THE LAX TYPE DIFFERENTIAL-ALGEBRAIC IDEALS AND THE DIFFERENTIAL SYSTEMS INTEGRABILITY

Anatoliy K. Prykarpatsky
(AGH University of Science and Technology of Krakow, Poland,
and the Ivan Franko State University of Drohobych, Lviv region, Ukraine)
jointly with:
Orest D. Artemovych
(Politechnical University of Krakow, Poland, and the Pre-Carpathian University, Ivano-Frankivsk, Ukraine)
Yarema A. Prykarpatsky
(Agrarian University of Krakow, Poland)
and
Maxim V. Pavlov
(Lebedev Institute of Physics at RAS and the Lomomosov State University, Moscow, Russian Federation)

Topics to be discussed:

- 1) **Differential rings, differentiations and invariant ideals;**
- 2) **Hydrodynamical differential system as constraints on differentiations;**
- 3) **Spectral analysis of differentiations and their representations;**
- 4) **Lax type differential ideal and its invariance condition;**
- 5) **The linear matrix representaion of differentials and the Lax type integrability for hydrodynamical systems;**
- **6) The KdV and Lax type invariant differential ideals and the KdV integrability problem;**
- **7) Conclusions.**

SHORT INTRODUCTION:

Differential ideals in Differential rings

2.1. The generalized Riemann type hydrodynamical equation: the case N=3. We will begin with considering [5, 6, 7, 8] the differential ring $\mathbb{R}((x,t))$ with two differentiations $D_x := \partial/\partial x$ and $D_t : \mathbb{R}((x,t)) \to \mathbb{R}((x,t))$, satisfying the following Lie-algebraic commutator relationship:

$$[D_x, D_t] = (D_x u) D_x, \tag{2.1}$$

where an element $u \in \mathbb{R}((x,t))$ is fixed.

The generalized Riemann type hydrodynamical equation at an arbitrary $N \in \mathbb{Z}_+$ reads as follows:

$$D_t^N u = 0. \tag{2.2}$$

Subject to the Riemann type hydrodynamical equation (2.2) at $N = 3$

$$D_t^3 u = 0 \tag{2.3}$$

we would like to construct analytically a so called "Lax differential ideal" $L[u]((x,t)) \subset \mathbb{R}((x,t))$, realizing its Lax type integrability and the related adjoint linear matrix representations of the differentiations D_x and $D_t : \mathbb{R}^p((x,t)) \to \mathbb{R}^p((x,t))$ in the space $\mathbb{R}^p((x,t))$ for some integer $p \in \mathbb{Z}_+$.

To proceed with analyzing the integrability problem of the generalized Riemann type equation (2.3) we construct an adjoint invariant, so called "Riemann differential ideal", $R[u]((x,t)) \subset \mathbb{R}((x,t))$ as

$$R[u]((x,t)) \quad : \quad = \{\lambda \sum_{n\in\mathbb{Z}_+} f_n^{(1)} D_x^n u - \sum_{n\in\mathbb{Z}_+} f_n^{(2)} D_t D_x^n u + \sum_{n\in\mathbb{Z}_+} f_n^{(3)} D_t^2 D_x^n u : D_t^3 u = 0,$$

$$f_n^{(k)} \quad \in \quad \mathbb{R}((x,t)), k = \overline{1,3}, n \in \mathbb{Z}_+\} \subset \mathbb{R}((x,t)) \tag{2.4}$$

and formulate the following simple but important lemma.

Lemma 2.1. *The kernel* $Ker\ D_t \subset R[u]((x,t))$ *of the differentiation* $D_t : \mathbb{R}((x,t)) \to \mathbb{R}((x,t))$, *reduced upon the Riemann differential ideal* $R[u]((x,t)) \subset \mathbb{R}((x,t))$, *is generated by elements, satisfying the following linear functional-differential relationships:*

$$D_t f^{(1)} = 0, \quad D_t f^{(2)} = \lambda f^{(1)}, \quad D_t f^{(3)} = f^{(2)}, \tag{2.5}$$

where, by definition, $f^{(k)} := f^{(k)}(\lambda) = \sum_{n\in\mathbb{Z}_+} f_n^{(k)} \lambda^n \in \mathbb{R}((x,t)), k = \overline{1,3}$, *and* $\lambda \in \mathbb{R}$ *is arbitrary.*

