Isoptics of pairs of open rosettes with common asymptotes

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Isoptics of a closed strictly convex curve

W. Cieślak, A. Miernowski, W. Mozgawa

origin 0 in the interior of C. Let $p(t)$, $t \in [0, 2\pi]$, be the distance from 0 to the support line $l(t)$ to C perpendicular to the vector e^{it}. The function p is called a support function of the curve C. It is well known (cf. [2]) that the support function is differentiable and that the parametrization of C in terms of this function is given by

(1.1)
$$z(t) = p(t)e^{it} + \dot{p}(t)ie^{it} \quad \text{for} \quad t \in [0, 2\pi].$$

Let C_α be a locus of vertices of a fixed angle $\pi - \alpha$ formed by two support lines of the curve C. The curve C_α will be called an alpha-isoptic of C.

Next we introduce the following notations:

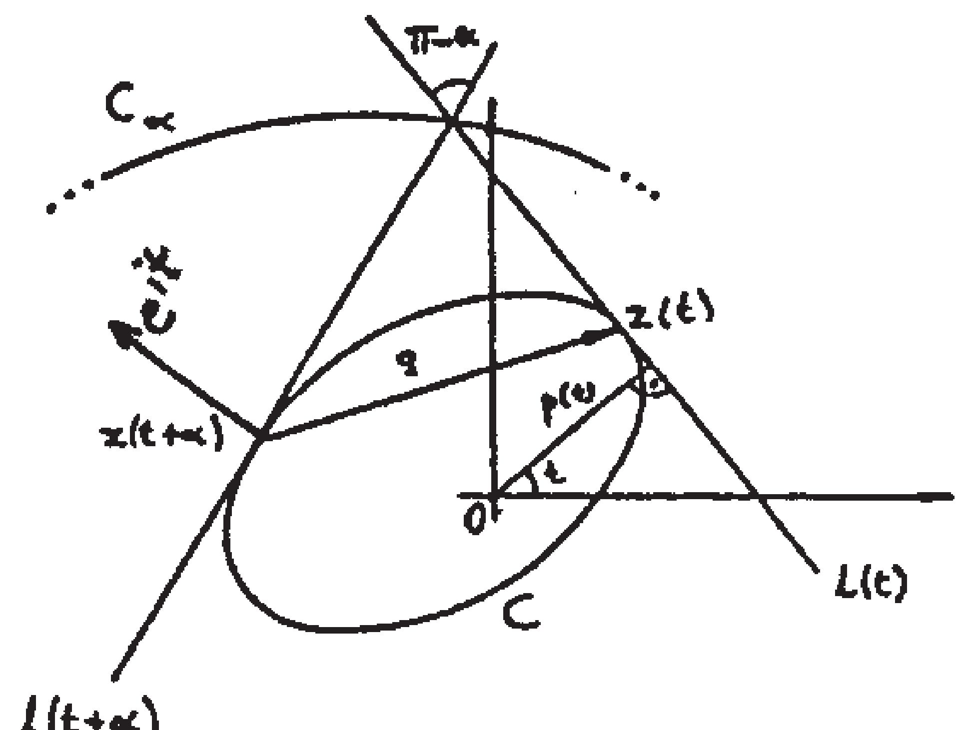

Figure 1

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Sine theorem for rosettes

S. Góźdź, A. Miernowski, W. Mozgawa

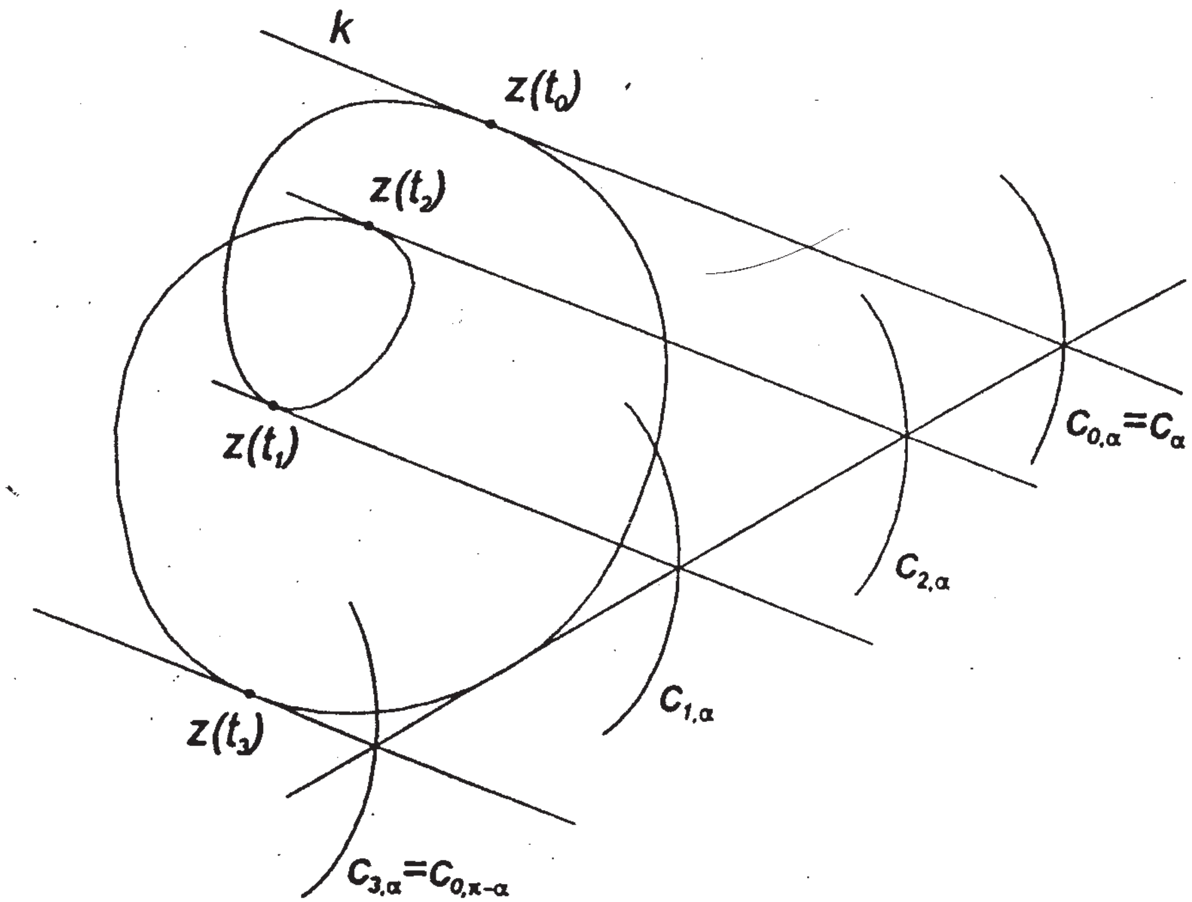
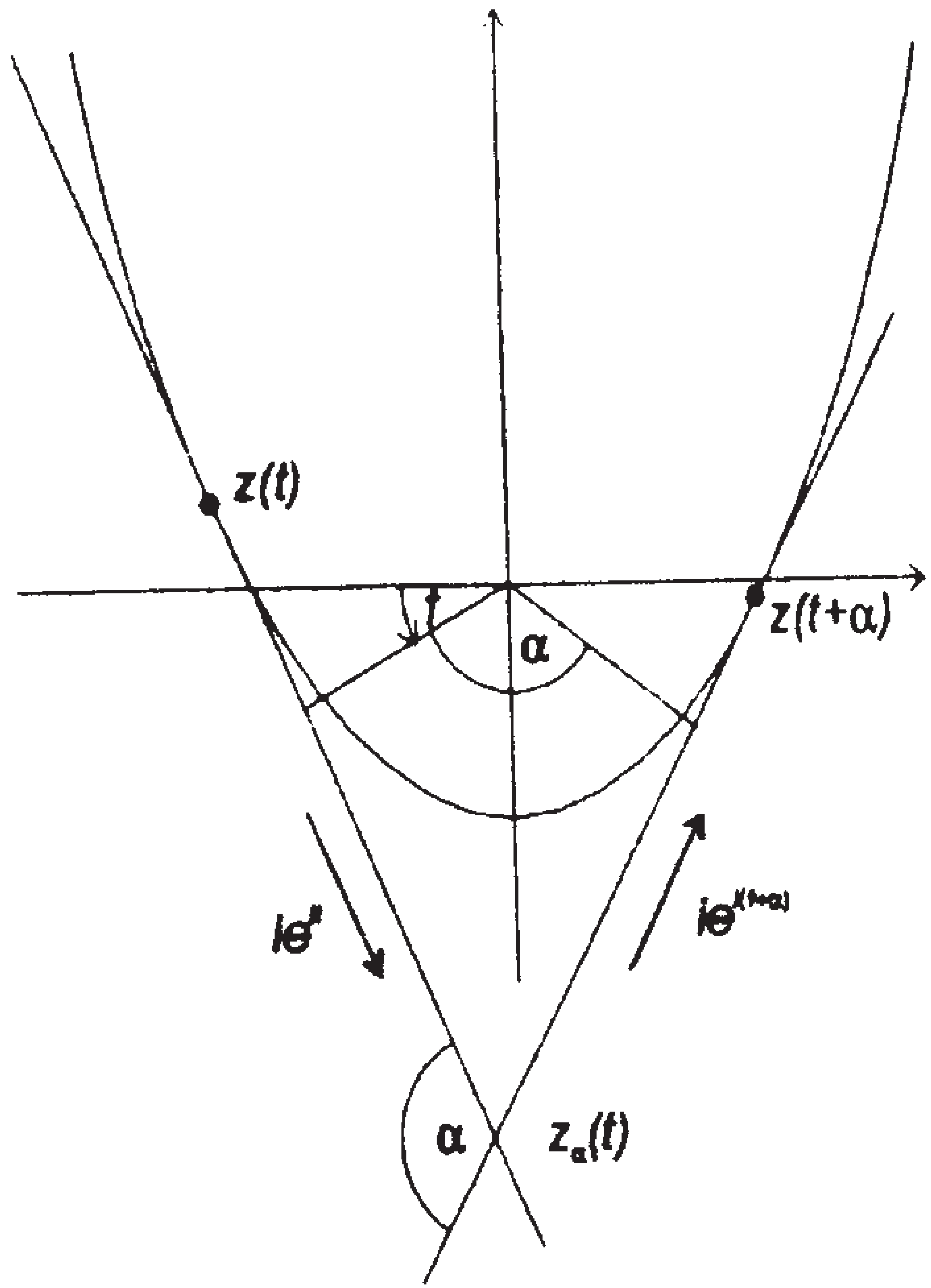


Fig. 2

Definition 2.1. The cut locus of the intersection points of the tangent lines at t and t_k is said to be an isoptic of the type (k, α) and is denoted $C_{k,\alpha}$.

