Wreath products as isometry groups of non standard metric products

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Let (X_i, d_i), $i = 1, \ldots, n$, be metric spaces. To define a metric on their cartesian products $X = \prod_{i=1}^{n} X_i$ one can use, for instance, one of the following equalities

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = d_1(x_1, y_1) + \ldots + d_n(x_n, y_n);$$

$$\widetilde{d}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{d_1^2(x_1, y_1) + \ldots + d_n^2(x_n, y_n)}.$$

There are different generalizations of these constructions. They include f-products (M. Moszynska, 1992), warped products (C.-H. Chen, 1999), μ-products (S. Avgustinovich, D. Fon-Der-Flaass, 2000), etc.
Following A. Bernig, T. Foertsch, V. Schroeder (2003) we consider non standard metric products or Φ-products of metric spaces.

Assume that $\Phi : [0, \infty)^n \to [0, \infty)$ be a function such that the following conditions hold
(A) $\Phi(p_1, p_2, \ldots, p_n) = 0$ iff $p_1 = p_2 = \ldots = p_n = 0$;
(B) for arbitrary $q_i, r_i, p_i \in [0, \infty)$ such that $q_i \leq r_i + p_i$, $1 \leq i \leq n$, the inequality

$$\Phi(q_1, \ldots, q_n) \leq \Phi(r_1, \ldots, r_n) + \Phi(p_1, \ldots, p_n)$$

holds.

Then the function

$$d_\Phi((x_1, \dots, x_n), (y_1, \dots, y_n)) = \Phi(d_1(x_1, y_1), \dots, d_n(x_n, y_n))$$

is a metric on X.

Definition

The metric space (X, d_Φ) is called a Φ-product or non standard metric product of metric spaces $X_1, \dots, X_n$.

Let q be a positive real number. It is easy to see that the function

$$\hat{\Phi}(p_1, p_2, \ldots, p_n) = \begin{cases} 0, & \text{if } p_1 = p_2 = \ldots = p_n = 0 \\ q, & \text{in other cases} \end{cases}$$

meets conditions (A) and (B).
The isometry group of $(X, d_{\hat{\Phi}})$ is isomorphic as a permutation group to the symmetric group $S_{|X|}$. This is the largest possible isometry group of Φ-products of $X_1, \ldots, X_n$.

Proposition 1 *Let X be a Φ-product of metric spaces $X_1, \dots, X_n$, $n \geq 2$. Then the transformation group $(Isom X, X)$ contains a subgroup isomorphic to the direct product of the transformation groups*

$$(Isom X_1, X_1) \times \dots \times (Isom X_n, X_n).$$

Let now $(X_1, d_1), \ldots, (X_n, d_n)$ be discrete spaces, i.e., for different points $u, v \in X_i$ $d_i(u, v) = 1$, $1 \leq i \leq n$. And let $|X_i| = k_i$, $1 \leq i \leq n$. We can introduce the function $\Phi_1 : [0, \infty)^n \to [0, \infty)$ putting

$$\Phi_1(q_1, \ldots, q_n) = \begin{cases} q_1, & \text{if } q_1 \neq 0; \\ \frac{1}{2} q_2, & \text{if } q_1 = 0 \text{ and } q_2 \neq 0; \\ \ldots \quad \ldots \quad \ldots & \\ \frac{1}{n} q_n, & \text{if } q_1 = 0,\ \ldots,\ q_{n-1} = 0,\ q_n \neq 0; \\ 0, & \text{if } q_1 = 0,\ \ldots,\ q_n = 0. \end{cases}$$

Let T be a finite n-levels rooted tree with root v_0. Assume, that a rooted tree T is level homogenous with level index $[k_1; k_2; \ldots, k_n]$, where k_i is the number of edges joining a vertex of the i-th level with vertices of the $(i+1)$-st level. The metric space δT is defined to be the set of all rooted path of T equipped with a natural ultrametric

$$\rho(\gamma_1, \gamma_2) = 1/(m+1),$$

where m is the length of the maximal common part of rooted paths γ_1 and γ_2.

The space δT of paths in the rooted level homogeneous tree T and the Φ_1-product of discrete metric spaces $X_1, \ldots, X_n$ are isometric. It is well known that the isometry group of the space δT is isomorphic as a permutation group to the wreath product of symmetric group S_{k_i}, $i = 1, \ldots, n$.
Therefore, the isometry group of the space $(X_1 \times \ldots \times X_n, d_{\Phi_1})$ is isomorphic as a permutation group to the wreath product of isometry groups of discrete spaces X_i, $i = 1, \ldots, n$.

