Loop Groups with Infinite Dimensional Targets and their Unitary Representations

Karl-Hermann Neeb

Department of Mathematics, FAU Erlangen-Nürnberg

Third W. Killing and K. Weierstraß Colloquium (Braniewo)
28-30 March 2012

Outline

1.1. From compact groups to Hilbert–Lie groups

Definition (Hilbert–Lie algebra)

A Hilbert–Lie algebra is a Lie algebra $\mathfrak{k}$ which is a real Hilbert space whose scalar product is invariant: $([x, y], z) = (x, [y, z])$.
A Lie group K is a Hilbert–Lie group if $\mathbf{L}(K) = \mathfrak{k}$ is a Hilbert–Lie algebra.

Finite dimensional Hilbert–Lie algebras are the compact Lie algebras.

Theorem (Schue, 1960/61; Structure of Hilbert–Lie algebrs)

$\mathfrak{k}$ is an orthogonal direct sum $\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus \widehat{\bigoplus}_{j\in J}\mathfrak{k}_j$, where $\mathfrak{k}_j$ is simple.
If $\mathfrak{k}$ is inf. dim. simple, then $\mathfrak{k} \cong \mathfrak{u}_2(\mathcal{H})$ (skew-herm. Hilbert–Schmidt ops) for a Hilbert space $\mathcal{H}$ over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ with $(x, y) = \mathrm{tr}_{\mathbb{R}}(xy^*) = -\mathrm{tr}_{\mathbb{R}}(xy)$.

Example

$\mathrm{U}_2(\mathcal{H}) = \{g \in \mathrm{U}(\mathcal{H}) \colon \|\mathbf{1} - g\|_2 < \infty\}$ is a Hilbert–Lie group with Lie algebra $\mathbf{L}(\mathrm{U}_2(\mathcal{H})) = \mathfrak{u}_2(\mathcal{H})$. Here $\|X\|_2 = \sqrt{\mathrm{tr}(X^*X)}$.

1.2. Root data of simple Hilbert–Lie algebras

$\mathfrak{k}$ simple Hilbert–Lie algebra
$\mathfrak{t} \subseteq \mathfrak{k}$ maximal abelian (Cartan subalgebra), $\mathfrak{t} \cong \ell^2(J, \mathbb{R})$
$\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \widehat{\bigoplus}_{\alpha \in \Delta} \mathfrak{k}_{\mathbb{C}}^{\alpha}$ (root decomposition), orthogonal direct sum
$\Delta = \Delta(\mathfrak{k}, \mathfrak{t})$ is a locally finite root system.

Theorem (Stumme '99, Classif. of infinite locally finite root systems)

$$A_J = \{\varepsilon_i - \varepsilon_j \colon i \neq j \in J\}, \qquad B_J = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \colon i \neq j \in J\}$$
$$C_J = \{\pm 2\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \colon i \neq j \in J\}, \qquad D_J = \{\pm\varepsilon_i \pm \varepsilon_j \colon i \neq j \in J\}.$$

$\Rightarrow$ 4 iso-classes of pairs $(\mathfrak{k}, \mathfrak{t})$ (for each cardinality $|J|$):

A_J: $\mathbb{K} = \mathbb{C}$, $\mathfrak{k} = \mathfrak{u}_2(\mathcal{H})$
B_J, D_J: $\mathbb{K} = \mathbb{R}$, $\mathfrak{k} = \mathfrak{u}_2(\mathcal{H}) =: \mathfrak{o}_2(\mathcal{H})$, $\dim(\ker(\mathfrak{t})) \in \{1, 0\}$
C_J: $\mathbb{K} = \mathbb{H}$, $\mathfrak{k} = \mathfrak{u}_2(\mathcal{H}) =: \mathfrak{sp}_2(\mathcal{H})$.

$\mathfrak{k} = \mathfrak{o}_2(\mathcal{H})$ has two conjugacy classes of Cartan subalgebras under $\mathrm{Aut}(\mathfrak{k})$.