It is easy to see that equations (2.5) can be equivalently rewritten both in the matrix form as

$$D_t f = q(\lambda) f, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.6}$$

where $f := (f^{(1)}, f^{(2)}, f^{(3)})^\intercal \in \mathbb{R}^3((x,t))$, $\lambda \in \mathbb{R}$ is an arbitrary "spectral" parameter, and in the compact scalar form as

$$D_t^3 f_3 = 0 \tag{2.7}$$

for an element $f_3 \in \mathbb{R}((x,t)$. Now we can construct by means of relationship (2.7) a new, so-called "Lax differential ideal" $L[u]((x,t)) \subset \mathbb{R}((x,t))$, isomorphic to the Riemann differential ideal $R[u]((x,t)) \subset \mathbb{R}((x,t))$ and realizing the Lax type integrability condition of the Riemann type hydrodynamical equation (2.3). Namely, based on the result of Lemma 2.1 the following proposition holds.

Proposition 2.2. *The expression (2.6) is an adjoint linear matrix representation in the space $\mathbb{R}^p((x,t))$ at $p = 3$ of the differentiation $D_t : \mathbb{R}((x,t)) \to \mathbb{R}((x,t))$, reduced to the ideal $R[u]((x,t)) \subset \mathbb{R}((x,t))$. The related D_x- and D_t-invariant Lax differential ideal $L[u]((x,t)) \subset \mathbb{R}((x,t))$, which is isomorphic to the invariant Riemann differential ideal $R[u]((x,t)) \subset \mathbb{R}((x,t))$, is generated by elements $f_3(\lambda) \in \mathbb{R}((x,t)), \lambda \in \mathbb{R}$, satisfying condition (2.7), and equals*

$$\begin{aligned} L[u]((x,t)) \quad : &= \{g_1 f_3(\lambda) + g_2 D_t f_3(\lambda) + g_3 D_t^2 f_3(\lambda) : D_t^3 f_3(\lambda) = 0, \\ \lambda \quad \in & \quad \mathbb{R}, g_j \in \mathbb{R}((x,t)), j = \overline{1,3}\} \subset \mathbb{R}((x,t)). \end{aligned} \tag{2.8}$$

Proceed now to constructing a related adjoint linear matrix representation in the space $\mathbb{R}^3((x,t))$ for the differentiation $D_x : \mathbb{R}((x,t)) \to \mathbb{R}((x,t))$, reduced upon the Lax differential ideal $L[u]((x,t)) \subset \mathbb{R}((x,t))$. For this problem to be solved, we need to take into account the commutator relationship (2.1) and the important invariance condition of the Lax differential ideal $L[u]((x,t)) \subset \mathbb{R}((x,t))$ with respect to the differentiation $D_x : \mathbb{R}((x,t)) \to \mathbb{R}((x,t))$. As a result of simple but slightly tedious calculations one obtains the following matrix representation:

$$D_x f = \ell[u,v,z;\lambda] f, \quad \ell[u,v,z;\lambda] := \begin{pmatrix} \lambda u_x & -v_x & z_x \\ 3\lambda^2 & -2\lambda u_x & \lambda v_x \\ 6\lambda^2 r[u,v,z] & -3\lambda & \lambda u_x \end{pmatrix}, \tag{2.9}$$

where, by definition, $v := D_t u, z := D_t v$, $(...)_x := D_x(...)$, a vector $f \in \mathbb{R}^3((x,t))$, $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter and a smooth mapping $r : \mathbb{R}^3((x,t)) \to \mathbb{R}$ solves the following functional-differential equation

$$D_t r + r D_x u = 1. \tag{2.10}$$

Moreover, the matrix $\ell := \ell[u,v,z;\lambda] : \mathbb{R}^3((x,t)) \to \mathbb{R}^3((x,t))$ satisfies the following determining functional-differential equation:

$$D_t \ell + \ell D_x u = [q(\lambda), \ell], \tag{2.11}$$

where $[\cdot,\cdot]$ denotes the usual matrix commutator in the space $\mathbb{R}^3((x,t))$. Thereby, the following proposition solving the problem, posed above, holds.