Let now (X_i, d_i), $i = 1, \ldots, n$, be arbitrary metric spaces. And let as before C_i be the set of values of metric d_i, $1 \leq i \leq n$. Assume that there exist functions $f_i : [0, \infty) \to [0, \infty)$, $1 \leq i \leq n$, such that

$$\Phi(q_1, \ldots, q_n) = \begin{cases} f_1(q_1), & \text{if } q_1 \neq 0; \\ f_2(q_2), & \text{if } q_1 = 0 \text{ and } q_2 \neq 0; \\ \ldots \quad \ldots \quad \ldots & \\ f_n(q_n), & \text{if } q_1 = 0,\ \ldots,\ q_{n-1} = 0,\ q_n \neq 0; \\ 0, & \text{if } q_1 = 0,\ \ldots,\ q_n = 0 \end{cases} \tag{1}$$

for arbitrary $q_i \geq 0$, $1 \leq i \leq n$.

For each $i, 1 \leq i \leq n$, denote by $\widehat{X}_i$ the space $(X_i, \widehat{d}_i)$, where for arbitrary $u, v \in X_i$

$$\widehat{d}_i(u,v) = \begin{cases} f_i(d_i(u,v)), & \text{if } u \neq v \\ 0, & \text{in other cases} \end{cases}.$$

Assume that for all i, $1 \leq i \leq n-1$, the inequalities

$$\inf_{q_i \in C_i, q_i \neq 0} f_i(q_i) > \sup_{q_{i+1} \in C_{i+1}} f_{i+1}(q_{i+1}) \tag{2}$$

hold.

Theorem

Let $\Phi : [0, \infty)^n \to [0, \infty)$ be a function such that conditions (A),(B), (1) and (2) hold. Then the isometry group of the Φ-product of metric spaces X_1, X_2, ..., X_n is isomorphic as a permutation group to the wreath product of isometry groups of spaces $\widehat{X}_i$, $i = 1, \ldots, n$,

$$(Isom(X, d_\Phi), X) \simeq \wr_{i=1}^{n} (Isom\widehat{X}_i, X_i).$$

Corollary

Let $\Phi : [0, \infty)^n \to [0, \infty)$ be a function such that the conditions (A),(B), (1) and (2) hold. If

$$Isom(X_i, f_i(d_i)) = Isom(X_i, d_i)$$

for all i, $1 \leq i \leq n$, then

$$(Isom X, X) \simeq \wr_{i=1}^{n} (Isom X_i, X_i).$$

Example

Let $X_i = \mathbb{Z}$ and d_i be the Euclidean distance, $1 \leq i \leq n$. The function

$$\Phi_5(q_1, \ldots, q_n) = \begin{cases} n+1-\frac{1}{q_1+1}, & \text{if } q_1 \neq 0; \\ n-\frac{1}{q_2+1}, & \text{if } q_1 = 0 \text{ and } q_2 \neq 0; \\ \ldots \quad \ldots \quad \ldots & \\ 2-\frac{1}{q_n+1}, & \text{if } q_1 = 0, \ldots, q_{n-1} = 0, q_n \neq 0; \\ 0, & \text{if } q_1 = 0, \ldots, q_n = 0. \end{cases}.$$

meets conditions (A),(B), (1) and (2). Therefore, one can consider the Φ_5-product $(\mathbb{Z} \times \ldots \times \mathbb{Z}, d_{\Phi_5})$ of X_i, $1 \leq i \leq n$.

The isometry group of $(\mathbb{Z} \times \ldots \times \mathbb{Z}, d_{\Phi_5})$ is isomorphic as a permutation group to the wreath product of n infinite dihedral groups D_∞.

$$(Isom(\mathbb{Z} \times \ldots \times \mathbb{Z}, d_{\Phi_5}), \mathbb{Z} \times \ldots \times \mathbb{Z}) \simeq \wr_{i=1}^{n} (D_\infty, \mathbb{Z}).$$

Example

Let (X_1, d_1) and (X_2, d_2) be metric spaces of finite diameters D_1, D_2. Assume that there exists a positive number r such that for arbitrary points $x_1, x_2 \in X_1$, $x_1 \neq x_2$, the inequality $d_1(x_1, x_2) \geq r$ holds. Let $\Phi_3(q_1, q_2) = max(q_1, q_2)$.