2.1. Loop groups and twisted loop groups

Definition (Twisted loop groups)

For a Hilbert–Lie group K,

$$\mathcal{L}(K) := \{f \in C^\infty(\mathbb{R}, K) \colon (\forall t \in \mathbb{R})\, f(t + 2\pi) = f(t)\}$$

is called the corresponding loop group. For an automorphism $\varphi \in \mathrm{Aut}(K)$,

$$\mathcal{L}_\varphi(K) := \{f \in C^\infty(\mathbb{R}, K) \colon (\forall t \in \mathbb{R})\, f(t + 2\pi) = \varphi^{-1}(f(t))\}$$

is called the corresponding twisted loop group.

$\mathcal{L}_\varphi(K)$ is a Fréchet–Lie group with Lie algebra

$$\mathcal{L}_\varphi(\mathfrak{k}) := \{\xi \in C^\infty(\mathbb{R}, \mathfrak{k}) \colon (\forall t \in \mathbb{R})\, \xi(t + 2\pi) = \mathbf{L}(\varphi)^{-1}\xi(t)\}.$$

Note: $\mathcal{L}_\varphi(K)$ is the group of smooth sections of a K-Lie group bundle $\mathcal{K} = (\mathbb{R} \times K)/\sim$ over $\mathbb{S}^1 \cong \mathbb{R}/2\pi\mathbb{Z}$, where $(t + 2\pi, k) \sim (t, \varphi(k))$.

2.2. Loop groups with Hilbert targets

K-Lie group bundles over $\mathbb{S}^1$ correspond to $\mathrm{Aut}(K)$ principal bundles, hence are classified by

$$\pi_0(\mathrm{Aut}(K))/\mathrm{conj}.$$

Note: $\mathrm{Aut}(K) \cong \mathrm{Aut}(\mathfrak{k})$ if K is 1-connected.

Theorem

The automorphism groups of the infinite dimensional simple Hilbert–Lie algebras are given by the connected groups

$$\mathrm{Aut}(\mathfrak{o}_2(\mathcal{H})) \cong \mathrm{O}(\mathcal{H})/\{\pm\mathbf{1}\}, \quad \mathrm{Aut}(\mathfrak{sp}_2(\mathcal{H})) \cong \mathrm{Sp}(\mathcal{H})/\{\pm\mathbf{1}\}$$

(real and quaternionic case) and the 2-component group (complex case)

$$\mathrm{Aut}(\mathfrak{u}_2(\mathcal{H})) = \mathrm{PU}(\mathcal{H}) \rtimes \{\mathbf{1}, \sigma\}, \quad \sigma\colon \mathcal{H} \to \mathcal{H} \textit{ antilin. isom. involution.}$$

We thus obtain 4 iso-classes of twisted loop algebras

$$\mathcal{L}(\mathfrak{o}_2(\mathcal{H})), \quad \mathcal{L}(\mathfrak{u}_2(\mathcal{H})), \quad \mathcal{L}(\mathfrak{sp}_2(\mathcal{H})) \quad \text{and} \quad \mathcal{L}_\sigma(\mathfrak{u}_2(\mathcal{H})).$$

2.3. Double extensions

Definition (Double extensions)

For a Lie algebra $\mathfrak{g}$ with invariant symmetric bilinear form κ (quadratic Lie algebra) and a κ-skew-symmetric derivation D on $\mathfrak{g}$, the corresponding double extension is the quadratic Lie algebra $(\widehat{\mathfrak{g}}, \widehat{\kappa})$, where

$$\widehat{\mathfrak{g}} := \mathbb{R} \times \mathfrak{g} \times \mathbb{R}$$

$$[(z_1, x_1, t_1), (z_2, x_2, t_2)] := \big(\kappa(Dx_1, x_2), t_1 Dx_2 - t_2 Dx_1 + [x_1, x_2], 0\big)$$

$$\widehat{\kappa}((z_1, x_1, t_1), (z_2, x_2, t_2)) := z_1 t_2 + z_2 t_1 + \kappa(x_1, x_2).$$