Istanbul Univ. Fen Fak. Mat. Dergisi 57-58 (1998-1999), 33-39
Isoptics of open, convex curves and Crofton-type formulas
A. Miernowski, W. Mozgawa



Beitrage zur Algebra und Geometrie
Volume 42 (2001), No. 1, 281-288
Isoptics of pairs of nested closed strictly convex curves and Crofton-type formulas
A. Miernowski, W. Mozgawa

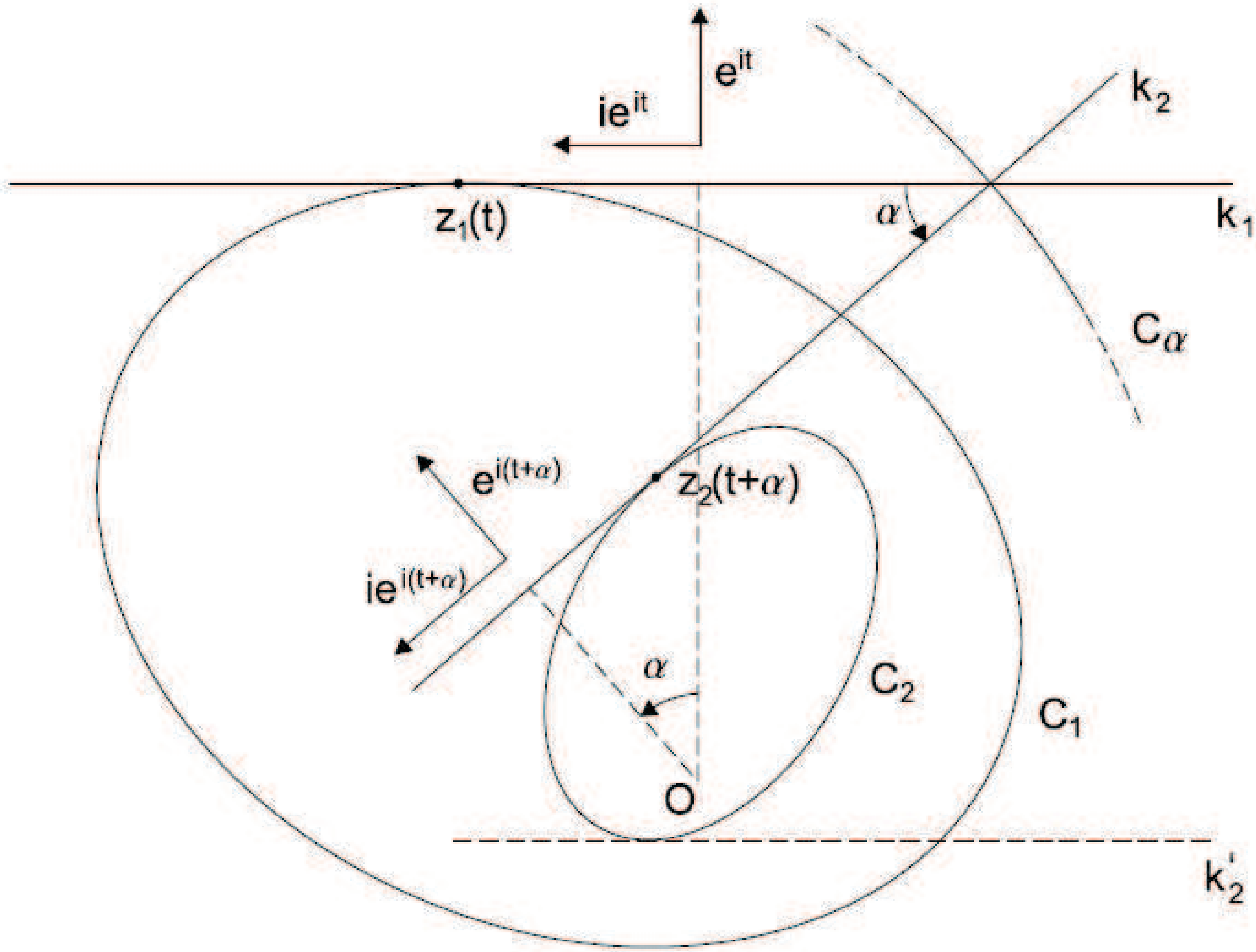


Figure 1.1

Annales Universitatis Mariae Curie-Skłodowska, Lublin –
Polonia,
Vol. LIX, 2005, Sectio A, 119-128
Isoptics of open rosettes
D. Szałkowski

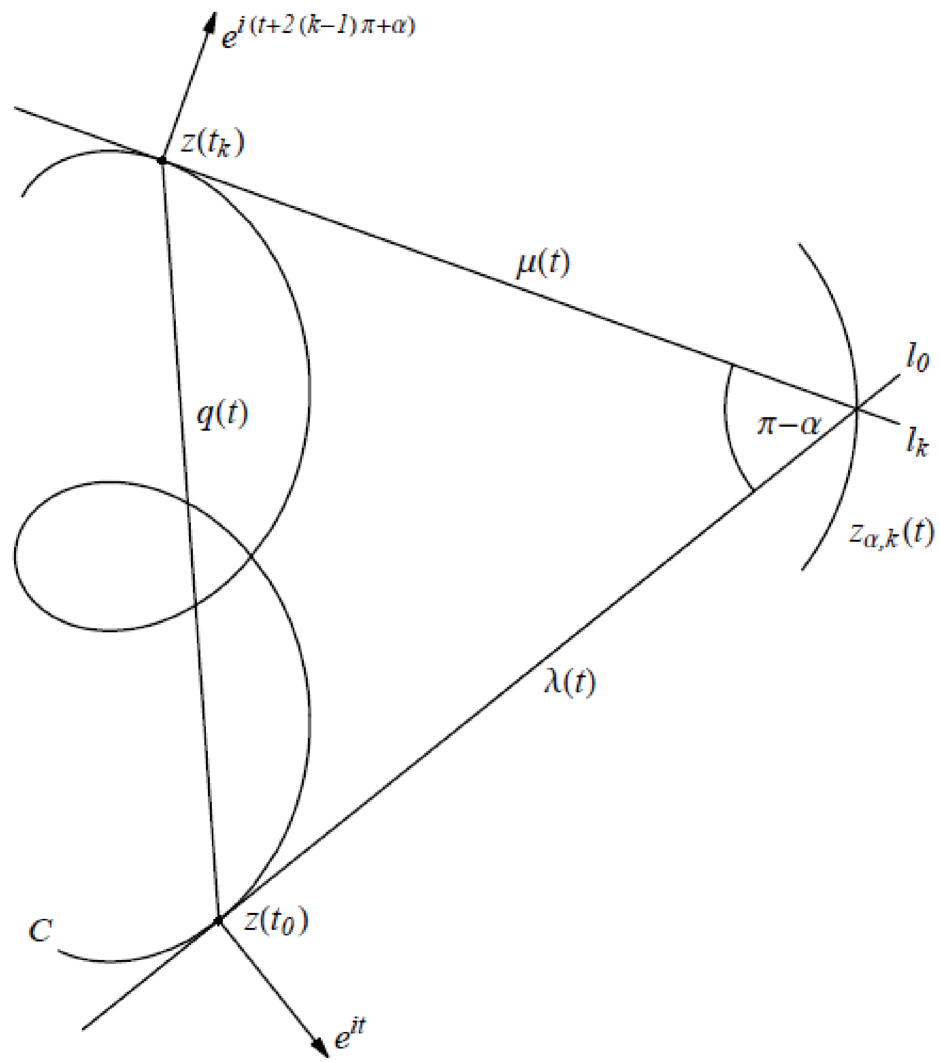


FIGURE 6. Parametrization of an k, α-isoptic.

To appear

Singular points of isoptics of open rosettes

D. Szałkowski

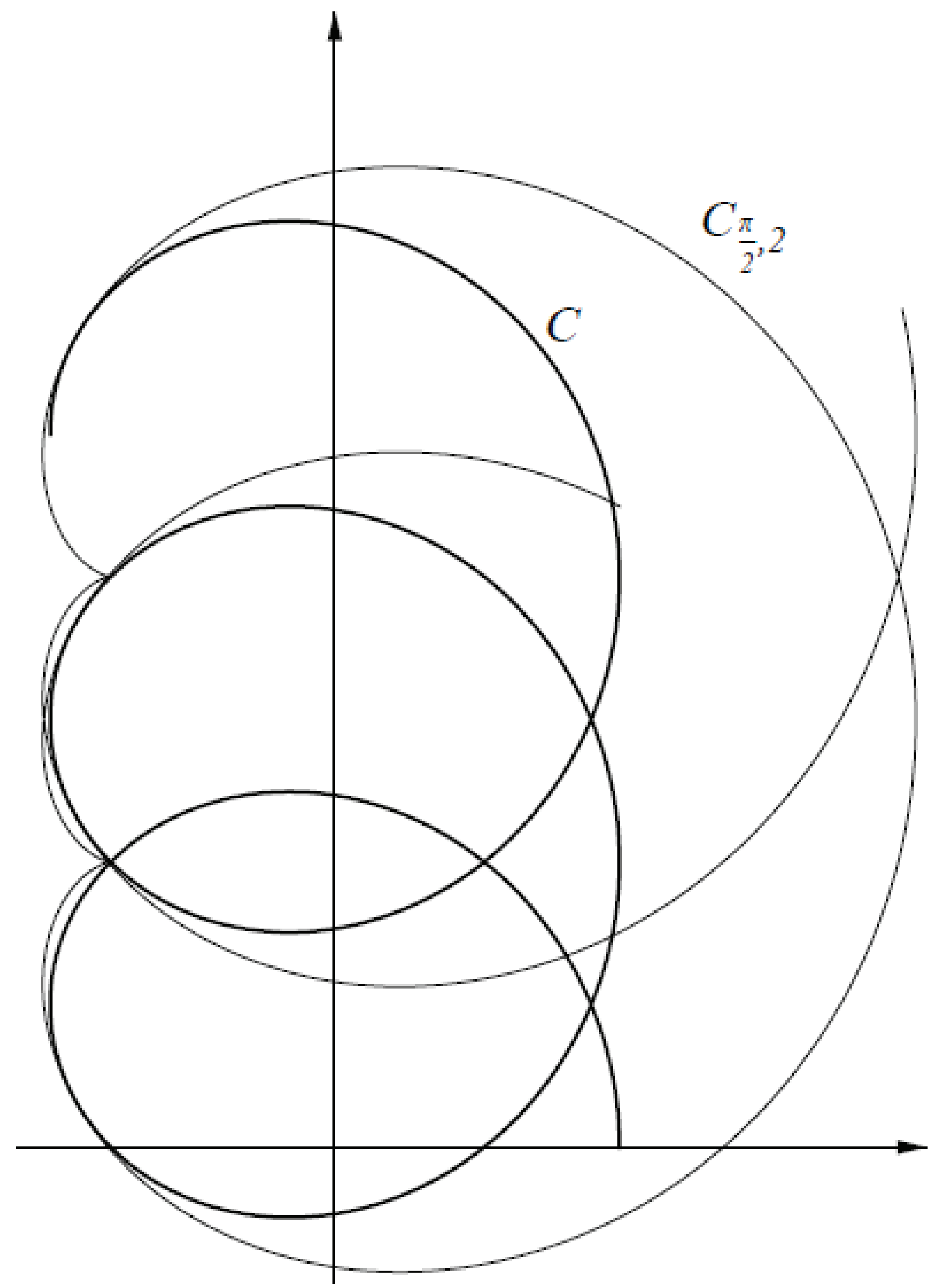


FIGURE 4. Open rosette and its orthoptic of second order from example 1

Definition

An open rosette without self intersections is said to be a simple open rosette.

Definition

A pair of open rosettes (C_1, C_2) which satisfy conditions:

- rosettes C_1 and C_2 are simple open rosettes,
- asymptotes A_1, A_2 are given by $y = kx$ and $y = -kx$, $k \in (0, \infty)$,
- rosette C_1 lies between asymptotes A_1 and A_2 in II and III quadrant of the coordinate system, rosette C_2 lies between asymptotes A_1 and A_2 in I and IV quadrant of the coordinate system,
- $p_1(t)$ and $p_2(t)$ are support functions of C_1 i C_2 and they are defined on the same interval $(-\beta, \beta)$, where $\beta = \dfrac{\pi}{2} - \operatorname{arc\,tg} k$,

is said to be a pair of open rosettes with common asymptotes.