Proposition 2.3. *The expression (2.9) is an adjoint linear matrix representation in the space $\mathbb{R}^3((x,t))$ of the differentiation $D_x : \mathbb{R}((x,t)) \to \mathbb{R}((x,t))$, reduced upon the invariant Lax differential ideal $L[u]((x,t)) \subset \mathbb{R}((x,t))$, given by (2.8).*

Remark 2.4. It is here necessary to mention that matrix representation (2.6) coincides completely with that obtained before in the work [10] by means of completely different methods, based mainly on the gradient-holonomic algorithm, devised in [17, 18, 19]. The presented derivation of these representations (2.6) and (2.9) is much more easier and simpler that can be, eventually, explained by a deeper insight into the integrability problem, devised above within the differential algebraic approach.

2.2. The solution set analysis of the functional-differential equation $D_t r + r D_x u = 1$.

Below we will describe all functional solutions to equation (2.15), making use of the lemma, following from the results of [10].

Lemma 2.6. *The following functions*

$$B_0 = \xi(z), \; B_1 = u - tv + zt^2/2, \; B_2 = v - zt, \quad B_3 = x - tu + vt^2/2 - zt^3/6, \tag{2.16}$$

where $\xi : \mathbb{R}((x,t)) \to \mathbb{R}((x,t))$ is an arbitrary smooth mapping, are the main invariants of the Riemann type dynamical system (2.13), satisfying the determining condition

$$D_t B = 0. \tag{2.17}$$

As a simple inference of relationships (2.16) the next lemma holds.

Lemma 2.7. *The local functionals*

$$b_0 := \xi(z), b_1 := \frac{u}{z} - \frac{v^2}{2z^2}, b_2 := \frac{u_x}{z_x} - \frac{v_x^2}{2z_x^2}, b_3 := x - \frac{uv}{z} + \frac{v^3}{3z^2} \tag{2.18}$$

and

$$\tilde{b}_1 := \frac{v}{z}, \tilde{b}_2 := \frac{v_x}{z_x}$$

on the functional manifold $\mathcal{M}$ *are the basic functional solutions* $b_j : \mathcal{M} \to \mathbb{R}((x,t)), j = \overline{0,3}$, *and* $\tilde{b}_k : \mathcal{M} \to \mathbb{R}((x,t)), k = \overline{1,2}$, *to the determining functional-differential equations*

$$D_t b = 0 \tag{2.19}$$

and

$$D_t \tilde{b} = 1, \tag{2.20}$$

respectively.

Now one can formulate the following theorem about the general solution set to the functional-differential equation (2.19).

Theorem 2.8. *The following infinite hierarchies*

$$\eta_{1,j}^{(n)} := (\alpha D_x)^n b_j, \quad \eta_{2,k}^{(n)} := (\alpha D_x)^{n+1} \tilde{b}_k, \tag{2.21}$$

where $\alpha := 1/z_x$, $j = \overline{0,3}$, $k = \overline{1,2}$ *and* $n \in \mathbb{Z}_+$, *are the basic functional solutions to the functional-differential equation (2.19), that is*

$$D_t \eta_{s,j}^{(n)} = 0 \tag{2.22}$$

for $s = \overline{1,2}$, $j = \overline{0,3}$ *and all* $n \in \mathbb{Z}_+$.

Proceed now to analyzing the solution set to functional-differential equation (2.15), making use of the following transformation:

$$r := \frac{a}{\alpha\eta}, \tag{2.23}$$

where $\eta : \mathcal{M} \to \mathbb{R}((x,t))$ is any solution to equation (2.22) and a smooth functional mapping $a : \mathcal{M} \to \mathbb{R}((x,t))$ satisfies the following determining functional-differential equation:

$$D_t a = \alpha\eta. \tag{2.24}$$

Then any solution to functional-differential equation (2.15) reads as

$$r = \frac{a}{\alpha\eta} + \eta_0, \tag{2.25}$$

where $\eta_0 : \mathcal{M} \to \mathbb{R}((x,t))$ is any smooth solution to the functional-differential equation (2.22).

To find solutions to equation (2.24), we make use of the following linear α-expansion in the corresponding Riemann differential ideal $R[\alpha]((x,t)) \subset \mathbb{R}((x,t))$:

$$a = c_3 + c_0\alpha + c_1\dot{\alpha} + c_2\ddot{\alpha} \in R[\alpha]((x,t)), \tag{2.26}$$

where $\dot{\alpha} := D_t\alpha, \ddot{\alpha} := D_t^2\alpha$ and taking into account that all functions $\alpha, \dot{\alpha}$ and $\ddot{\alpha}$ are functionally independent owing to the fact that $\dddot{\alpha} := D_t^3\alpha \equiv 0$. As a result of substitution (2.26) into (2.24) we obtain the relationships

$$\dot{c}_1 + c_0 = 0,\ \dot{c}_0 = \eta,\ \dot{c}_2 + c_1 = 0,\ \dot{c}_3 + c_2 = 0. \tag{2.27}$$