Note: $\widetilde{\mathfrak{g}} := \mathbb{R} \times \mathfrak{g}$ is a central ext. with cocycle $\omega_D(x_1, x_2) := \kappa(Dx_1, x_2)$.
$\widehat{\mathfrak{g}} \cong \widetilde{\mathfrak{g}} \rtimes_{\widetilde{D}} \mathbb{R}$ for $\widetilde{D}(z, x) := (0, Dx)$.

Ex: $\mathfrak{g} = \mathfrak{u}_2(\mathcal{H})$, $Dx = [T, x]$, $T \in \mathfrak{u}(\mathcal{H}), \kappa(Dx, y) = -\operatorname{tr}(T[x, y])$

Rem: $\mathfrak{k}$ Hilbert–Lie algebra $\Rightarrow$ Any 2-cocycle ω can be written as $\omega(x, y) = (Dx, y)$ with $D \in \operatorname{der}(\mathfrak{k}) \Rightarrow$ double extension $\widehat{\mathfrak{k}}_D$.

2.4. Affine Kac–Moody groups

If $\mathfrak{k}$ is a Hilbert–Lie algebra and $\varphi \in \mathrm{Aut}(\mathfrak{k})$, then the loop algebra $\mathcal{L}_\varphi(\mathfrak{k})$ carries the scalar product $(\xi, \eta) := \int_0^{2\pi} (\xi(t), \eta(t))\, dt$ and the derivation $D\xi = \xi'$ is skew-symmetric. This leads to the double extension

$$\widehat{\mathcal{L}}_\varphi(\mathfrak{k}) = \mathbb{R} \oplus \mathcal{L}_\varphi(\mathfrak{k}) \oplus \mathbb{R}$$

$$[(z_1, \xi_1, t_1), (z_2, \xi_2, t_2)] := \big((\xi_1', \xi_2), t_1\xi_2' - t_2\xi_1' + [\xi_1, \xi_2], 0\big)$$

Theorem (N., '02, N./Wockel '09; Integrability Theorem)

If $\mathfrak{k}$ is simple, then there exists a simply connected Fréchet–Lie group $\widehat{\mathcal{L}}_\varphi(K)$ with Lie algebra $\widehat{\mathcal{L}}_\varphi(\mathfrak{k})$ and center $\mathbb{T}$.

Definition

We call $\widehat{\mathcal{L}}_\varphi(K)$ the corresponding (affine) Kac–Moody group.

Goal: Understand unitary rep's of $\widehat{\mathcal{L}}_\varphi(K) \Rightarrow$ We need root data.

2.5. Root systems for Kac–Moody groups

Here are the candidates for root systems of $\widehat{\mathcal{L}}_\varphi(\mathfrak{k})$:

Theorem (Y. Yoshii, 2006)

The irreducible reduced locally affine root systems of infinite rank are the following $X_J^{(1)} := X_J \times \mathbb{Z}$ *for* $X_J \in \{A_J, B_J, C_J, D_J\}$*, J an infinite set, and*

$B_J^{(2)} := (B_J \times 2\mathbb{Z}) \dot\cup ((B_J)_{\rm sh} \times (2\mathbb{Z}+1))$*, where* $(B_J)_{\rm sh} = \{\pm\varepsilon_j \colon j \in J\}$.