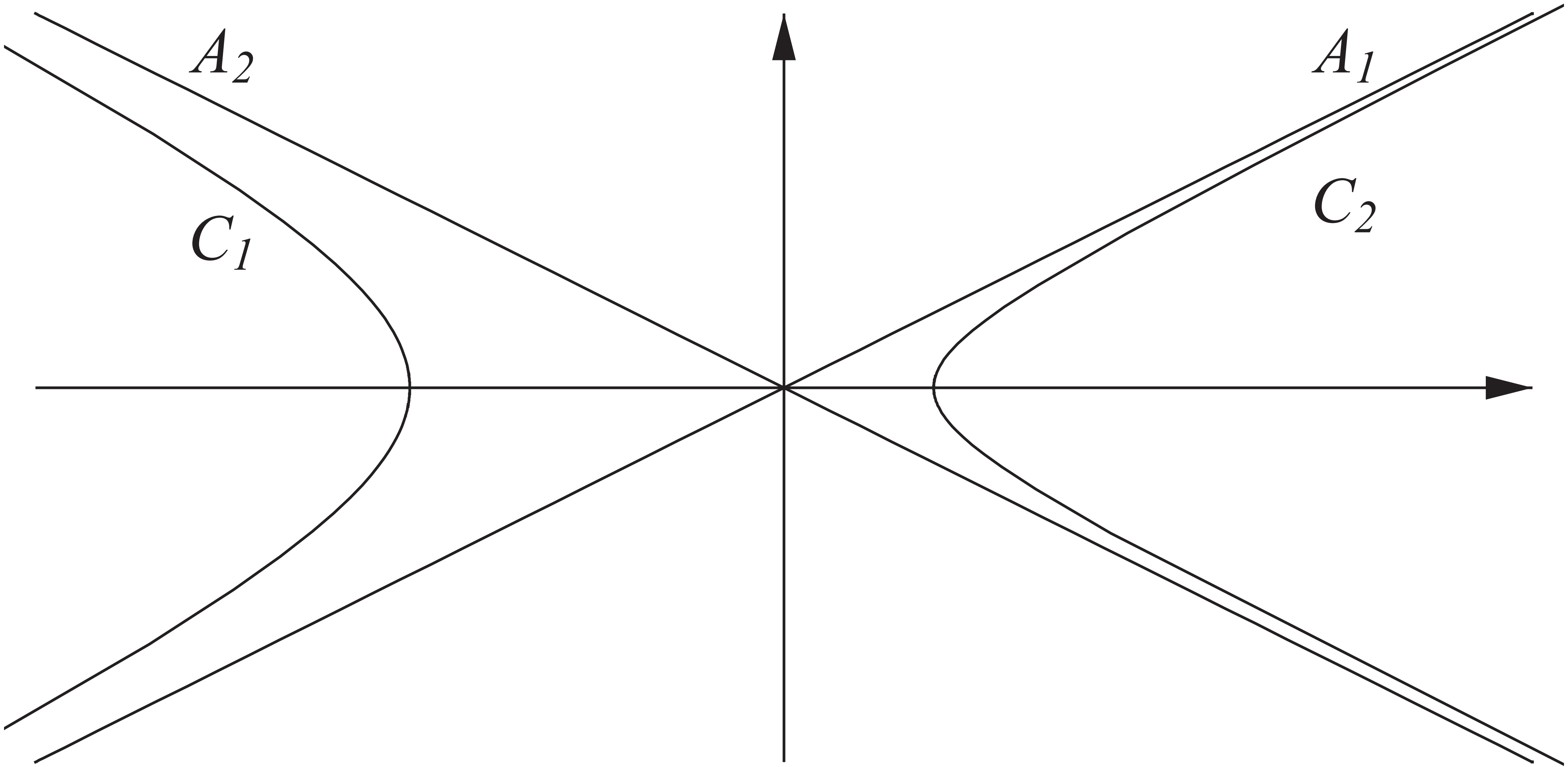


Figure: A pair of open rosettes with common asymptotes

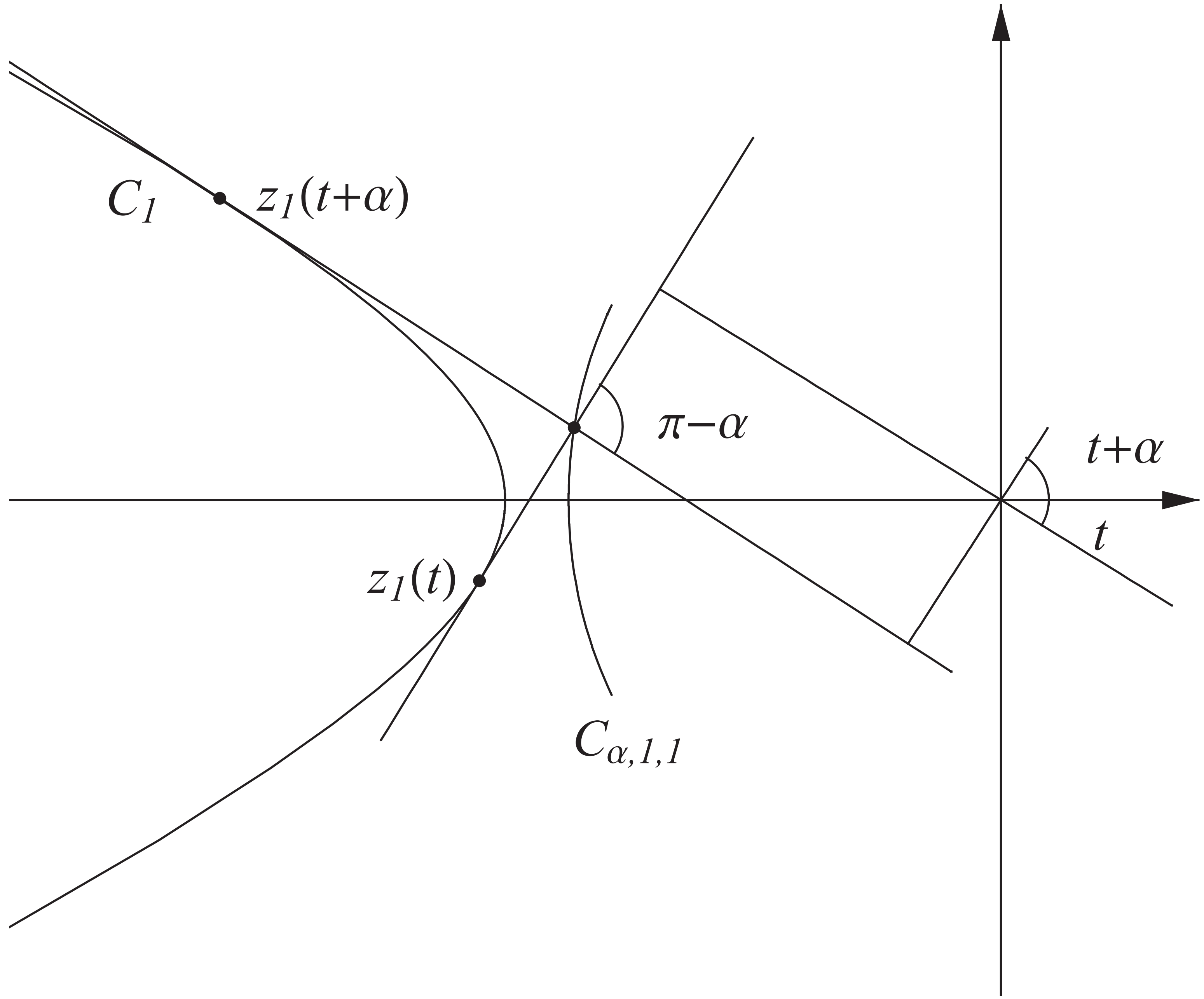


Figure: Section $C_{\alpha,1,1}$ of α-isoptic

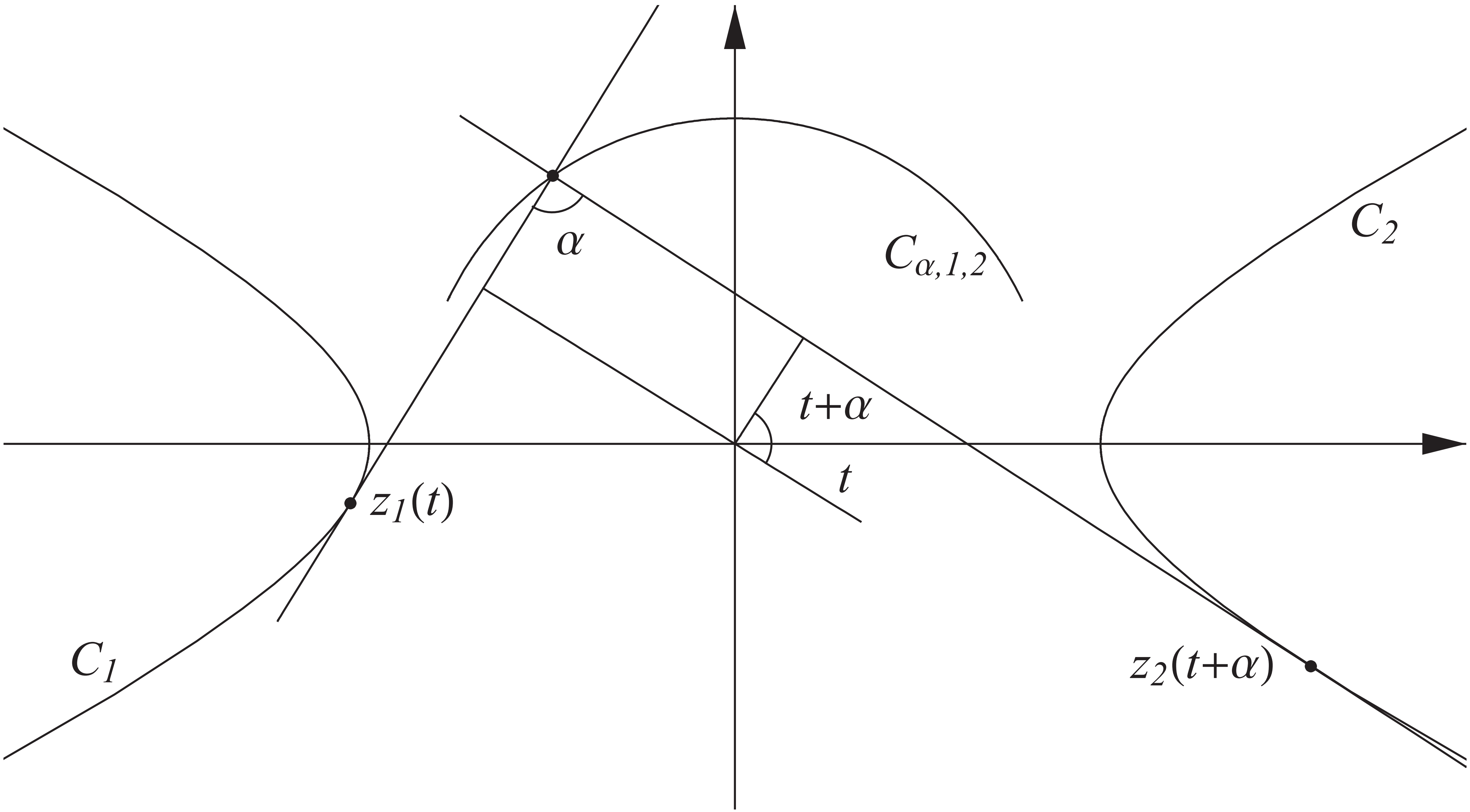


Figure: Section $C_{\alpha,1,2}$ of α-isoptic