The latter allow, owing to (2.20), at the special solution $\eta = 1$ to equation (2.22) two functional solutions for the mapping $c_0 : \mathcal{M} \to \mathbb{R}((x,t))$:

$$c_0^{(1)} = \frac{v}{z}, \quad c_0^{(2)} = \frac{v_x}{z_x}. \tag{2.28}$$

As a result, we obtain, solving the recurrent functional equations (2.27), that

$$\begin{aligned} a_2^{(1)} &= [(xv - u^2/2)/z]_x,\ a_2 = \frac{v_x}{z_x^2} - \frac{u_x^2}{2z_x^2}, \\ a_2^{(1)} &= \frac{v_x v^3}{6z_x z^3} - \frac{u_x v^2}{2z_x z^2} + \frac{u(uz - v^2)}{6z^3} + \frac{v}{zz_x}, \end{aligned} \tag{2.29}$$

giving rise to the following three functional solutions to (2.15):

$$r_1^{(1)} = \frac{v_x v^3}{6z^3} - \frac{u_x v^2}{2z^2} + \frac{u(uz - v^2)z_x}{6z^3} + \frac{v}{z}, \tag{2.30}$$

$$r_1^{(2)} = (xv - u^2/2)/z]_x, \; r_2 = \frac{v_x}{z_x} - \frac{u_x^2}{2z_x}.$$

Having now chosen the next special solution $\eta := b_2 = \frac{u_x}{z_x} - \frac{v_x^2}{2z_x^2}$ to equation (2.22), one easily obtains from (2.27) that the functional expression

$$r_3 = (\frac{u_x^3}{6z_x^2} - \frac{u_x v_x}{2z_x^2} + \frac{3}{4z_x})/(\frac{u_x}{z_x} - \frac{v_x^2}{2z_x^2}) \tag{2.31}$$

also solves the functional-differential equation (2.27). Doing further the same way as above, one can construct an infinite set $\mathcal{R}$ of the searched solutions to the functional-differential equation (2.27) on the manifold $\mathcal{M}$. Thereby one can formulate the following theorem.

Theorem 2.9. *The complete set $\mathcal{R}$ of functional-differential solutions to equation (2.15) on the manifold $\mathcal{M}$ is generated by functional solutions in the form (2.25) to the reduced functional-differential equations (2.22) and (2.24).*

In particular, the subset

$$
\begin{aligned}
(2.32)\qquad \tilde{\mathcal{R}} &= \{r_1^{(1)} = \frac{v_x v^3}{6z^3} - \frac{u_x v^2}{2z^2} + \frac{u(uz - v^2)z_x}{6z^3} + \frac{v}{z},\ r_1^{(2)} = [(xv - u^2/2)/z]_x, \\
r_2 &= \frac{v_x}{z_x} - \frac{u_x^2}{2z_x}, r_3 = (\frac{u_x^3}{6z_x^2} - \frac{u_x v_x}{2z_x^2} + \frac{3}{4z_x})/(\frac{u_x}{z_x} - \frac{v_x^2}{2z_x^2})\} \subset \mathcal{R}
\end{aligned}
$$

coincides exactly with that found before in article [10].

2.3. The generalized Riemann type hydrodynamical equation: the case N=4. Now consider the generalized Riemann type differential equation (2.2) at $N = 4$

$$D_t^4 u = 0 \tag{2.33}$$

on an element $u \in \mathbb{R}((x,t))$ and construct the related invariant Riemann differential ideal $R[u]((x,t)) \subset \mathbb{R}((x,t))$ as follows:

$$\begin{aligned} R[u]((x,t)) \quad : \quad &= \{\lambda^3 \sum_{n\in\mathbb{Z}_+} f_n^{(1)} D_x^n u - \lambda^2 \sum_{n\in\mathbb{Z}_+} f_n^{(2)} D_t D_x^n u + \lambda \sum_{n\in\mathbb{Z}_+} f_n^{(3)} D_t^2 D_x^n u - \\ &- \sum_{n\in\mathbb{Z}_+} f_n^{(4)} D_t^3 D_x^n u \quad : \quad D_t^4 u = 0, \lambda \in \mathbb{R}, f_n^{(k)} \in \mathbb{R}((x,t)), k = \overline{1,4}, n \in \mathbb{Z}_+\} \end{aligned} \tag{2.34}$$