$C_J^{(2)} := (C_J \times 2\mathbb{Z}) \dot\cup (D_J \times (2\mathbb{Z}+1))$

$BC_J^{(2)} := (B_J \times 2\mathbb{Z}) \dot\cup (BC_J \times (2\mathbb{Z}+1)), \quad BC_J := B_J \cup C_J.$

These root systems contain no root bases $\Rightarrow$ No Dynkin diagrams.
To obtain root decompositions of $\widehat{\mathcal{L}}_\varphi(\mathfrak{k})$, we **assume:**
$\mathfrak{k}$ is simple, $\varphi \in \mathrm{Aut}(\mathfrak{k})$ involution,

$$\mathfrak{t} \subseteq \mathfrak{k}^\varphi := \{x \in \mathfrak{k} \colon \varphi(x) = x\} \quad \text{is maximal abelian.}$$

Then $\widehat{\mathfrak{t}} := \mathbb{R} \times \mathfrak{t} \times \mathbb{R} \subseteq \widehat{\mathcal{L}}_\varphi(\mathfrak{k})$ is maximal abelian.
For $\alpha\colon \mathfrak{t} \to i\mathbb{R}$ and $n \in \mathbb{Z}$ we define $(\alpha, n)\colon \widehat{\mathfrak{t}} \to i\mathbb{R}$ by

$$(\alpha, n)(z, x, t) := \alpha(x) + int.$$

Then the (anisotropic) root system $\widehat{\Delta} := \Delta(\widehat{\mathcal{L}}_\varphi(\mathfrak{k}), \widehat{\mathfrak{t}})$ is

$$\widehat{\Delta} = (\Delta_+ \times 2\mathbb{Z}) \dot\cup (\Delta_- \times (2\mathbb{Z}+1)) \quad \text{with} \quad \Delta_\pm := \Delta(\mathfrak{k}^{\pm\varphi}, \mathfrak{t}) \quad \mathfrak{t}\text{-(weights)}.$$

Theorem (Realization of the 7 locally affine root systems)

For $\Delta(\mathfrak{k}, \mathfrak{t}) = X_J$ we obtain $\widehat{\Delta} = \Delta(\widehat{\mathcal{L}}(\mathfrak{k}), \widehat{\mathfrak{t}}) = X_J^{(1)}$,
and $\widehat{\Delta} = \Delta(\widehat{\mathcal{L}}_\varphi(\mathfrak{k}), \widehat{\mathfrak{t}}) = X_J^{(2)}$ is obtained for $\varphi(x) = \sigma x \sigma^{-1}$ as follows:
- $B_J^{(2)}$ *for $\mathfrak{k} = \mathfrak{o}_2(\mathcal{H})$, σ orth. reflection in a hyperplane, $\mathfrak{t}$ of type B_J.*
- $C_J^{(2)}$ *for $\mathfrak{k} = \mathfrak{u}_2(\mathcal{H})$, σ antilinear with $\sigma^2 = -\mathbf{1}$.*
- $BC_J^{(2)}$ *for $\mathfrak{k} = \mathfrak{u}_2(\mathcal{H})$, σ antilinear with $\sigma^2 = \mathbf{1}$ and $\ker(\mathfrak{t}) \neq \{0\}$.*

Three root systems for $\widehat{\mathcal{L}}(\mathfrak{o}_2(\mathcal{H})) \cong \widehat{\mathcal{L}}_\varphi(\mathfrak{o}_2(\mathcal{H}))$: $B_J^{(1)}, D_J^{(1)}$ and $B_J^{(2)}$.
Two root systems for $\widehat{\mathcal{L}}_\sigma(\mathfrak{u}_2(\mathcal{H}))$ (σ antilinear): $C_J^{(2)}$ and $BC_J^{(2)}$.

3.1. Bounded and semibounded representations

Definition

A unitary representation $\pi\colon G \to \mathrm{U}(\mathcal{H})$ is called smooth if the space $\mathcal{H}^\infty := \{v \in \mathcal{H}\colon G \to \mathcal{H}, g \mapsto \pi(g)v \text{ smooth}\}$ of smooth vectors is dense.

The derived representation: $\mathtt{d}\pi(x)v = \frac{d}{dt}\big|_{t=0}\pi(\exp tx)v, \quad v \in \mathcal{H}^\infty, x \in \mathfrak{g}$.