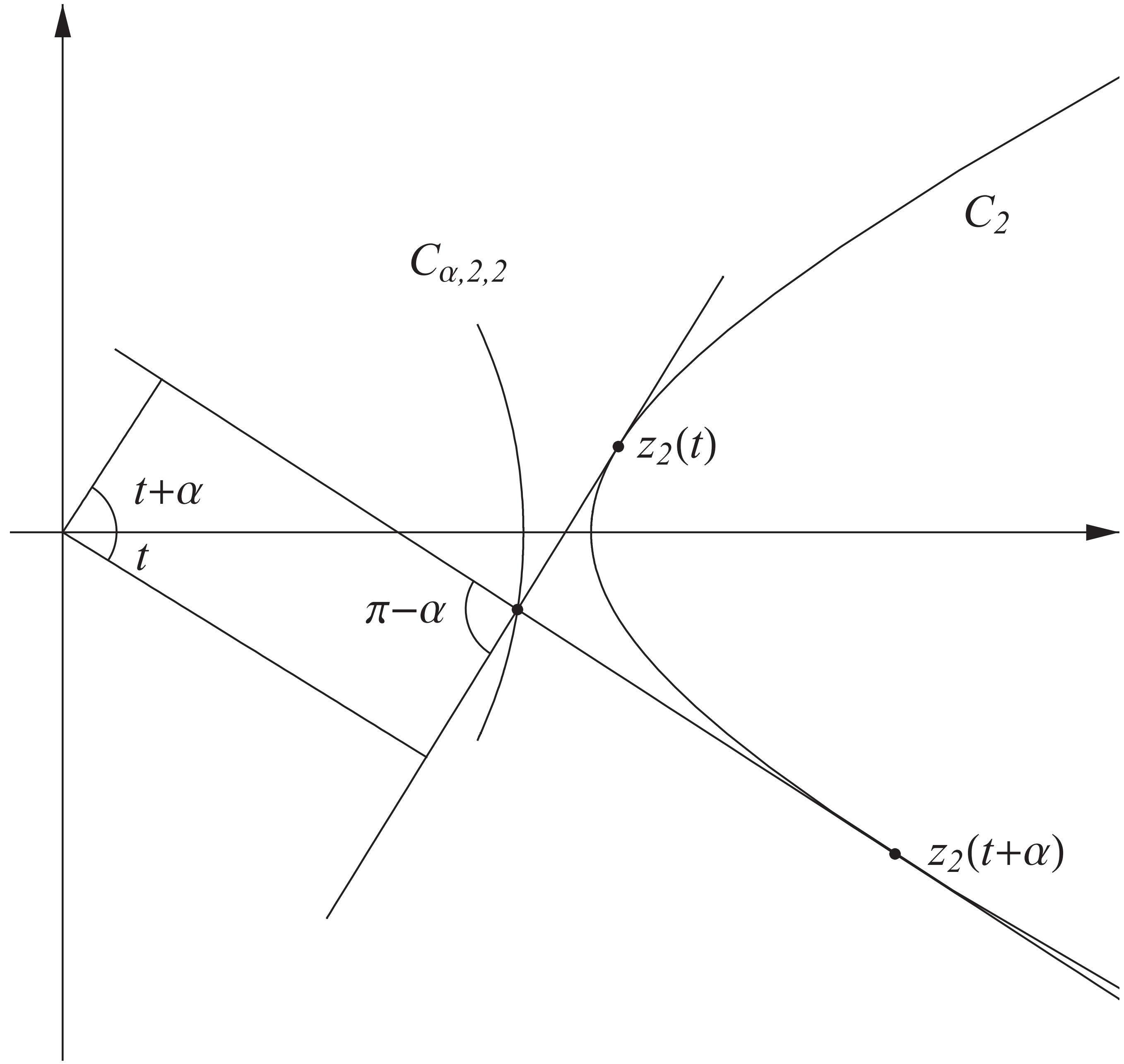


Figure: Section $C_{\alpha,2,2}$ of α-isoptic

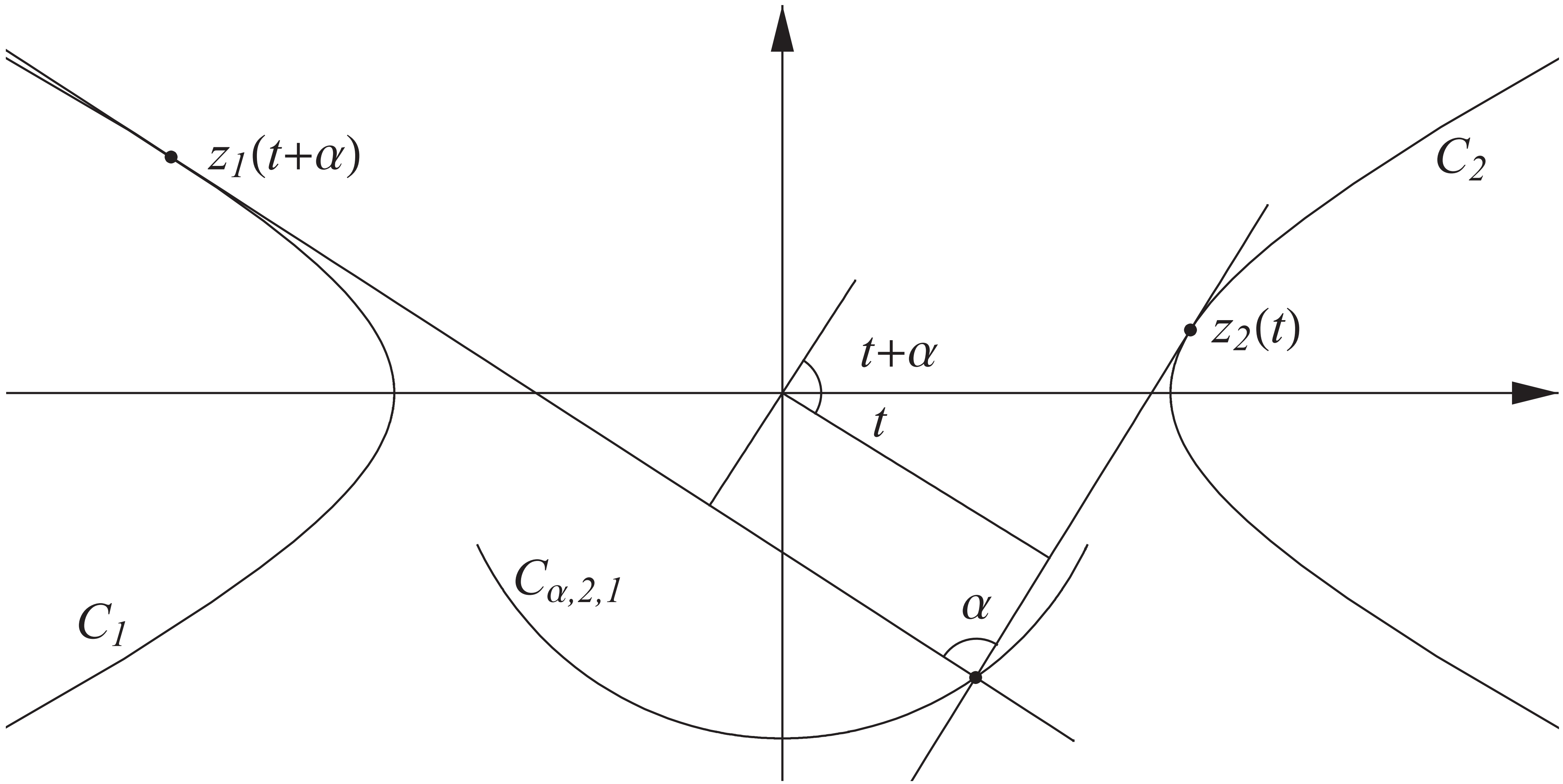


Figure: Section $C_{\alpha,2,1}$ of α-isoptic

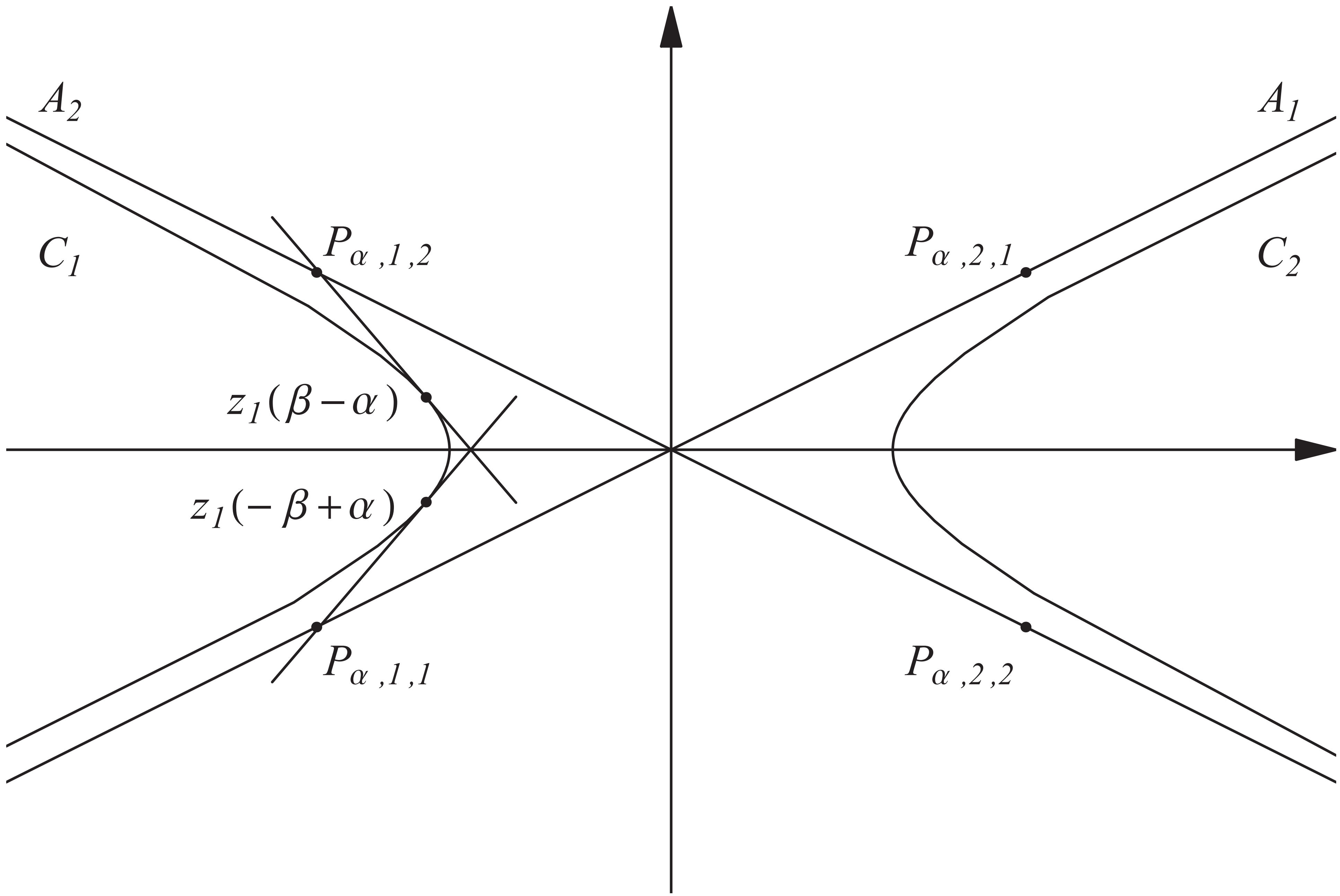


Figure: Asymptotic points of α-isoptic of pair of open rosettes with common asymptotes