If the inequality

$$r > D_2$$

holds then

$$Isom(X_1 \times X_2, d_{\Phi_3}) \simeq Isom X_1 \wr Isom X_2.$$

Now we consider Φ-products $(X_1 \times X_2, d_\Phi)$ of two metric spaces (X_1, d_1), (X_2, d_2). For each $a_1 \in X_1$, $a_2 \in X_2$ let

$$X^2_{a_1} = \{(a_1, x_2) \mid \ x_2 \in X_2\}, \quad X^1_{a_2} = \{(x_1, a_2) \mid \ x_1 \in X_1\}$$

be subspaces of $(X_1 \times X_2, d_\Phi)$. The points of spaces $X^2_{a_1}$, $a_1 \in X_1$ are in natural one-to-one correspondence with the points of the space X_2, while the points of spaces $X^1_{a_2}$, $a_2 \in X_2$ are in natural one-to-one correspondence with the points of X_1. Hence we can assume that the group $Isom X^2_{a_1}$ acts on the set X_2 and the group $Isom X^1_{a_2}$ acts on the set X_1.

Denote by C_i the set of values of the metric d_i, $i = 1, 2$. Assume that inequalities

$$\inf_{q_1 \in C_1, q_1 \neq 0} \Phi(q_1, 0) > \sup_{q_2 \in C_2} \Phi(0, q_2),$$
$$\inf_{q_2 \in C_2, q_2 \neq 0} \Phi(0, q_2) > \frac{1}{2} \sup_{q_1 \in C_1} \Phi(q_1, 0). \quad (3)$$

hold.

Theorem

Let $\Phi : [0, \infty)^2 \to [0, \infty)$ be a function such that conditions (A),(B) and inequalities (3) hold. Assume that

$$\Phi(q_1, q_2) = \Phi(q_1, 0) + \Phi(0, q_2). \quad (4)$$

Then

$$(Isom X, X) \simeq (Isom X^1_{a_2}, X_1) \times (Isom X^2_{a_1}, X_2)$$

for any $(a_1, a_2) \in X_1 \times X_2$.

Corollary

Let $\Phi : [0,\infty)^2 \to [0,\infty)$ be a function such that conditions (A),(B) and inequalities (3) hold. Assume that

$$\Phi(q_1, q_2) = \Phi(q_1, 0) + \Phi(0, q_2).$$

If $\mathit{Isom} X^1_{a_2} = \mathit{Isom} X_1$, $\mathit{Isom} X^2_{a_1} = \mathit{Isom} X_2$ for some $(a_1, a_2) \in X_1 \times X_2$, then

$$(\mathit{Isom} X, X) \simeq (\mathit{Isom} X_1, X_1) \times (\mathit{Isom} X_2, X_2).$$

Example

Let (X_1, d_1) and (X_2, d_2) be uniformly discrete metric spaces of finite diameters D_1, D_2 correspondingly. And let r_1, r_2 be positive numbers, such that for arbitrary points $x_1, x_2 \in X_i$, $x_1 \neq x_2$, the inequalities $d_i(x_1, x_2) \geq r_i$ hold, $i = 1, 2$. Denote $\Phi_2(q_1, q_2) = q_1 + q_2$. Then the function $\Phi_2(q_1, q_2)$ meets conditions (A) and (B). If the inequalities

$$r_1 > D_2 \geq r_2 > \frac{1}{2} D_1 \text{ or } r_2 > D_1 \geq r_1 > \frac{1}{2} D_2$$

hold then the inequalities (3) hold as well. Therefore

$$Isom(X_1 \times X_2, d_{\Phi_2}) \simeq Isom X_1 \times Isom X_2.$$

Example

Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let

$$\Phi_1(q_1, q_2) = \begin{cases} 0, & \text{if } p_1 = p_2 = 0 \\ 4, & \text{if } p_1 \neq 0,\ p_2 = 0 \\ 3, & \text{if } p_1 = 0,\ p_2 \neq 0 \\ q_1 + q_2, & \text{in other cases} \end{cases}.$$

Then

$$Isom(X_1 \times X_2, d_{\Phi_1}) \simeq S_{|X_1|} \times S_{|X_2|}.$$

Let $(G_1, X_1), \ldots, (G_n, X_n)$ be a sequence of transformation groups. Following Kaloujnine L.A., Beleckij P.M., Feinberg V.T the transformation group $(G, \prod_{i=1}^{n} X_i)$ is called the *wreath products of groups* $(G_1, X_1), \ldots, (G_n, X_n)$ if for all elements $u \in G$ the following conditions hold:
1) if $(x_1, \ldots, x_n)^u = (y_1, \ldots, y_n)$, then for all i, $1 \leq i \leq n$, the value of y_i depends only on x_1, ..., x_i;
2) for fixed $x_1, \ldots, x_{i-1}$ the mapping $g_i(x_1, \ldots, x_{i-1})$ defined by the equality

$$g_i(x_1, \ldots, x_{i-1})(x_i) = y_i, \quad x_i \in X_i$$

is a permutation on the set X_i which belongs to G_i. Denote the wreath products of groups $(G_1, X_1), \ldots, (G_n, X_n)$ by $\wr_{i=1}^{n} (G_i, X_i)$.