The support function: $s_\pi\colon \mathfrak{g} \to \mathbb{R} \cup \{\infty\}, s_\pi(x) := \sup \mathrm{Spec}(i\mathtt{d}\pi(x))$

Cone of semiboundedness: $W_\pi := \{x \in \mathfrak{g}\colon s_\pi \text{ bounded in a nbhd of } x\}$.

Definition

A smooth representation is called semibounded if $W_\pi \neq \emptyset$.
It is called bounded if $W_\pi = \mathfrak{g}$.

Theorem (N. '08)

π bounded iff $\mathtt{d}\pi\colon \mathfrak{g} \to \mathfrak{u}(\mathcal{H})$ continuous iff $\pi\colon G \to \mathrm{U}(\mathcal{H})$ norm-cont.

3.2. Automatic boundedness

Example

If K is compact, then every continuous unitary representation is a direct sum of irreducible ones and irreducible reps are bounded.

Remark

(a) If π is semibounded, then W_π is an open $\mathrm{Ad}(G)$-invariant convex cone in $\mathfrak{g}$.

(b) If all open invariant cones in $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ are trivial, then every semibounded irreducible representation of G is bounded.

(c) All open invariant cones in $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ are trivial iff all open invariant cones in $\mathfrak{g}$ intersect $\mathfrak{z}(\mathfrak{g})$ (=fixed points of $\mathrm{Ad}(G)$).

Examples

Lie algebras $\mathfrak{g}$ for which open invariant cones in $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ are trivial:

(a) $\mathfrak{g}$ semisimple Hilbert–Lie algebra (Bruhat–Tits Fixed Point Thm)

(b) $\mathfrak{u}(\mathcal{H})$, $\mathcal{H}$ Hilbert space over $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

4. Bounded representations of Hilbert–Lie groups

$\mathfrak{k}$ simple Hilbert–Lie algebra, $\mathfrak{t} \subseteq \mathfrak{k}$ maximal abelian, Δ corresp. roots
Coroots: $\check{\alpha} \in i\mathfrak{t} \cap [\mathfrak{k}_{\mathbb{C}}^{\alpha}, \mathfrak{k}_{\mathbb{C}}^{-\alpha}]$ with $\alpha(\check{\alpha}) = 2$, for $\alpha \in \Delta$
K the 1-connected Lie group with Lie algebra $\mathfrak{k}$; $T := \exp(\mathfrak{t})$
$\mathcal{P}_T := \{\lambda \in i\mathfrak{t}' \colon (\forall \alpha \in \Delta)\, \lambda(\check{\alpha}) \in \mathbb{Z}\} \cong \mathrm{Hom}(T, \mathbb{T}) \subseteq i\mathfrak{t}'$ (T-weights)
Weyl group: $\mathcal{W} = \langle r_\alpha \colon \alpha \in \Delta \rangle \subseteq \mathrm{GL}(\mathfrak{t}_{\mathbb{C}})$, $r_\alpha(x) = x - \alpha(x)\check{\alpha}$.

Theorem (Classification Theorem, N. '98, '11)

Bounded unitary representations of K are direct sums of irreducible ones. The irreducible bounded reps π_λ are characterized by their T-weight set

$$\mathrm{conv}(\mathcal{W}\lambda) \cap (\lambda + \mathcal{Q}), \quad \mathcal{Q} = \langle \Delta \rangle_{\mathrm{grp}} \subseteq \mathcal{P}_T \text{ (root group).}$$

Classification of bounded irreps by $\mathcal{P}_T/\mathcal{W}$ (every $\lambda \in \mathcal{P}_T$ occurs).