at a fixed function $u \in \mathbb{R}((x,t))$. The corresponding kernel $Ker\ D_t \subset R[u]((x,t))$ of the differentiation $D_t : \mathbb{R}((x,t)) \to \mathbb{R}((x,t))$, reduced upon the Riemann differential ideal (2.34), is given by the following linear differential relationships:

$$D_t f^{(1)} = 0,\ D_t f^{(2)} = \lambda f^{(1)},\ D_t f^{(3)} = \lambda f^{(2)},\ D_t f^{(4)} = \lambda f^{(3)}, \tag{2.35}$$

where $f^{(k)} := f^{(k)}(\lambda) = \sum_{n\in\mathbb{Z}_+} f_n^{(k)} \lambda^n \in \mathbb{R}((x,t))$, $k = \overline{1,4}$ and $\lambda \in \mathbb{R}$ is arbitrary. The linear relationships (2.35) can be easily represented in the space $\mathbb{R}^4((x,t))$ in the following matrix form:

$$D_t f = q(\lambda) f, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix}, \tag{2.36}$$

where $f := (f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)})^\intercal \in \mathbb{R}^4((x,t))$, and $\lambda \in \mathbb{R}$. Moreover, it is easy to observe that relationships (2.35) can be equivalently rewritten in the compact scalar form as

$$D_t^4 f^{(4)} = 0, \tag{2.37}$$

where an element $f_4 \in \mathbb{R}((x,t))$. Thus, now one can construct the invariant Lax differential ideal, isomorphically equivalent to (2.34), as follows:

$$\begin{aligned} L[u]((x,t)) \quad : &= \{g_1 f^{(4)} + g_2 D_t f^{(4)} + g_3 D_t^2 f^{(4)} + g_4 D_t^3 f^{(4)} : D_t^4 f^{(4)} = 0, \\ g_j \quad &\in \quad \mathbb{R}((x,t)), j = \overline{1,4}\} \subset \mathbb{R}((x,t)), \end{aligned} \tag{2.38}$$

whose D_x-invariance should be checked separately. The latter gives rise to the representation

$$D_x f = \ell[u,v,w,z;\lambda] f, \quad \ell[u,v,w,z;\lambda] := \begin{pmatrix} -\lambda^3 u_x & \lambda^2 v_x & -\lambda w_x & z_x \\ -4\lambda^2 & 3\lambda^3 u_x & -2\lambda^2 v_x & \lambda w_x \\ -10\lambda^5 r_1 & 6\lambda^4 & -3\lambda^3 u_x & \lambda^2 v_x \\ -20\lambda^6 r_2 & 10\lambda^5 r_1 & -4\lambda^4 & \lambda^3 u_x \end{pmatrix}, \tag{2.39}$$

where we put, by definition,

$$D_t u := v, D_t v := w, D_t w := z, D_t z := 0, \tag{2.40}$$

$(u,v,w,z)^\intercal \in \mathbb{R}((x,t))^3 \simeq \mathcal{M}$, and the mappings $r_j : \mathcal{M} \to \mathbb{R}((x,t)), j = \overline{1,2}$, satisfy the following functional-differential equations:

$$D_t r_1 + r_1 D_x u = 1, \qquad D_t r_2 + r_2 D_x u = r_1, \tag{2.41}$$

similar to (2.10), considered already above. The equations (2.41) possess a lot of different solutions, amongst which there are functional expressions:

$$
\begin{aligned}
r_1 &= D_x\left(\frac{uw^2}{2z^2} - \frac{vw^3}{3z^3} + \frac{vw^4}{24z^4} + \frac{7w^5}{120z^4} - \frac{w^6}{144z^5}\right), \\
r_2 &= D_x\left(\frac{uw^3}{3z^3} - \frac{vw^4}{6z^4} + \frac{3w^6}{80z^5} + \frac{vw^5}{120z^5} - \frac{w^7}{420z^6}\right).
\end{aligned} \tag{2.42}
$$

As a result, we can formulate the following proposition.

