq 0$, $|\omega| = 1$.

Definition of S, continuation

4. (*Ramanujan hypothesis*) For every $\varepsilon > 0$ we have $a(n) \ll n^{\varepsilon}$.
5. (*Euler product*) For $\sigma > 1$ we have

$$\log F(s) = \sum_n b(n) n^{-s},$$

where $b(n) = 0$ unless $n = p^m$ and $b(n) \ll n^{\theta}$ for some $\theta < 1/2$.

1. Remark: $r = 0$ is possible — the functional equation takes form

$$Q^s F(s) = \omega Q^{1-s} \overline{F}(1-s).$$

2. The *extended Selberg class* $S^{\#}$ consists of $F(s)$ not identically zero satisfying axioms (1), (2) and (3).
3. $\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$ — the *gamma factor* of $F \in S^{\sharp}$.

EXAMPLES

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1. The Riemann zeta function $\zeta(s)$
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3. $\zeta_K(s)$, Dedekind zeta function of an algebraic number field K
4. $L_K(s, \chi)$, Hecke L-function to a primitive Hecke character $\chi(\mathrm{mod}\,\mathfrak{f})$, $\mathfrak{f}$ is an ideal of the ring of integers of K

EXAMPLES, continuation

5. L-function associated with a holomorphic newform of a congruence subgroup of $SL_2(\mathbb{Z})$ (after suitable normalization)
6. Rankin-Selberg convolution of any two normalized holomorphic newforms.
7. $F, G \in S$ implies $FG \in S$ (the same for $S^\#$)
8. If $F \in S$ is entire then the *shift* $F_\theta(s) = F(s + i\theta)$ is in S for every real θ

Conditional examples

1. Artin *L*-functions for irreducible representations of Galois groups (modulo Artin's conjecture: holomorphy is missing).
2. *L*-functions associated with nonholomorphic newforms (Ramanujan hypothesis is missing, exceptional eigenvalue problem).

Conditional examples, continuation

3. Symmetric powers (for normalized holomorphic newforms, say):

$$L(s) = \prod_p \left(1 - \frac{a_p}{p^s}\right)^{-1} \left(1 - \frac{b_p}{p^s}\right)^{-1}$$

r-th symmetric power:

$$L_r(s) = \prod_p \prod_{j=0}^{r} (1 - a_p^j b_p^{r-j} p^{-s})^{-1}$$

(modulo Langlands functoriality conjecture).

4. In general: $GL_n(K)$ automorphic L functions (Ramanujan hypothesis is missing).

Examples, continuation

General examples of *L*-functions from the extended Selberg class: linear combinations of solutions of the same functional equation as for instance the Davenport-Heilbronn *L*-function.

$L(s) = \overline{\lambda} L(s, \chi_1) + \lambda L(s, \overline{\chi}_1)$,

$\chi_1(\mathrm{mod} 5)$ such that $\chi_1(2) = i$,

$\lambda = \frac{1}{2}\left(1 + i\frac{\sqrt{10-2\sqrt{5}}-2}{\sqrt{5}-1}\right)$.

Functional equation

$$\left(\frac{\pi}{5}\right)^{\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s) = \left(\frac{\pi}{5}\right)^{\frac{1-s}{2}} \Gamma\left(\frac{2-s}{2}\right) L(1-s).$$

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⋆ There is no simple way to deduce PNT from Mertens' Theorem.

MOTIVATIONS

NC: if $F \in S$ has degree $d_F > 0$ then

$$\sum_{p \le x} \frac{|a(p)|^2}{p} \sim n_F \log\log x \qquad x \to \infty$$

with some constant $n_F > 0$, and $n_F \le 1$ if $F(s)$ is primitive.

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$\star\star$ THEOREM: NC $\implies$ PNT for S

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Non-Vanishing Conjecture: *For every entire L-function G belonging to the Selberg class we have* $G(1) \neq 0$.

THEOREM Suppose NVC. Then for every entire $F \in S$, every algebraic number field K, and every positive N there exists a non-trivial zero ρ of the Dedekind zeta function $\zeta_K(s)$ of K such that $m(\rho, \zeta_K) > Nm(\rho, F)$.