Remark

(a) Bounded reps of K behave like reps of a compact group.
(b) The continuous representation theory of K is not type I (Boyer '80)

5.Semibounded representations of Kac–Moody groups

$G = \widehat{\mathcal{L}}_\varphi(K)$ (1-connected) as above (7 types), $d := (0,0,-i) \in i\widehat{\mathcal{L}}_\varphi(\mathfrak{k})$

- π irreducible semibounded rep of $G \Rightarrow i \cdot d \in W_\pi \cup -W_\pi$ (positive/negative energy representations if $\pm \mathtt{d}\pi(d)$ bounded below).
- We use that $\mathfrak{k}^\varphi$ is simple, hence all its open inv. cones are trivial.
- On the minimal/maximal eigenspace of $\mathtt{d}\pi(d)$ we find a bounded irreducible representation ρ_λ of $Z_G(d) \cong \mathbb{T} \times K^\varphi \times \mathbb{R}$

Theorem (Classification Theorem, Part 1)

Irreducible semibounded representations π_λ of $\widehat{\mathcal{L}}_\varphi(K)$ are extremal weight representations characterized by their $\widehat{\mathfrak{t}}$-weight set

$$\mathcal{P}_\lambda := \mathrm{conv}(\widehat{\mathcal{W}}\lambda) \cap (\lambda + \widehat{\mathcal{Q}}) \quad \textit{with} \quad \mathrm{Ext}(\mathrm{conv}(\mathcal{P}_\lambda)) = \widehat{\mathcal{W}}\lambda$$

($\widehat{\mathcal{W}}$ is the Weyl group of $\widehat{\Delta}$). The set of occurring extremal weights λ is

$$\mathcal{P}^\pm := \{\mu \in \mathcal{P}_{\widehat{T}^\varphi} \colon \pm(\widehat{\mathcal{W}}\mu)(d) \textit{ bounded from below}\}.$$

Let $\mathcal{P}_d^\pm \subseteq \mathcal{P}^\pm$ denote those elements μ for which $\mu(d)$ is minimal/maximal in $\widehat{\mathcal{W}}\mu$. With $c := \mu(i,0,0)$ (central charge), the elements $\mu \in \mathcal{P}_d^+$ are characterized by:

$$c \geq 0, \quad |\mu(\check{\alpha})| \leq \frac{2c}{(\alpha,\alpha)}, \quad |\mu(\check{\beta})| \leq \frac{4c}{(\beta,\beta)} \quad \text{for} \quad (\alpha,1),(\beta,2) \in \widehat{\Delta}.$$

Theorem (Classification Theorem, Part 2)

Classification of semibounded irreps: $\mathcal{P}^\pm/\widehat{\mathcal{W}} \cong \mathcal{P}_d^\pm/\mathcal{W}$.

Methods:

- Convex geometry of $\widehat{\mathcal{W}}$-orbits (local Coxeter theory).
- Complex geometry: Realization of π_λ in holomorphic sections of a complex Hilbert bundle with fiber representation ρ_λ of $Z_G(d)$ over the complex manifold $G/Z_G(d) \cong \mathcal{L}_\varphi(K)/K^\varphi$ (holomorphic induction).
- Harmonic analysis: Locally defined operator-valued analytic positive definite functions; automatic extension.

6. Semibounded projective reps of Hilbert–Lie groups

Problem: Boundedness of representations of a Hilbert–Lie group K is rather restrictive. It excludes important representations like "infinite wedge representations". These lead to projective representations, hence to central extensions and further to double extensions.

Setup: $\mathfrak{k}$ simple Hilbert–Lie algebra, $\mathfrak{t} \subseteq \mathfrak{k}$ maximal abelian, $\check{\Delta} \subseteq i\mathfrak{t}$ coroots
A $\mathfrak{t}$-invariant continuous cocycle $\omega(x, y)$ on $\mathfrak{k}$ can be represented by $D \in \mathrm{der}(\mathfrak{k})$ via $\omega(x, y) = (Dx, y)$ and there exists a linear functional $\lambda \colon \mathfrak{t} \cap [\mathfrak{k}, \mathfrak{k}] \to i\mathbb{R}$ with

$$\omega(x, y) = i\lambda([x, y]) \quad \text{for} \quad x, y \in \mathfrak{k}.$$

We call λ a bounded weight if $\lambda(\check{\alpha}) \in \mathbb{Z}$ for $\alpha \in \Delta$; $\mathcal{P}_b$ set of bd weights.