Proposition 2.10. *The expressions (2.36) and (2.39) are the linear matrix representations in the space* $\mathbb{R}^4((x,t))$ *of the differentiations* $D_t : \mathbb{R}((x,t)) \to \mathbb{R}((x,t))$ *and* $D_x : \mathbb{R}((x,t)) \to \mathbb{R}((x,t))$*, respectively, reduced upon the invariant Lax differential ideal* $L[u]((x,t)) \subset \mathbb{R}((x,t))$*, given by (2.8).*

The following Lax type integrability theorem holds.

Theorem 2.11. *The dynamical system (2.43), equivalent to the generalized Riemann type hydrodynamical system (2.33), possesses the Lax type representation*

$$f_x = \ell[u,v,z,w;\lambda]f, \quad f_t = p(\ell)f, \quad p(\ell) := -u\ell[u,v,w,z;\lambda] + q(\lambda), \tag{2.44}$$

where

$$\ell[u,v,w,z;\lambda] := \begin{pmatrix} -\lambda^3 u_x & \lambda^2 v_x & -\lambda w_x & z_x \\ -4\lambda^4 & 3\lambda^3 u_x & -2\lambda^2 v_x & \lambda w_x \\ -10\lambda^5 r_1 & 6\lambda^4 & -3\lambda^3 u_x & \lambda^2 v_x \\ -20\lambda^6 r_2 & 10\lambda^5 r_1 & -4\lambda^4 & \lambda^3 u_x \end{pmatrix}, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix},$$

$$p(\ell) = \begin{pmatrix} \lambda u u_x & -\lambda^2 u v_x & \lambda u w_x & -u z_x \\ \lambda + 4\lambda^4 u & -3\lambda^3 u u_x & 2\lambda^2 u v_x & -\lambda u w_x \\ 10\lambda^5 u r_1 & \lambda - 6\lambda^4 u & 3\lambda^3 u u_x & -\lambda^2 u v_x \\ 20\lambda^6 u r_2 & -10\lambda^5 u r_1 & \lambda + 4\lambda^4 u & -\lambda^3 u u_x \end{pmatrix}, \tag{2.45}$$

thereby being a Lax type integrable dynamical system on the functional manifold $\mathcal{M}$.

The result obtained above can be easily generalized on the case of an arbitrary integer $N \in \mathbb{Z}_+$, thereby proving the Lax type integrability of the whole hierarchy of the Riemann type hydrodynamical equation (2.2). The related calculations will be presented and discussed in other work. Here we only do the next remark.

Remark 2.12. The Riemann type hydrodynamical equation (2.2) as $N \to \infty$ can be equivalently rewritten as the following Benney type [14, 15, 4] chain

$$D_t u^{(n)} = u^{(n+1)}, \quad D_t := \partial/\partial t + u^{(0)}\partial/\partial x, \tag{2.46}$$

for the suitably constructed moment functions $u^{(n)} := D_t^n u^{(0)}, u^{(0)} := u \in \mathbb{R}((x,t)), n \in \mathbb{Z}_+$.

This aspect of the problem, looking very interesting, we also expect to treat in detail by means of the differential-geometric tools elsewhere later.

3. The differential-algebraic analysis of the Lax type integrability of the Korteweg-de Vries dynamical system

3.1. The differential-algebraic problem setting. We consider the well known Korteweg-de Vries equation in the following differential-algebraic form:

$$D_t u - D_x^3 u = 0, \tag{3.1}$$

where $u \in \mathbb{R}((x,t))$ and the differentiations $D_t := \partial/\partial t + u\partial/\partial x, D_x := \partial/\partial x$ satisfy the commutation condition (2.1):

$$[D_x, D_t] = (D_x u)D_x. \tag{3.2}$$

We will also interpret relationship (3.1) as a nonlinear dynamical system

$$D_t u = D_{xxx} u \tag{3.3}$$

on a suitably chosen functional manifold $\mathcal{M} \subset \mathbb{R}((x,t))$.

Based on the expression (3.1) we can easily construct a suitable invariant KdV-differential ideal $KdV[u]((x,t)) \subset \mathbb{R}((x,t))$ as follows:

$$KdV[u]((x,t)) \quad : \quad = \{ \sum_{k=\overline{0,2}} \sum_{n \in \mathbb{Z}_+} f_n^{(k)} D_x^k D_t^n u \in \mathbb{R}((x,t)) : \; D_t u - D_x^3 u = 0,$$

$$(3.4) \qquad f_n^{(k)} \quad \in \quad \mathbb{R}((x,t)), k = \overline{0,2}, n \in \mathbb{Z}_+ \} \subset \mathbb{R}((x,t)).$$