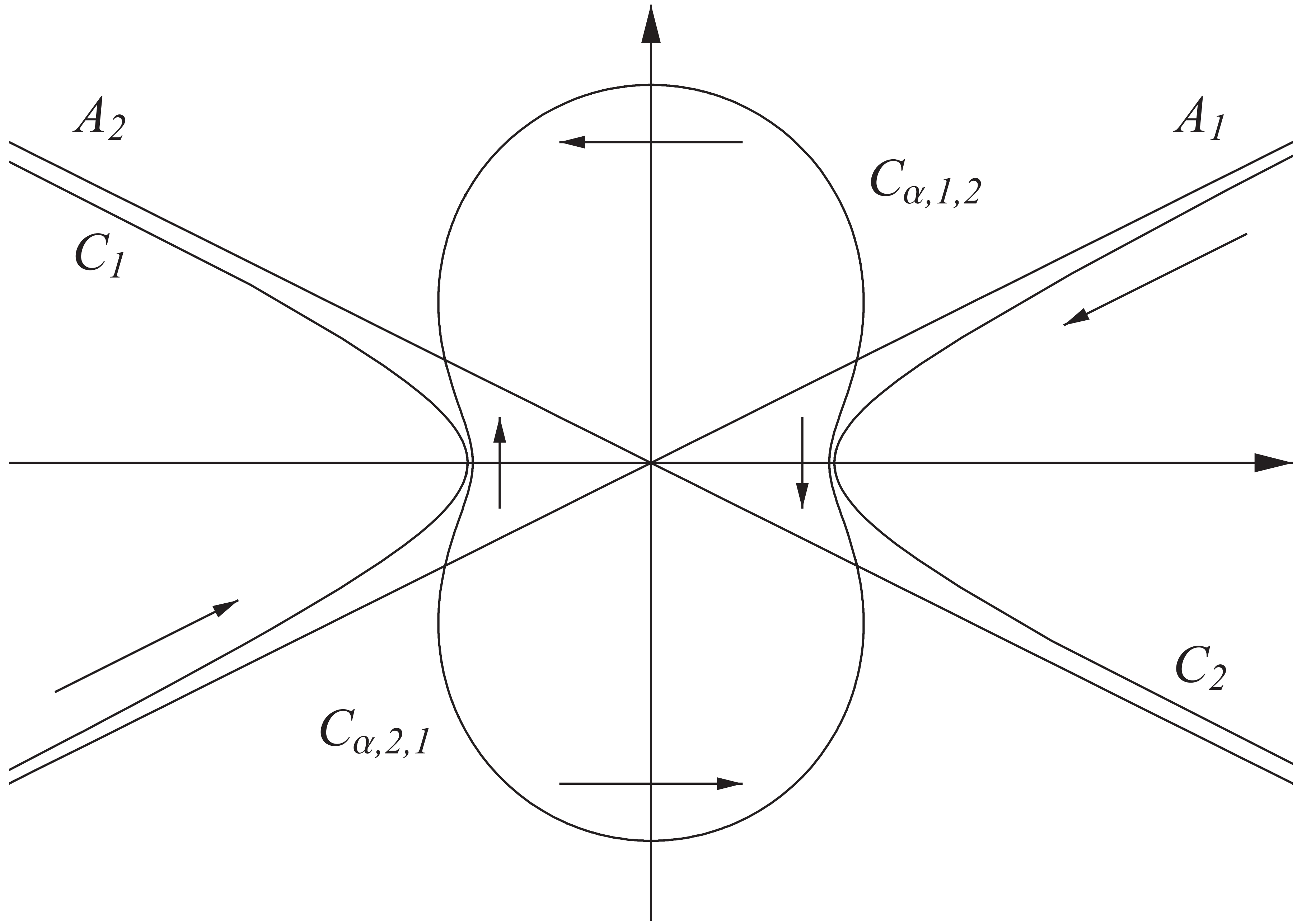


Figure: Orientation of rosettes C_1, C_2 and sections of C_α

Theorem

Let (C_1, C_2) *be a pair of open rosettes with common asymptotes and let* C_α *be its* α*-isoptic for* $\alpha \in (0, 2\beta)$. *Then the parametrization of section* $C_{\alpha,m,n}$ *of isoptic is given by*

$$C_{\alpha,m,n}\colon z_{\alpha,m,n}(t) = p_m(t)e^{it} + \left(-p_m(t)\cot\alpha + \frac{p_n(t+\alpha)}{\sin\alpha}\right) ie^{it},$$

where $t \in (-\beta, \beta - \alpha)$, $m = 1, 2$, $n = 1, 2$.

Some useful functions

$$
\begin{aligned}
q_{\alpha,m,n}(t) = {} & p_m(t)\cos t - p'_m(t)\sin t \\
& - p_n(t+\alpha)\cos(t+\alpha) + p'_n(t+\alpha)\sin(t+\alpha) \\
& + i(p_m(t)\sin t + p'_m(t)\cos t \\
& - p_n(t+\alpha)\sin(t+\alpha) - p'_n(t+\alpha)\cos(t+\alpha), \\
\lambda_{\alpha,m,n}(t) = {} & -p'_m(t) - p_m(t)\cot\alpha + \frac{p_n(t+\alpha)}{\sin\alpha}, \\
\mu_{\alpha,m,n}(t) = {} & -\frac{p_m(t)}{\sin\alpha} + p_n(t+\alpha)\cot\alpha - p'_n(t+\alpha), \\
\varrho_{\alpha,m,n}(t) = {} & p_m(t) - p'_m(t)\cot\alpha + \frac{p'_n(t+\alpha)}{\sin\alpha}.
\end{aligned}
$$

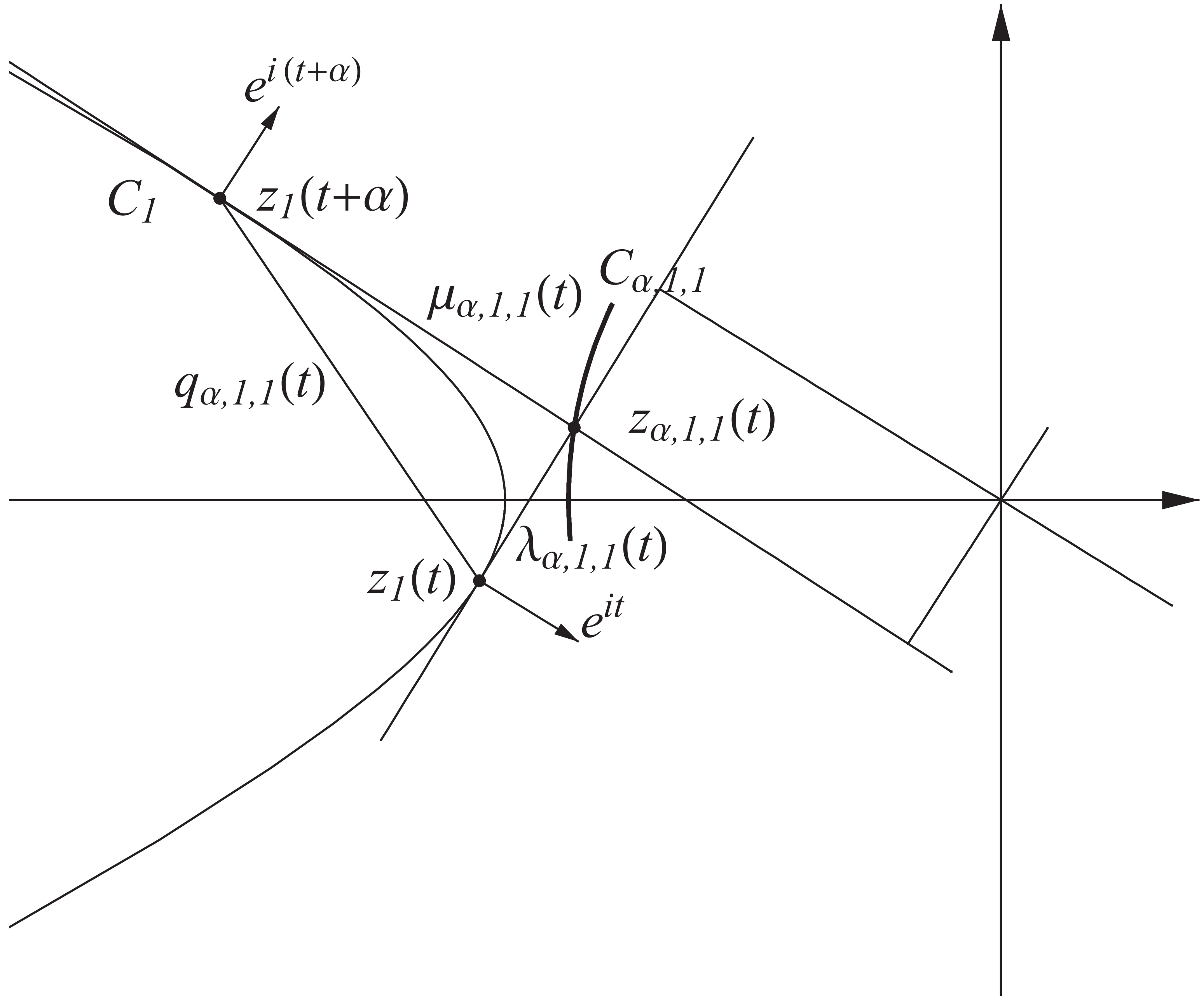


Figure: Parametrization of section $C_{\alpha,1,1}$ of α-isoptic

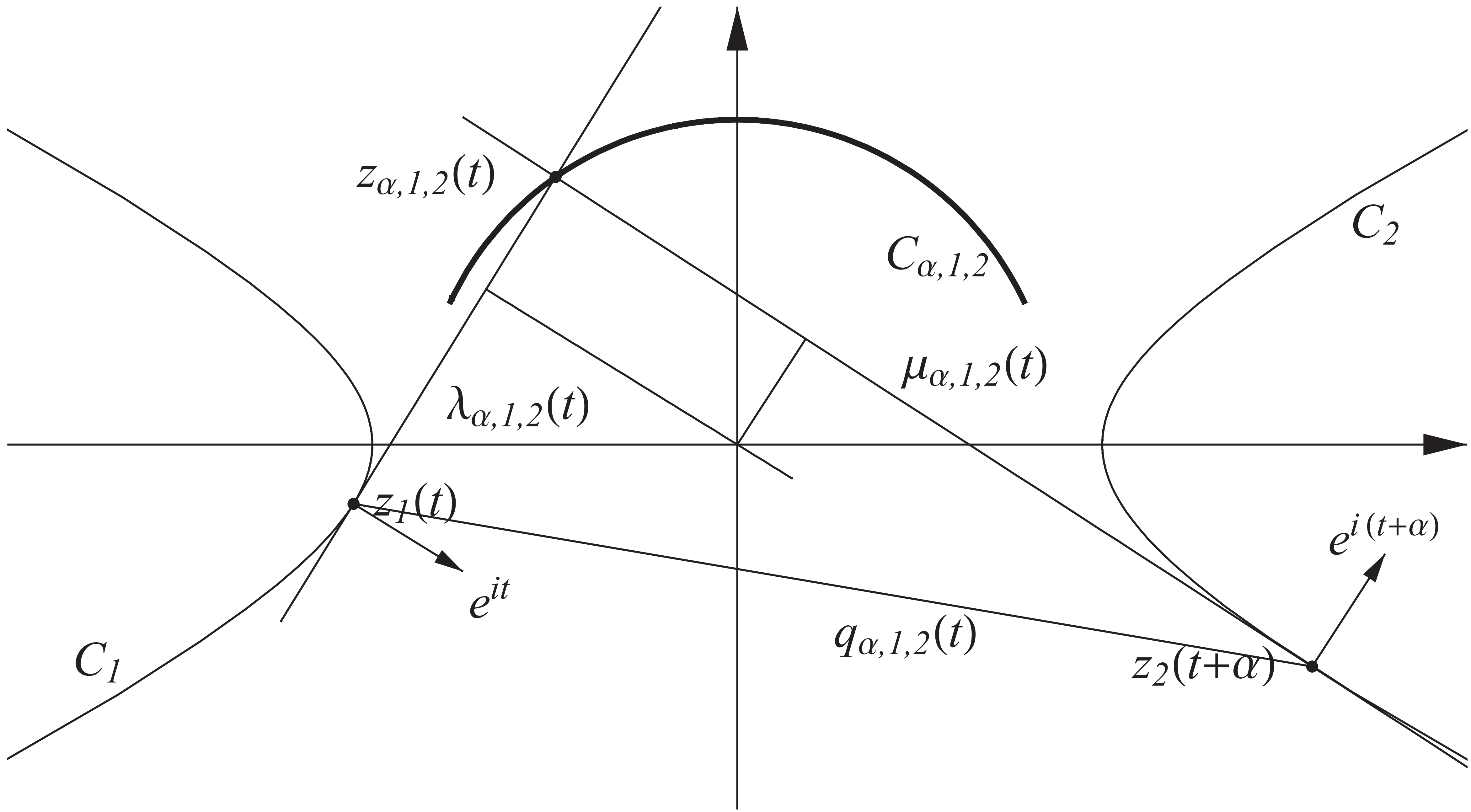


Figure: Parametrization of section $C_{\alpha,1,2}$ of α-isoptic

Theorem

Let (C_1, C_2) *be a pair of open rosettes with common asymptotes and let* C_α *be its isoptic for* $\alpha \in (0, 2\beta)$. *Asymptotic points of the isoptic have the following coordinates*

$$P_{\alpha,1,1}\left(\frac{p_1(-\beta+\alpha)}{\sqrt{1+k^2}\sin\alpha}, k\frac{p_1(-\beta+\alpha)}{\sqrt{1+k^2}\sin\alpha}\right),$$
$$P_{\alpha,1,2}\left(\frac{p_1(\beta-\alpha)}{\sqrt{1+k^2}\sin\alpha}, -k\frac{p_1(\beta-\alpha)}{\sqrt{1+k^2}\sin\alpha}\right),$$
$$P_{\alpha,2,1}\left(\frac{p_2(-\beta+\alpha)}{\sqrt{1+k^2}\sin\alpha}, k\frac{p_2(-\beta+\alpha)}{\sqrt{1+k^2}\sin\alpha}\right),$$
$$P_{\alpha,2,2}\left(\frac{p_2(\beta-\alpha)}{\sqrt{1+k^2}\sin\alpha}, -k\frac{p_2(\beta-\alpha)}{\sqrt{1+k^2}\sin\alpha}\right).$$