Definition

For $\lambda \in \mathcal{P}_b$ we write $\widehat{\mathfrak{k}}_\lambda = \mathbb{R} \oplus \mathfrak{k} \oplus \mathbb{R}$ for the corresp. double extension.
$\widehat{\mathfrak{t}} := \mathbb{R} \oplus \mathfrak{t} \oplus \mathbb{R} \subseteq \widehat{\mathfrak{k}}_\lambda$ is maximal abelian.
$\widehat{K}_\lambda$ is the corresponding 1-connected group; $\widehat{T} := \exp \widehat{\mathfrak{t}} \subseteq \widehat{K}_\lambda$.
$\mathcal{P}_{\widehat{T}} \subseteq \mathrm{Hom}(\widehat{\mathfrak{t}}, i\mathbb{R})$ (group of $\widehat{T}$-weights).

Theorem (Classification Theorem, N. '11)

Irreducible semibounded representations π_μ of $\widehat{K}_\lambda$ are extremal weight representations characterized by their $\widehat{\mathfrak{t}}$-weight set

$$\mathcal{P}_\mu := \mathrm{conv}(\mathcal{W}\mu) \cap (\mu + \mathcal{Q}) \quad \textit{with} \quad \mathrm{Ext}(\mathrm{conv}(\mathcal{P}_\mu)) = \mathcal{W}\mu.$$

Put $d := (0, 0, -i) \in i\widehat{\mathfrak{t}}$. The set of occurring extremal weights is

$$\mathcal{P}^\pm := \{\mu \in \mathcal{P}_{\widehat{T}} \colon \pm(\mathcal{W}\mu)(d) \text{ bounded from below}\}.$$

By minimizing/maximizing, we get the d-extremal weights

$$\mathcal{P}_d^\pm = \{\mu \in \mathcal{P}_{\widehat{T}} \colon (\forall \alpha \in \Delta)\, \lambda(\check{\alpha}) > 0 \Rightarrow \pm\mu(\check{\alpha}) \geq 0\}.$$

Classification: $\mathcal{P}^\pm/\mathcal{W} \cong \mathcal{P}_d^\pm/\mathcal{W}_\lambda$, where $\mathcal{W}_\lambda \subseteq \mathcal{W}$ is the stabilizer of λ.

Remark: (a) Representations of $\widehat{K}_\lambda$ are projective representations of K.
(b) For $K = \mathrm{U}_2(\mathcal{H})$ we cover in particular infinite wedge representations.

7. Semibounded projective reps of Kac–Moody groups

Again, projective representations of $\widehat{\mathcal{L}}_\varphi(K)$ lead to double extensions of $\mathfrak{g} = \widehat{\mathcal{L}}_\varphi(\mathfrak{k})$, hence to iterated double extensions $\widehat{\widehat{\mathcal{L}}}_\varphi(\mathfrak{k})$. Here the cocycle is of the form

$$\omega((z_1, \xi_1, t_1), (z_2, \xi_2, t_2)) := \frac{1}{2\pi} \int_0^{2\pi} i\lambda([\xi_1(t), \xi_2(t)])\, dt$$

for some bounded weight $\lambda \in \mathcal{P}_b$ for $(\mathfrak{k}, \mathfrak{t})$. Corresponding Lie groups $\widehat{\widehat{\mathcal{L}}}_\varphi(K)$ exist, and for $d = (0, 0, -i) \in i\widehat{\mathcal{L}}_\varphi(\mathfrak{k})$ we have

$$Z_\mathfrak{g}(d) = \mathbb{R} \oplus (\mathbb{R} \oplus \mathfrak{k}^\varphi \oplus \mathbb{R}) \oplus \mathbb{R} = \mathbb{R} \oplus \widehat{\mathfrak{k}}^\varphi_\lambda \oplus \mathbb{R}.$$

Semibounded representations of $\widehat{\widehat{\mathcal{L}}}_\varphi(K)$ now lead to semibounded representations of the double extension $(\widehat{K^\varphi})_\lambda$. These representations are classified!

