Automatic Continuity in Lie Groups

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The Main Problem

Suppose that S is a simple centerless Lie group and that Γ is a locally compact and σ-compact group. Suppose that

$$\phi : \Gamma \longrightarrow S$$

is an 'abstract' isomorphism.

What can be said about ϕ and Γ?

Equivalently, how unique is the group topology on S?

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Example 1 — The geometry of SO(3)

Suppose that $S = SO(3)$. The 1-parameter subgroups are centralizers of certain group elements, so they are definable in the 'abstract' group. This group acts regularly on the real 3-dimensional projective space $\mathbb{R}P^3$.

The projective lines/geodesics in $\mathbb{R}P^3$ are the cosets of the 1-parameter subgroups.

By the (topological) Fundamental Theorem of Projective Geometry there is just one compact topology on $SO(3)$ such that the projective lines are closed. Hence the compact topology on S is unique.

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Example 2 — The complex numbers cause problems

Suppose that $S = \mathrm{PSL}(2,\mathbb{C})$. The field of complex numbers $\mathbb{C}$ admits

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non-continuous field automorphisms. Therefore $\mathrm{PSL}(2,\mathbb{C})$ carries $2^{2^{\aleph_0}}$ different Lie group topologies.

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This follows by choosing a transcendence basis of $\mathbb{C}/\mathbb{Q}$.

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In the case of $\mathrm{SO}(3)$ the topology was unique. This is related to the fact that the field $\mathbb{R}$ has a trivial automorphism group.

The early pioneers of Lie theory

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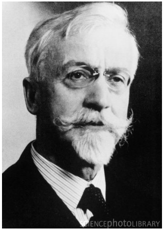

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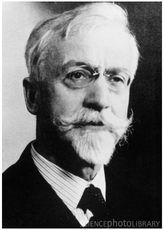

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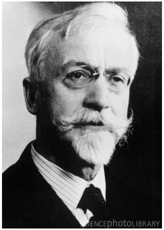

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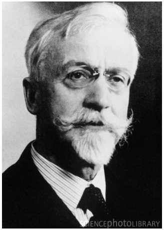

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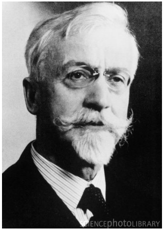

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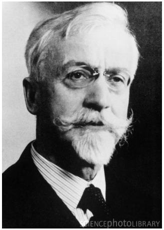

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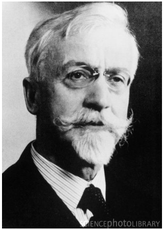

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made already in the first half of the 20st century fundamental contributions to this problem.

Theorem [Cartan, van der Waerden]

Every abstract isomorphism between compact simple Lie groups is continuous. (Comment. Math. Helv. 1930, Math. Z. 1933)

A real Lie group S is *absolutely simple* if the complexification of its Lie algebra $\mathrm{Lie}(S) \otimes_{\mathbb{R}} \mathbb{C}$ is simple.

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Remarks

Borel and Tits generalized Freudenthal's Theorem later and classified the 'abstract' isomorphisms between isotropic absolutely simple algebraic groups over arbitrary fields. They also proved a version of Freudenthal's continuity result for simple Lie groups over local fields, such as $SL(n, \mathbb{Q}_p)$. (Ann. Math. 1973)

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The results by Cartan, van der Waerden, Freudenthal, Borel and Tits all deal with the uniqueness of the Lie topology. The following result generalizes the Theorems of Cartan and van der Waerden in a different direction.

Theorem [Kallman]

A compact simple Lie group S admits only one locally compact and σ-compact group topology. (Adv. Math. 1974)

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Uniqueness Theorem [K]

Suppose that S is a simple Lie group.

(a) If S is absolutely simple, then S admits only one locally compact and σ-compact group topology.

(b) If S is complex, then all locally compact and σ-compact group topologies on S are conjugate under the automorphism group of $\mathbb{C}$. (Adv. Math. 2011)

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We have seen before that such a result is not true for the group $\mathrm{PSL}(2, \mathbb{C})$, which is not absolutely simple.

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The continuity results by Cartan, van der Waerden and Freudenthal are special cases of this result.

The main ingredient in our proof is the following result.

Technical Lemma

Let S be a simple Lie group. Let $\mathcal{L}$ be its topology as a Lie group and let $\mathcal{T}$ be a locally compact and σ-compact group topology on S. Suppose that there exists a subvariety $C \subseteq S$ of positive dimension which is compact in the Lie topology $\mathcal{L}$ and which is σ-compact in the unknown topology $\mathcal{T}$. Then

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Proof of the Technical Lemma.