Theorem

Let (C_1, C_2) be a pair of open rosettes with common asymptotes and let C_α be its α-isoptic, $\alpha \in (0, 2\beta)$. Then

$$\lim_{t \to -\beta^+} z_{\alpha,1,1}(t) = P_{\alpha,1,1} = \lim_{t \to -\beta^+} z_{\alpha,2,1}(t),$$

$$\lim_{t \to (\beta-\alpha)^-} z_{\alpha,1,1}(t) = P_{\alpha,1,2} = \lim_{t \to (\beta-\alpha)^-} z_{\alpha,1,2}(t),$$

$$\lim_{t \to -\beta^+} z_{\alpha,2,2}(t) = P_{\alpha,2,1} = \lim_{t \to -\beta^+} z_{\alpha,1,2}(t),$$

$$\lim_{t \to (\beta-\alpha)^-} z_{\alpha,2,2}(t) = P_{\alpha,2,2} = \lim_{t \to (\beta-\alpha)^-} z_{\alpha,2,1}(t).$$

Theorem

Let (C_1, C_2) *be a pair of open rosettes with common asymptotes and let* C_α *be its* α*-isoptic,* $\alpha \in (0, 2\beta)$*. Then*

$$\lim_{t\to-\beta^+} \frac{z'_{\alpha,1,1}(t)}{\left|z'_{\alpha,1,1}(t)\right|} = - \lim_{t\to-\beta^+} \frac{z'_{\alpha,2,1}(t)}{\left|z'_{\alpha,2,1}(t)\right|},$$

$$\lim_{t\to(\beta-\alpha)^-} \frac{z'_{\alpha,1,1}(t)}{\left|z'_{\alpha,1,1}(t)\right|} = - \lim_{t\to(\beta-\alpha)^-} \frac{z'_{\alpha,1,2}(t)}{\left|z'_{\alpha,1,2}(t)\right|},$$

$$\lim_{t\to-\beta^+} \frac{z'_{\alpha,2,2}(t)}{\left|z'_{\alpha,2,2}(t)\right|} = - \lim_{t\to-\beta^+} \frac{z'_{\alpha,1,2}(t)}{\left|z'_{\alpha,1,2}(t)\right|},$$

$$\lim_{t\to(\beta-\alpha)^-} \frac{z'_{\alpha,2,2}(t)}{\left|z'_{\alpha,2,2}(t)\right|} = - \lim_{t\to(\beta-\alpha)^-} \frac{z'_{\alpha,2,1}(t)}{\left|z'_{\alpha,2,1}(t)\right|}.$$

Theorem

Let (C_1, C_2) *be a pair of open rosettes with common asymptotes and let* C_α *be its* α*-isoptic, where* $\alpha \in (0, 2\beta)$*. Curvature of section* $C_{\alpha,m,n}$ *is given by*

$$\kappa_{\alpha,m,n}(t) =$$

$$= \frac{\lambda^2_{\alpha,m,n}(t) + \varrho^2_{\alpha,m,n}(t) + \lambda'_{\alpha,m,n}(t)\varrho_{\alpha,m,n}(t) - \lambda_{\alpha,m,n}(t)\varrho'_{\alpha,m,n}(t)}{(\lambda^2_{\alpha,m,n}(t) + \varrho^2_{\alpha,m,n}(t))^{3/2}}$$

where $t \in (-\beta, \beta - \alpha)$*,* $m = 1, 2$*,* $n = 1, 2$*.*

Theorem

Let (C_1, C_2) *be a pair of open rosettes with common asymptotes and let* C_α *be its* α*-isoptic, where* $\alpha \in (0, 2\beta)$*. Curvature of section* $C_{\alpha,m,n}$ *is given by*

$$\kappa_{\alpha,m,n}(t) = \frac{2|q_{\alpha,m,n}(t)|^2 - [q_{\alpha,m,n}(t), q'_{\alpha,m,n}(t)]}{|q_{\alpha,m,n}(t)|^3} \sin\alpha,$$

where $t \in (-\beta, \beta - \alpha)$*,* $m = 1, 2$*,* $n = 1, 2$*.*

Corollary

Let (C_1, C_2) *be a pair of open rosettes with common asymptotes and let* C_α *be its* α*-isoptic, where* $\alpha \in (0, 2\beta)$*. Then:*

- *section* $C_{\alpha,m,n}$ *of* α*-isoptic for* $m \neq n$ *is convex curve, if and only if the following condition holds*

$$2|q_{\alpha,m,n}(t)|^2 > [q_{\alpha,m,n}(t), q'_{\alpha,m,n}(t)],$$

- *section* $C_{\alpha,m,n}$ *of* α*-isoptic for* $m = n$ *is convex curve, if and only if the following condition holds*

$$2|q_{\alpha,m,n}(t)|^2 < [q_{\alpha,m,n}(t), q'_{\alpha,m,n}(t)],$$

for every $t \in (-\beta, \beta - \alpha)$*, where* $m = 1, 2,\ n = 1, 2.$

Theorem

Let (C_1, C_2) *be a pair of open rosettes with common asymptotes and let* C_α *be its* α*-isoptic, where* $\alpha \in (0, 2\beta)$*. Then*

$$\frac{|q_{\alpha,m,n}(t)|}{\sin\alpha} = \frac{|\lambda_{\alpha,m,n}(t)|}{\sin\xi_{m,n}} = \frac{|\mu_{\alpha,m,n}(t)|}{\sin\eta_{m,n}},$$

for $m = 1, 2$*,* $n = 1, 2$*. Angle* $\xi_{m,n}$ *is the angle between vectors* $z'_{\alpha,k}(t)$ *and* ie^{it}*, and angle* $\eta_{m,n}$ *is the angle between vectors* $z'_{\alpha,k}(t)$ *and* $ie^{i(t+\alpha)}$*.*

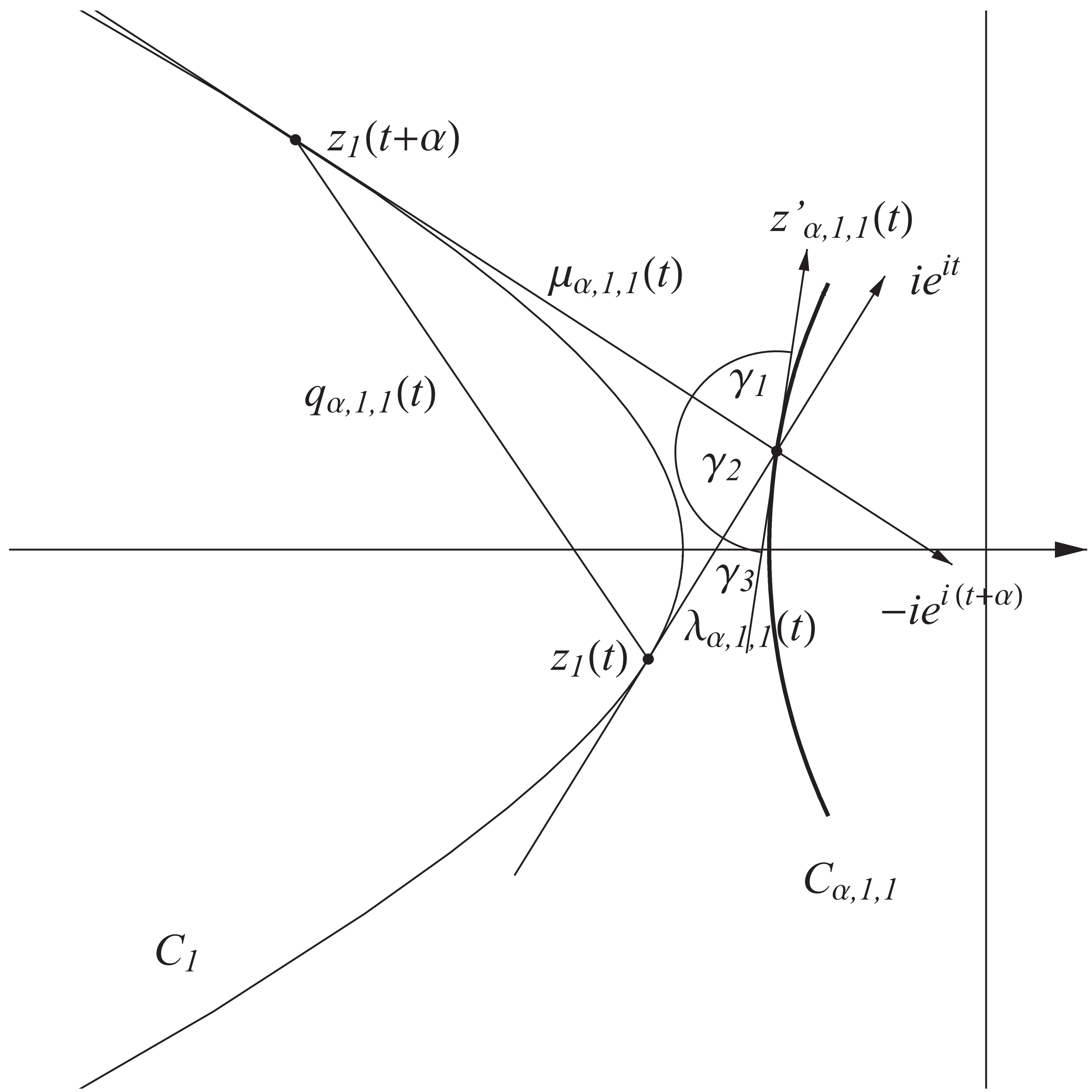


Figure: Sine theorem for section $C_{\alpha,1,1}$ of α-isoptic, $\gamma_1 = \eta_{1,1}$, $\gamma_2 = \pi - \alpha$, $\gamma_3 = \xi_{1,1}$