Conjecture

Irreducible semibounded representations π_μ of $\widehat{\widehat{\mathcal{L}}}_\varphi(K)$ are extremal weight representations characterized by their $\widehat{\widehat{\mathfrak{t}}}$-weight set

$$\mathcal{P}_\mu := \mathrm{conv}(\widehat{\widehat{\mathcal{W}}}\mu) \cap (\mu + \widehat{\widehat{\mathcal{Q}}}) \quad \textit{with} \quad \mathrm{Ext}(\mathrm{conv}(\mathcal{P}_\mu)) = \widehat{\widehat{\mathcal{W}}}\mu.$$

The set of occurring extremal weights is

$$\mathcal{P}^\pm := \{\mu \in \mathcal{P}_{\widehat{\widehat{T}}} \colon \pm(\widehat{\widehat{\mathcal{W}}}\mu)(d) \text{ bounded from below}\}.$$

By minimizing/maximizing, we get the d-extremal weights $\mathcal{P}_d^\pm$.

Classification of semibounded irreps: $\mathcal{P}^\pm/\widehat{\widehat{\mathcal{W}}} \cong \mathcal{P}_d^\pm/\widehat{\mathcal{W}}_d$.

Problems: (a) The complex geometric Banach methods (holomorphic induction) break down because the representations of $\widehat{K}_\lambda$ are unbounded. We need a weaker notion of a complex Hilbert bundle.
(b) The iterated double extension creates 2 "d-elements", but semiboundedness should be controlled by the first one. This requires refined information on convexity properties of coadjoint orbits.

Positive energy vs. semiboundedness

- Semiboundedness is stronger than the positive energy condition $\mathrm{d}\pi(d)$ bd. below. It is crucial that semiboundedness implies boundedness of the K-representation on the minimal energy space. This is automatic if K is compact. In general K has many irreducible unbounded representations which are harder to control, f.i., Boyer's factor representations of $\mathrm{U}_2(\mathcal{H})$. We do not expect that the positive energy condition implies semiboundedness in general.
- Semiboundedness is intrinsic, it does not refer to the specification of an element $d \in \mathfrak{g}$, such as the positive energy condition. It also does not refer to a specific Cartan subalgebra.
- Our classification results hold for each of the 7 types of root systems of the 4 classes of Lie algebras. For different root systems, resp., conjugacy classes of Cartan subalgebras, we obtain different parameters for the same representations.

Concluding remarks

- Non-connected loop groups: $\pi_0(\mathcal{L}(K)) \cong \pi_1(K)$ is non-trivial in general. Which semibounded representations extend to non-connected groups?
- We need a better understanding of the concept of a Cartan subalgebra for $\widehat{\mathcal{L}}_\varphi(\mathfrak{k})$. Are there finitely many conjugacy classes?
- Describe the automorphism group of $\mathcal{L}_\varphi(\mathfrak{k})$.
- Are there also semibounded representations for double extensions of mapping groups $C^\infty(M, K)$, where $\dim M > 1$? The corresponding derivations should correspond to divergence free vector fields on M. Possibly one has to consider n-fold iterated double extensions, where $n = \dim M$. Here $M = \mathbb{T}^2$ is the natural testing case.
- For $K = \mathrm{U}_2(\mathcal{H})$, $\mathcal{H}$ complex, we have $\mathrm{Aut}(K)_0 \cong \mathrm{PU}(\mathcal{H})$, so that K-group bundles with this structure group over X are classified by their Dixmier–Douady classes in

$$[X, B\,\mathrm{PU}(\mathcal{H})] = [X, K(\mathbb{Z}, 3)] \cong H^3(X, \mathbb{Z}).$$