Put $m = \dim(S)$. Recall that $\mathcal{L}$ denotes the Lie group topology and that C is a compact subvariety of positive dimension.

(i) There exist elements $a_0, \ldots, a_m \in S$ such that $D = a_0 C a_1 C \cdots C a_m$ is an $\mathcal{L}$-neighborhood of 1.

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Proof. *There is a smooth curve in* C *whose tangent vector we may translate to* 1. *Since* S *acts irreducibly on its Lie algebra, we find* m *conjugates of this vector which span the Lie algebra. The claim follows now from the inverse function theorem.*

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Proof. By continuity we have $E \subseteq U$ if the b_k are close enough to 1. A similar argument as in the proof of Step (i), using the inverse function theorem and the irreducibility of the adjoint representation, shows that we can choose the b_k at the same time in such a way that the image is a neighborhood of 1. (This argument is due to van der Waerden.)

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(v) Borel homomorphisms are in this situation continuous and, by the open mapping theorem, open. Therefore $\mathcal{L} = \mathcal{T}$.

We use this to prove Kallman's Theorem. Recall the following:

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Let S be a simple Lie group. Let $\mathcal{L}$ be its topology as a Lie group and let $\mathcal{T}$ be a locally compact and σ-compact group topology on S. Suppose that there exists a subvariety $C \subseteq S$ of positive dimension which is compact in the Lie topology $\mathcal{L}$ and which is σ-compact in the unknown topology $\mathcal{T}$. Then $\mathcal{L} = \mathcal{T}$.

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Now we consider the noncompact case.

Uniqueness Theorem — Part 1

An absolutely simple noncompact Lie group S admits only one locally compact and σ-compact group topology.

Outline of the proof.
We fix an Iwasawa decomposition $S = KAN$, with K maximal compact, A diagonal and N nilpotent. We put $M = \mathrm{Cen}_K(A)$ and $L = \mathrm{Cen}_S(A)$. Then $L = M \times A$.
If M is not abelian (i.e. if S is not quasi-split) we choose $c \in M$ such that the class $C = c^M = c^L$ is compact and of positive dimension.
By construction C is σ-compact in the unknown topology $\mathcal{T}$.
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Part 2 — The case where S is quasi-split.

Suppose that $S = KAN$ is quasi-split and has real rank 1, i.e. that A is 1-dimensional. There are only three such groups:

$$\mathrm{PSL}(2,\mathbb{R}), \quad \mathrm{PSU}_{2,1}(\mathbb{C}), \quad \text{and} \quad \mathrm{PSL}(2,\mathbb{C})$$

In the first two groups, K has a 1-dimensional center. This center can be used to construct the compact subvariety C.
If $\dim(A) > 1$ and if S is absolutely simple and quasi-split, one can find inside of S a copy of one of the first two groups, and this suffices to construct C. Now we apply again the Technical Lemma.

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We use the action on of the group on the projective line $\mathbb{C}\mathrm{P}^1$. The field $(\mathbb{C},+,\cdot,0,1)$ is visible in $\mathbb{C}\mathrm{P}^1$, and all locally compact σ-compact field topologies on $\mathbb{C}$ are conjugate under the group $\mathrm{Aut}(\mathbb{C})$. It follows that the subgroup

$$\mathrm{U} = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\} \cong (\mathbb{C},+)$$

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This follows from Weil's classification of local fields.

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For complex Lie groups of higher rank we use the rank 1 case and the action of S on the Tits building of S.

The following results about totally disconnected groups still have to be written up

Rupert McCallum and I look presently at the case of totally disconnected groups.

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The automorphism groups of two non-isomorphic locally finite regular trees are not isomorphic.

The vertices of the tree correspond to maximal compact subgroups of the automorphism group. Hence the tree is encoded in the group topology.

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Theorem [K-McCallum 2012]

A simple isotropic algebraic group over $\mathbb{Q}_p$, such as $PSL(n, \mathbb{Q}_p)$, admits only one locally compact and σ-compact group topology.

Summary

- Absolutely simple Lie groups over $\mathbb{R}$ and $\mathbb{Q}_p$ are rigid: they admit a unique locally compact and σ-compact group topology.
- In the case of complex simple Lie groups, non-continuous field automorphisms have to be taken into account, but the topology is still unique up to conjugation.
- We conjecture that all simple locally compact and σ-compact groups are rigid in this sense.
- All known proofs in this area use a mixture of advanced structure theory of the groups, functional analysis, and projective geometry/buildings.
- The infinite-dimensional situation, for example the case of Kac-Moody groups, seems to be different.

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