As the next step we need to find the kernel $Ker \; D_t \subset KdV[u]((x,t))$ of the differentiation D_t : $\mathbb{R}((x,t)) \to \mathbb{R}((x,t))$, reduced upon the KdV-differential ideal (3.4), we obtain by means of easy calculations that it is generated by the following differential relationships:

$$D_t f^{(0)} \quad = \quad -\lambda f^{(0)}, \;\; D_t f^{(2)} = -\lambda f^{(2)} \; + 2 f^{(2)} D_x u,$$

$$(3.5) \qquad D_t f^{(1)} \quad = \quad -\lambda f^{(1)} \; + f^{(1)} D_x u + f^{(2)} D_{xx} u,$$

where, by definition, $f^{(k)} := f^{(k)}(\lambda) = \sum_{n \in \mathbb{Z}_+} f_n^{(k)} \lambda^n \in \mathbb{R}((x,t)), k = \overline{0,2}$, and $\lambda \in \mathbb{R}$ is an arbitrary parameter. Based on the relationships (3.5) the following proposition holds.

Proposition 3.1. *The differential relationships (3.5) can be equivalently rewritten in the following linear matrix form:*

$$D_t f = q(\lambda) f, \quad q(\lambda) := \begin{pmatrix} D_x u - \lambda & D_{xx} u \\ 0 & D_x u - \lambda \end{pmatrix}, \tag{3.6}$$

where $f := (f_1, f_2)^\intercal \in \mathbb{R}^2((x,t))$, $\lambda \in \mathbb{R}$, *giving rise to the corresponding linear matrix representation in the space* $\mathbb{R}^2((x,t))$ *of the differentiation* $D_t : \mathbb{R}((x,t)) \to \mathbb{R}((x,t))$, *reduced upon the KDV-differential ideal (3.4).*

3.2. **The Lax type representation.** Now, making use of the matrix differential relationship (3.6), we can construct the related with ideal (3.4) Lax differential ideal

$$\begin{aligned} L[u]((x,t)) \quad &: \quad = \{< g, f >_{\mathbb{E}^2} \in \mathbb{R}((x,t)) : D_t f = q(\lambda) f, \\ f, g \quad &\in \quad \mathbb{R}^3((x,t))\} \subset \mathbb{R}((x,t)), \end{aligned} \tag{3.7}$$

where $< \cdot, \cdot >_{\mathbb{E}^3}$ denotes the standard scalar product in the Euclidean real space $\mathbb{E}^2$. Since the Lax differential ideal (3.7) is, by construction, D_t-invariant and isomorphic to the D_t- and D_x-invariant KDV-differential ideal (3.4), it is necessary to check its D_x-invariance. As a result of this condition the following differential relationship

$$D_x f = \ell[u; \lambda] f, \ \ \ell[u; \lambda] := \begin{pmatrix} D_x \tilde{a} & 2 D_{xx} \tilde{a}. \\ 0 & -D_x \tilde{a} \end{pmatrix} \tag{3.8}$$

holds, where the mapping $\tilde{a} : \mathcal{M} \to \mathbb{R}((x,t))$ satisfies the functional-differential relationships

$$D_t \tilde{a} = 1, D_t u - D_x^3 u = 0, \tag{3.9}$$

and the matrix $\ell := \ell[u;\lambda] : \mathbb{R}^2((x,t)) \to \mathbb{R}^2((x,t))$ satisfies for all $\lambda \in \mathbb{R}$ the determining functional-differential equation

$$D_t\ell + \ell D_x u = [q(\lambda),\ell] + D_x q(\lambda), \tag{3.10}$$

generalizing the similar equation (2.11). The result obtained above we will formulate as the following proposition.

Theorem 3.2. *The differentials $D_t : \mathbb{R}((x,t)) \to \mathbb{R}((x,t))$ and $D_x : \mathbb{R}((x,t)) \to \mathbb{R}((x,t))$ of the differential ring $\mathbb{R}((x,t))$, reduced upon the Lax differential ideal $L[u]((x,t)) \subset \mathbb{R}((x,t))$, isomorphic to the KDV-differential ideal $KdV[u]((x,t)) \subset \mathbb{R}((x,t))$, allow the compatible Lax type representation*