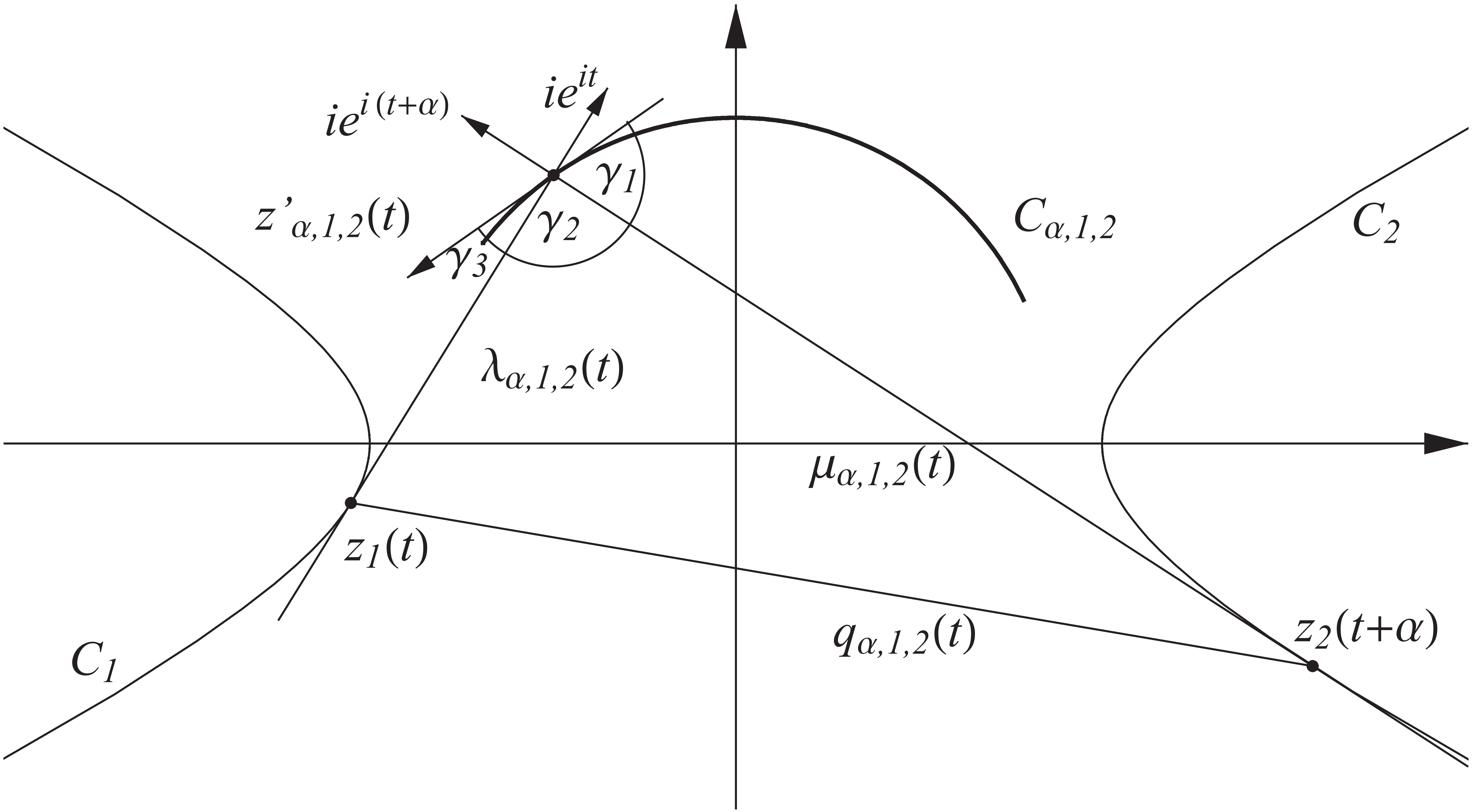


Figure: Sine theorem for section $C_{\alpha,1,2}$ α-isoptic, $\gamma_1 = \eta_{1,2}$, $\gamma_2 = \alpha$, $\gamma_3 = \pi - \xi_{1,2}$

Hyperbole

Let us consider hyperbola C given by the formula

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\ a > 0,\ b > 0.$$

By C_1 we denote left branch of hyperbola, and by C_2 its right branch.

The support function of C_1 is given by

$$p(t) = -\frac{1}{\sqrt{\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4}}} = -\sqrt{a^2 \cos^2 t - b^2 \sin^2 t}.$$

Since C_1 lies between asymptotes $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$, then the domain of its support function is the interval $(-\beta, \beta)$, where $\beta = \frac{\pi}{2} - \operatorname{arc\,tg} \frac{b}{a}$.

Theorem

Let C be such a hyperbola that $a > b$. Then the orthoptic of hyperbola exists and its a circle with center in the origin of the coordinate system and with radius $\sqrt{a^2 - b^2}$.

Theorem

Let C be a hyperbola and let C_α be its α-isoptic. For section $C_{\alpha,1,2}$ we have

$$\bigwedge_{\alpha \in (0,2\beta)} \kappa_{\alpha,1,2}\left(-\frac{\alpha}{2}\right) > 0.$$

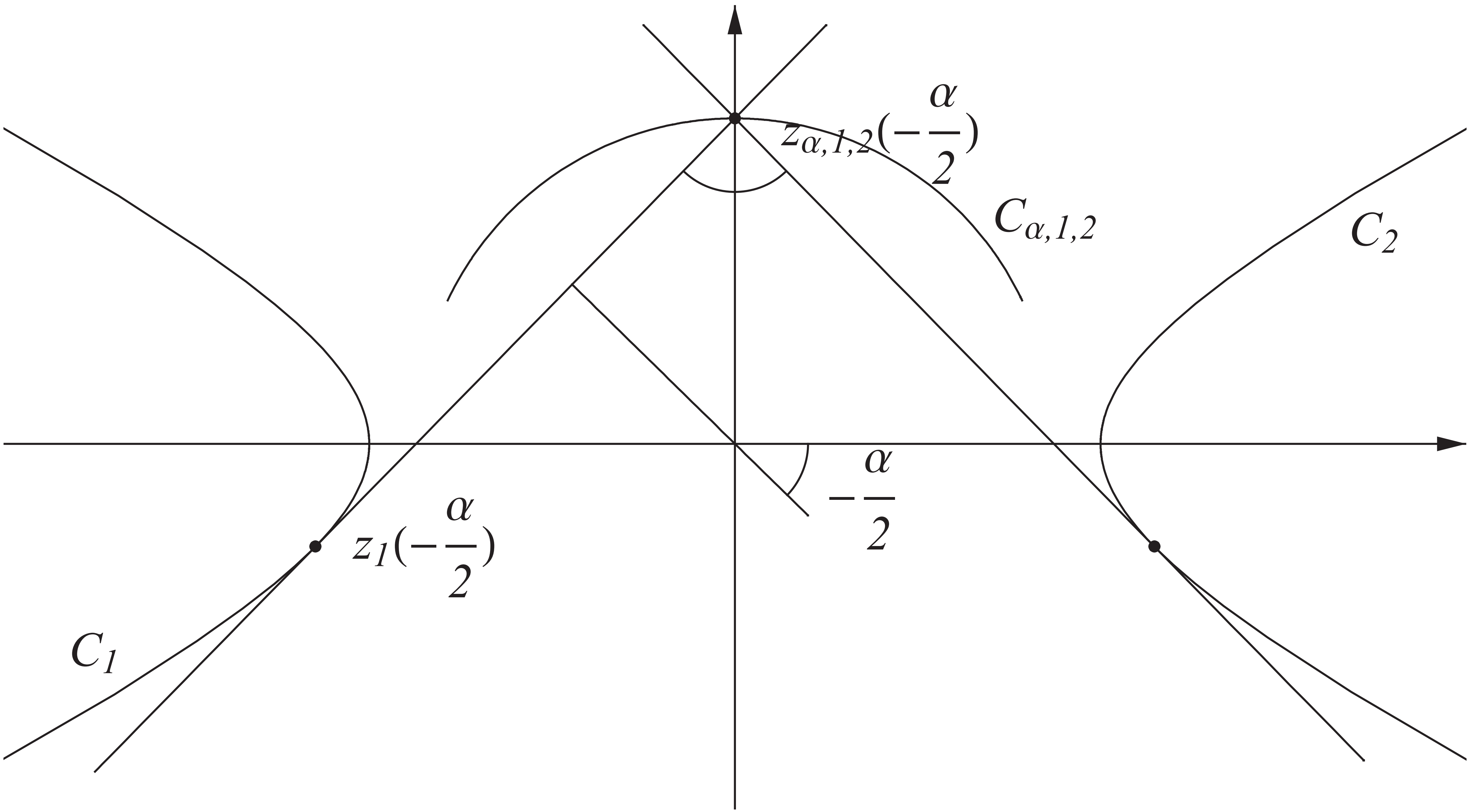


Figure: Point of section $C_{\alpha,1,2}$ of α-isoptic lying on the y-axis

Theorem

Let C be a hyperbola and let C_α be its α-isoptic. Then, for section $C_{\alpha,1,1}$ of α-isoptic there exist such angles $\alpha_1, \alpha_2 \in (0, 2\beta)$, that

$$\kappa_{1,1}\left(-\frac{\alpha_1}{2}, \alpha_1\right) > 0, \quad \kappa_{1,1}\left(-\frac{\alpha_2}{2}, \alpha_2\right) < 0.$$

Corollary

Let C be a hyperbola and let C_α be its α-isoptic. Let α_0 be an angle satysfying condition $\cos\alpha_0 = \dfrac{b^2}{a^2+b^2}$ (the critical angle). Then, for section $C_{\alpha,1,1}$ we have

$$\kappa_{1,1}\left(-\frac{\alpha}{2}, \alpha\right) > 0 \text{ for } \alpha < \alpha_0, \quad \kappa_{1,1}\left(-\frac{\alpha}{2}, \alpha\right) < 0 \text{ for } \alpha > \alpha_0.$$

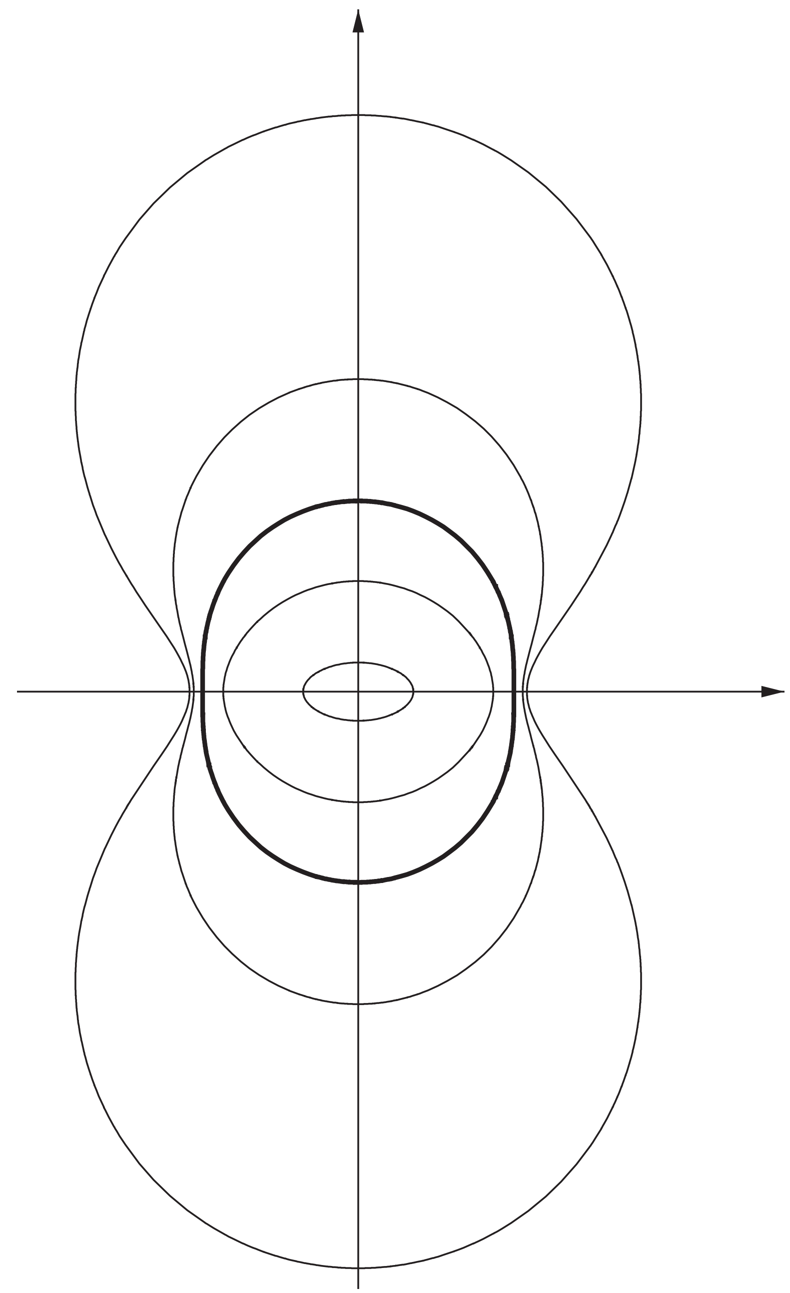


Figure: α-isoptics of hyperbolas for different angles. Bolded curve is the α-isoptic for critical angle

