Immersions of surfaces via Spin^c Killing spinors

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Spin Structures: intrinsic point of view

Let (M^n, g) be a compact Riemannian Spin manifold of positive scalar curvature S.

1. Lichnerowicz (1963):

$$\lambda^2 > \frac{1}{4} \inf_M \mathrm{S}.$$

2. Friedrich (1980):

$$\lambda^2 \geqslant \frac{n}{4(n-1)} \inf_M \mathrm{S}.$$

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Spin Structures: extrinsic point of view

- Friedrich (1998) proved that

$$(M^2, g) \hookrightarrow \mathbb{R}^3 \iff M^2 \text{ carries a generalized Killing spinor.}$$

- Hijazi-Montiel-Zhang (2000): on the compact boundary of a Spin manifold (M^n, g),

$$\lambda_1 \geqslant \frac{n-1}{2} \inf_M H,$$

where H denotes the mean curvature of the boundary.

Application of the limiting case: an elementary Spin proof of the Alexandrov theorem.

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The shift from Spin to Spin^c

Seiberg-Witten theory (1994)

$$\Downarrow$$

Donaldson theory (1982)

Applications

1. The calculus of the Yamabe invariant (LeBrun-Gursky 1997).
2. Topological restrictions on 4-dimensional Einstein manifolds (LeBrun 1995).

Spin^c Structures

- Spin, almost complex, complex, Kähler, Sasaki and some CR manifolds have a canonical Spin^c structure.
- Hijazi-Montiel-Urbano (2006): let (M^{2m}, g) be a Kähler Einstein manifold of nonnegative scalar curvature.

The restriction of Kählerian Spin^c Killing spinors to Lagrangian submanifolds

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Geometric and topological informations on these submanifolds.

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Definitions

Let (M^n, g) be an oriented (compact) Riemannian manifold.

- M has a Spin structure $\Longleftrightarrow \omega_2(M) = 0$.
- This condition is very restrictive ($\mathbb{C}P^2$ is not Spin but Spin^c).
- M has a Spin^c structure $\Longleftrightarrow$ there exists a complex line bundle L such that

$$\omega_2(M) = [c_1(L)]_{mod\ 2}.$$

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- The $\mathrm{Spin}^{\mathrm{c}}$ bundle can be written:

$$\Sigma M = \underbrace{\Sigma' M}_{\text{the Spin bundle}} \otimes L^{\frac{1}{2}}.$$

A section $\psi \in \Gamma(\Sigma M)$ is called a spinor field.

- Given a connection on the *auxiliary line bundle L*, we can define a (twisted) connection ∇ on ΣM. The (twisted) Dirac operator is then defined by

$$\begin{aligned} D : \Gamma(\Sigma M) &\longrightarrow \Gamma(\Sigma M) \\ \psi &\longrightarrow D\psi = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j}\psi. \end{aligned}$$

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3-dimensional homogeneous manifolds with 4-dimensional isometry group

- The manifolds $\mathbb{E}(\kappa,\tau)$ are Riemannian fibration over a simply connected 2-dimensional manifold $\mathbb{M}^2(\kappa)$ of curvature κ.
- These manifolds define the geometry of Thurston:

$$\underbrace{\mathbb{S}^2\times\mathbb{R}}_{\tau=0,\kappa=1},\ \underbrace{\mathbb{H}^2\times\mathbb{R}}_{\tau=0,\kappa=-1},\ \underbrace{\mathrm{Nil}_3}_{\tau\neq0,\kappa=0},\ \underbrace{\widetilde{PSL_2(\mathbb{R})}}_{\tau\neq0,\kappa<0}.$$

$$\underbrace{\text{Berger spheres}}_{\tau\neq0,\kappa>0}$$

Restriction to a surface

- The manifolds $\mathbb{E}(\kappa, \tau)$ are Spin^c manifolds carrying a Killing spinor ψ of Killing constant $\frac{\tau}{2}$, i.e., a spinor field ψ satisfying, for all vector fields X,

$$\nabla_X \psi = \frac{\tau}{2} X \cdot \psi.$$

- Using the Spin^c Gauss formula, the restriction of ψ to any oriented surface gives a spinor field ϕ satisfying

$$\nabla_X \varphi = -\frac{1}{2} II(X) \cdot \varphi + i\frac{\tau}{2} X \cdot \overline{\varphi},$$

where $\overline{\varphi} := \varphi_+ - \varphi_-$ is the conjugate of $\varphi = \varphi_+ + \varphi_-$ and II the second fundamental form of the immersion.

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Theorem (with J. Roth, 2011)

The following statements are equivalent:

1. (M^2, g) *is isometrically immersed into* $\mathbb{E}(\kappa, \tau)$ *with second fundamental form* A *and mean curvature* H.
2. *There exists on* M *a spinor field* φ *satisfying*

$$\begin{cases} \nabla_X \varphi = -\frac{1}{2}A(X)\cdot\varphi + i\frac{\tau}{2}X\cdot\overline{\varphi}, \\ i\Omega(e_1, e_2) = -i(\kappa - 4\tau^2) < \varphi, \frac{\overline{\varphi}}{|\varphi|2} > . \end{cases}$$

3. *There exists on* M *a spinor field* φ *satisfying*

$$\begin{cases} D\varphi = H\varphi - i\tau\overline{\varphi}, \\ |\varphi| = constant, \\ i\Omega(e_1, e_2) = -i(\kappa - 4\tau^2) < \varphi, \frac{\overline{\varphi}}{|\varphi|2} > . \end{cases}$$

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A Lawson type correspondence

Theorem (with J. Roth, 2011)

There exists an isometric correspondence between simply connected oriented surfaces minimal in Nil_3 *and simply connected oriented surfaces immersed into* $\mathbb{H}^2 \times \mathbb{R}$ *of mean curvature* $\frac{1}{2}$.

THANK YOU !