MOTIVATIONS

COROLLARY Assume NVC, and let for a fixed algebraic number field K with the class number $h_K > 1$,

$$E_K(x) = \sum_{N(a\mathcal{O}_K)\leq x} 1 - \frac{1}{2\pi i}\int_{\mathcal{C}} \zeta_K(s, M_K)\frac{x^s}{s}\, ds$$

denote the remainder term in the asymptotic formula for the number of irreducible elements of $\mathcal{O}_K$ with norms $\leq x$ counted modulo units. Then

$$E_K(x) = \Omega\left(\sqrt{x}\frac{(\log\log x)^{D_K - 1}}{\log x}\right)$$

as $x \to \infty$.

MOTIVATIONS

REMARK Unconditionally we know only

$$E_K(x) = \Omega(\sqrt{x}(\log x)^{-B_K}),$$

where B_K is a positive constant depending on K.

Invariants

Theorem: *For $F \in S^\sharp$ the gamma factor*

$$\gamma_F(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$$

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Main invariants:
1. degree: $d_F := 2\sum_{j=1}^{r} \lambda_j$
2. conductor: $q_F := (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}$

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For $d \geq 0$ let

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$d \notin \mathbb{N} \cup \{0\} \implies S_d^\# = S_d = \emptyset.$

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General Converse Conjecture:

1. DEGREE CONJECTURE:
   $d \notin \mathbb{N} \cup \{0\} \implies S_d^\# = S_d = \emptyset.$
2. $d \in \mathbb{N} \cup \{0\},\ F \in S_d \implies$
   $F - automorphic\ L - function.$

THEOREM *Let $Q > 0$, $\lambda_j > 0$, $\mu_j \in \mathbb{C}$, $\Re(\mu_j) \geq 0$, (j=1,...,r), and $\omega \in \mathbb{C}$, $|\omega| = 1$ be arbitrary. Moreover, put*

$$\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j).$$

Then the functional equation

$$\gamma(s)F(s) = \omega\overline{\gamma(1-\overline{s})F(\overline{s})}$$

has uncountably many linearly independent solutions in the set of generalized Dirichlet series

$$\sum_{n=1}^{\infty} a(n)e^{-\theta_n s} \qquad (\theta_n > 0).$$

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Corollary GCC badly fails in case of the general D-series.

A measure theoretic approach to the degree conjecture

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THEOREM *The sets of degrees of L-functions from S and $S^\sharp$*

$$d(S) = \{d_F : F \in S\}$$

$$d(S^\sharp) = \{d_F : F \in S^\sharp\}$$

are Lebesgue measurable. Moreover, $meas(d(S)) = 0$ or the set $d(S)$ contains a half-line. The same holds for $d(S^\sharp)$.

A measure theoretic approach to the conductor conjecture

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$$\text{CC: } F \in S \implies q_F \in \mathbb{N}$$

A measure theoretic approach to the conductor conjecture

CC: $F \in S \implies q_F \in \mathbb{N}$

THEOREM *The set*

$$q(S) = \{q_F : F \in S\}$$

has Lebesgue measure 0 *or contains a half-line. The same holds for* $q(S^\sharp)$.

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2. UNKNOWN for $d \geq 2$.

The present state of art (GCC)

THEOREM 1 $S^\sharp = \emptyset$ if $0 < d < 1$

Many authors including Richert, Bochner, Conrey-Ghosh, Molteni, J.K.&A.P ...

The present state of art (GCC)

THEOREM 2 $F \in S$, $d_F = 1 \implies F(s) = L(s + i\theta, \chi)$
(χ primitive)
J.K.& A.P. Acta Mathematica 182 (1999), no. 2, 207–241

The present state of art (GCC)

Main tool in the proof: the standard non-linear twist

$$F \in S_d^{\#},\ d > 0,\ \alpha > 0,\ \sigma > 1$$

$$F_d(s, \alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-n^{1/d}\alpha).$$

$$(e(\theta) := \exp(2\pi i\theta))$$

The present state of art (GCC)

THEOREM 3 $S^\sharp = \emptyset$ if $1 < d < 2$

J.K.& A.P. Ann. of Math. (2) 173 (2011), no. 3, 1397–1441

The present state of art (GCC)

Main tool in the proof: multidimensional twists

$$\sum_{n=1}^{\infty} \frac{a_F(n)}{n^s} \exp(-2\pi i \sum_{\nu=0}^{N} \alpha_\nu n^{\kappa_\nu})$$

$$\kappa_0 > \kappa_1 > \ldots > \kappa_N$$

$$\alpha_1, \ldots, \alpha_N \in \mathbb{R}, \qquad \alpha_1 > 0$$

Next step $d = 2$

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Big challenge!

$$T_h(a, n) = k(s)!$$