$$\begin{aligned} D_t f &= q(\lambda)f, \;\; q(\lambda) := \begin{pmatrix} D_x u - \lambda & D_{xx}u \\ 0 & D_x u - \lambda \end{pmatrix}, \\ D_x f &= \ell[u;\lambda]f, \;\; \ell[u;\lambda] := \begin{pmatrix} D_x\tilde{a} & 2D_{xx}\tilde{a}. \\ 0 & -D_x\tilde{a} \end{pmatrix}, \end{aligned} \tag{3.11}$$

where the mapping $\tilde{a} : \mathcal{M} \to \mathbb{R}((x,t))$ satisfies the functional-differential relationships (3.9), $f \in \mathbb{R}^2((x,t))$, $\lambda \in \mathbb{R}$, generated by the invariant Lax differential ideal $L[u]((x,t)) \subset \mathbb{R}((x,t))$.

It is interesting to mention that the Lax type representation (3.11) strongly differs from that given by the well known [16] classical expressions

$$D_t f = q_{cl}(\lambda) f, \; q_{cl}(\lambda) := \begin{pmatrix} D_x u/6 & -(2u/3 - 4\lambda) \\ D_{xx} u/6 - (u/6-\lambda)\times \\ \times(2u/3-4\lambda) & -11 D_x u/6 \end{pmatrix},$$

$$D_x f = \ell_{cl}[u;\lambda] f, \; \ell_{cl}[u;\lambda] := \begin{pmatrix} 0 & 1 \\ u/6-\lambda & 0 \end{pmatrix}, \tag{3.12}$$

where, as above, the following functional-differential equation

$$D_t \ell_{cl} + \ell_{cl} D_x u = [q_{cl}(\lambda), \ell_{cl}] + D_x q_{cl}(\lambda), \tag{3.13}$$

holds for any $\lambda \in \mathbb{R}$, being exactly equivalent to the nonlinear dynamical system (3.3) on the functional manifold $\mathcal{M}$. Thus, a problem of constructing a suitable KDV-differential ideal $KdV[u]((x,t)) \subset \mathbb{R}((x,t))$, generating the corresponding Lax type differential ideal $L[u]((x,t)) \subset \mathbb{R}((x,t))$, invariant with respect to the differential representations (3.12), naturally arises, what we expect to treat in detail elsewhere later. There also is a very interesting problem of the differential-algebraic analysis of the related symplectic structures on the functional manifold $\mathcal{M}$, with respect to which the dynamical system (3.3) is Hamiltonian and suitably integrable.

4. Conclusion

As one could be convinced by the results obtained in this work, the differential-algebraic tools, when applied to a given set of differential relationships, based on the basic differentiations D_t and $D_x : \mathbb{R}((x,t)) \to \mathbb{R}((x,t))$ in the differential ring $\mathbb{R}((x,t))$ and parameterized by a fixed element $u \in \mathbb{R}((x,t))$, make it possible to construct the corresponding Lax type representation as that, realizing the linear matrix representations of the mentioned above basic differentiations. This scheme was elaborated in detail for the generalized Riemann type differential equation (2.2) and for the classical Korteweg-de Vries equation (3.3). As these equations are equivalent to the corresponding Hamiltonian systems with respect to suitable symplectic structures, this aspect presents a very interesting problem from the differential-algebraic point of view and is planned to be studied elsewhere.

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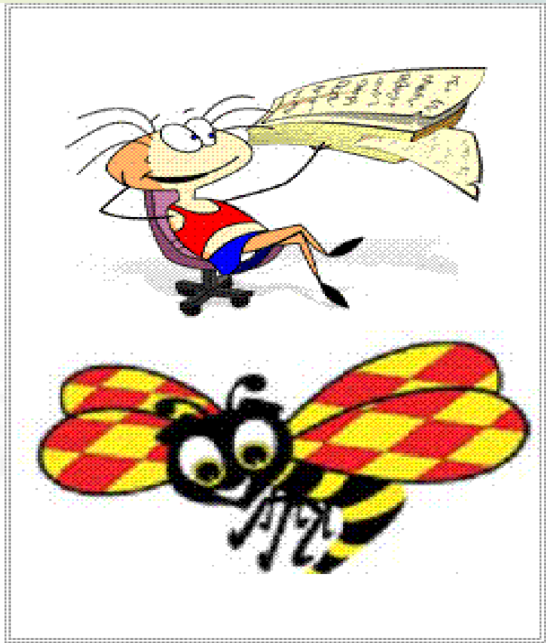
All the best in your mathematical aspirations!



Thanks for attention!