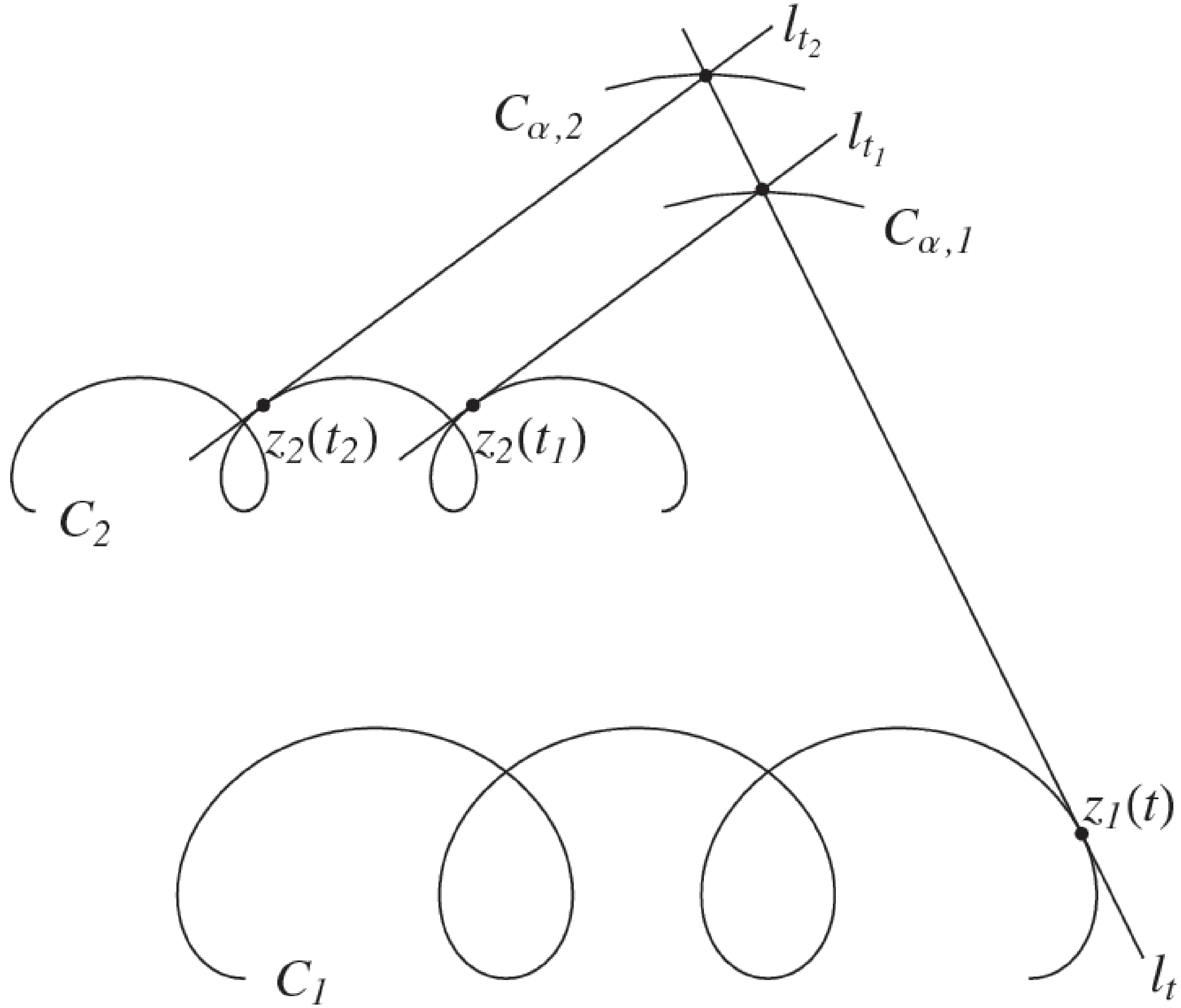


Figure: Isoptics of pair of open rosettes

$$t_k = t + 2(k-1)\pi + \alpha$$

Consider open rosettes

$$C_1 : p_1(t) = 4 - t\cos t,$$
$$C_2 : p_2(t) = 4 - t\sin t$$

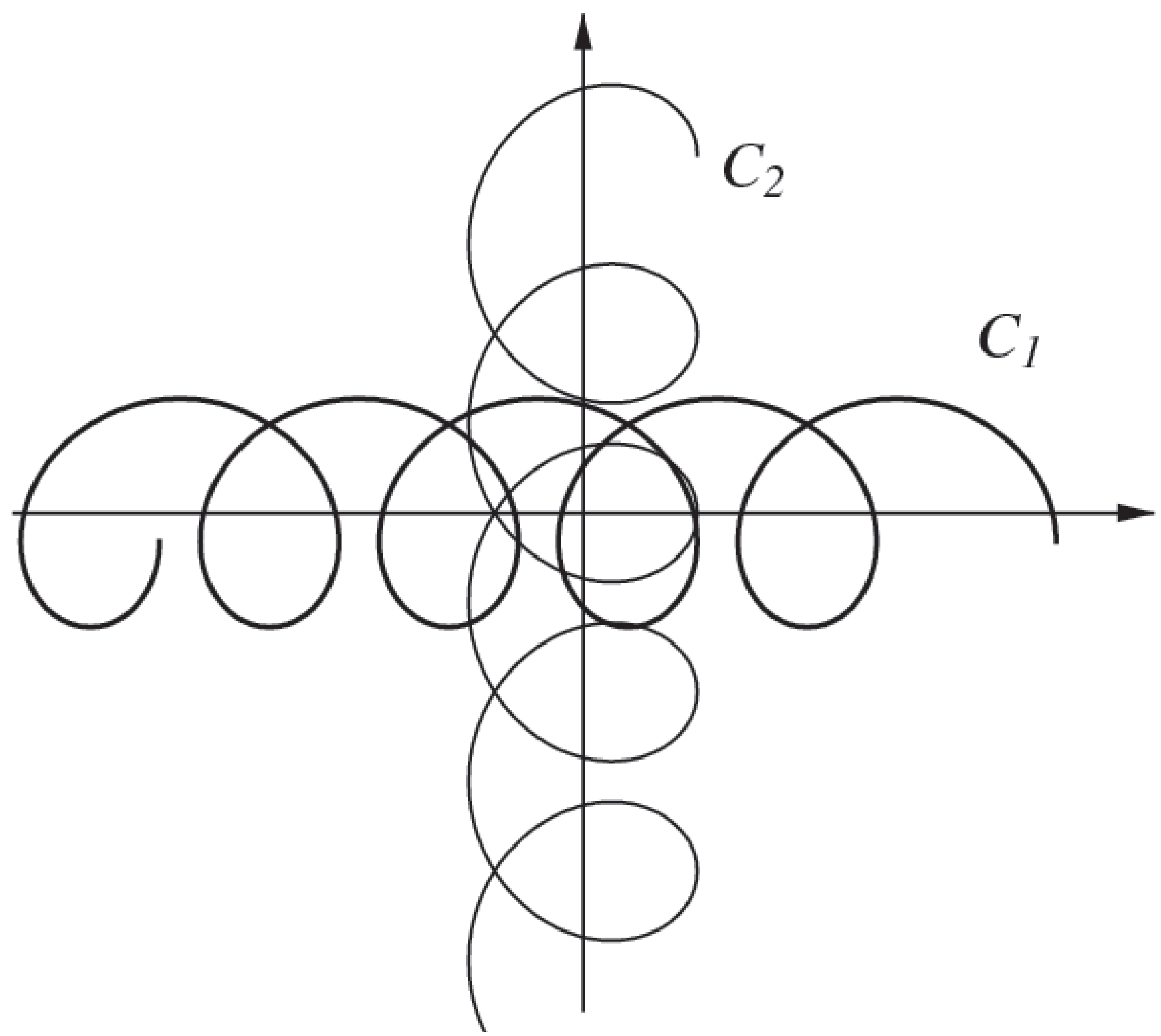


Figure: Pair of open rosettes

$$
\begin{aligned}
z_{\frac{3\pi}{4},1}(t) =& (4 - t\cos t)e^{it} \\
&+ \left(4 + 4\sqrt{2} - 2t\cos t + t\sin t + \frac{3\pi}{4}(\sin t - \cos t)\right) ie^{it}.
\end{aligned}
$$

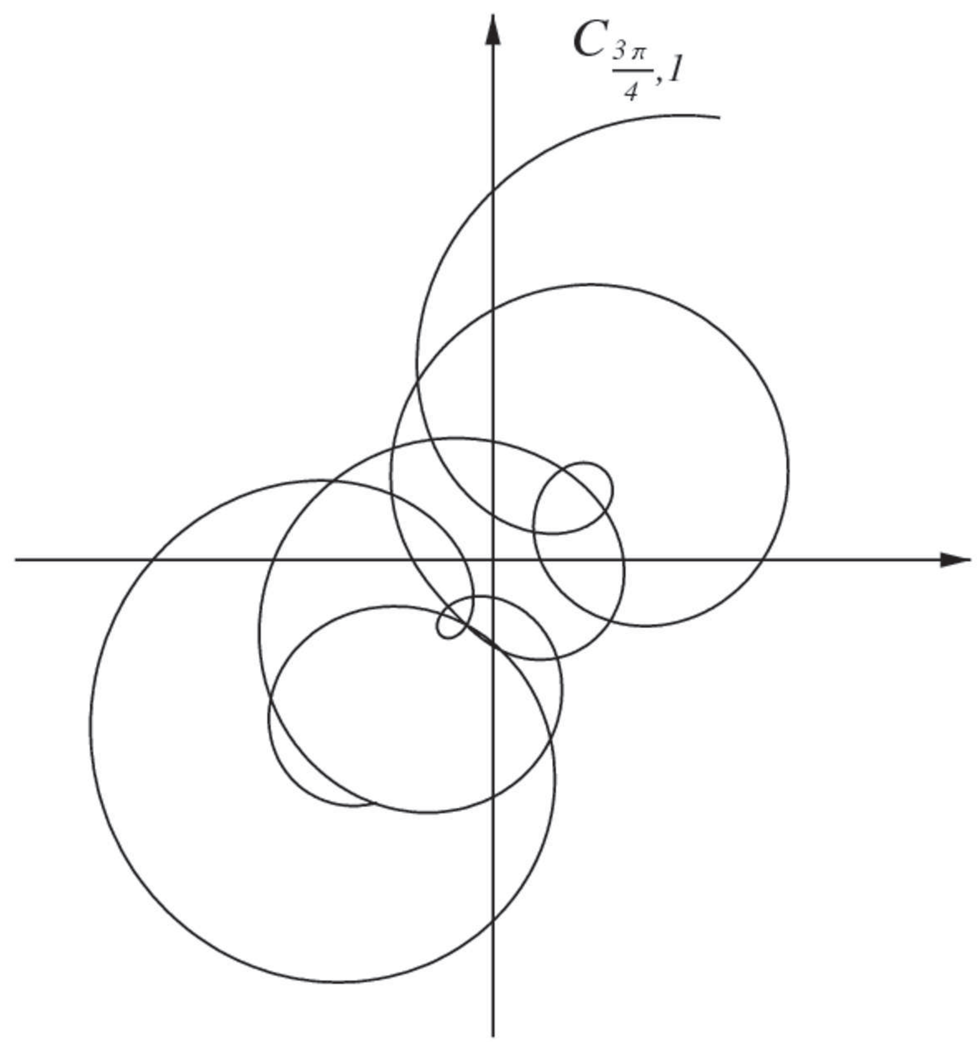


Figure: Isoptic of pair of open rosettes