LOCAL CHERN CLASSES AND RELATIONS AMONG RATIONAL FUNCTIONS

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Topological setup

- Torus $T = (S^1)^n$ or $(\mathbb{C}^*)^n$ acts on a topological space M.
- Denote by $T^\vee = \mathrm{Hom}(T, S^1)$ the group of characters.
- Equivariant cohomology $H_T^*(M)$ is a modul over

$$H_T^*(pt) = \mathbb{Q}[T^\vee] = \mathbb{Q}[t_1, t_2, \dots t_n]$$

If M is an algebraic manifold and T acts algebraically then

$$H_T^*(M) \otimes_{H_T^*(pt)} \mathbb{Q} = H^*(M)$$

Localization

(Almost) everything about equivariant cohomology can be read from some data concentrated at the fixed points.

Borel localization theorem

The restriction to the fixed set

$$H^*_T(M) \longrightarrow H^*_T(M^T)$$

is an isomorphism after inverting $T^\vee - \{0\}$.

Atiyah-Bott or Berline-Vergne formula

- Assumption about fixed points:
  $M^T = \{p_0, p_1, \dots p_n\}$ is discrete.

- For $p \in M^T$:

Define the Euler class e_p as the product of weights of T appearing in the tangent representation.

Integration Formula

The integral can be expressed by the local data.
For $a \in H^*_T(M)$ it is the sum of fractions

$$\int_M a = \sum_{p \in M^T} \frac{a_{|p}}{e_p}$$

Goal

- We use Localization Theorem to compute some invariants of T-invariant singular varieties $X \subset M$.
- The main interest:
  M is the Grassmannian $G_m(\mathbb{C}^n)$
  and X is a Schubert variety.
- The invariant:
  Chern-Schwartz-MacPherson class.

Chern-Schwartz-MacPherson class

- For a singular variety $c_{SM}(X) \in H_*(X)$
- If X is smooth, then $c_{SM}(X)$ is the Poincaré dual of the usual Chern class
- It is functorial (in some sense)
- It can be computed via resolution $\pi : \widetilde{X} \to X$
  $c_{SM}(X) = \pi_*(c_*(\widetilde{X})) + \textit{correction terms}$
- Correction terms are supported by singulatities.
- There is an equivariant version of CSM classes.

Chern-Schwartz-MacPherson class

From Localization theorem it follows that equivariant Chern-Schwartz-MacPherson classes are determined by local Chern classes at the fixed points.

Before computing equivariant Chern classes of Schubert varieties let us first see some computations based on Localization Theorem.

Example of computation

- $M = \mathbb{P}^n$, $T = (\mathbb{C}^*)^{n+1}$
- $M^T = \{p_0, p_1, \ldots p_n\}$ fixed points
- $T_{p_k}M = \bigoplus_{\ell \neq k} L_{t_\ell - t_k}$ decomposition into lines
- $e_k = \prod_{\ell \neq k}(t_\ell - t_k)$ Euler class
- $c_1 := c_1(\mathcal{O}(-1))$ Chern class of the tautological bundle

$$\int_{\mathbb{P}^n} c_1^m = 0 \qquad \text{for} \quad m < n$$

$$= (-1)^n \qquad \text{for} \quad m = n$$

- Applying Berline-Vergne formula we get an identity

$$\sum_{k=0}^{n} \frac{t_k^m}{\prod_{\ell \neq k}(t_\ell - t_k)} = 0$$

for $m < n$

- For example : $n = 2, \;\; m = 0$

$$\frac{1}{(t_1 - t_0)(t_2 - t_0)} + \frac{1}{(t_0 - t_1)(t_2 - t_1)} + \frac{1}{(t_0 - t_2)(t_1 - t_2)} = 0$$

- *(Please, check it by hand!)*

Integral of higher powers

- What do we get for $m > n$?

$$\int_{\mathbb{P}^n} c_1^m \;=\; ?$$

- For example $n = 2, \;\; m = 4$

$$\frac{t_0^4}{(t_1 - t_0)(t_2 - t_0)} + \frac{t_1^4}{(t_0 - t_1)(t_2 - t_1)} + \frac{t_2^4}{(t_0 - t_2)(t_1 - t_2)} =$$

$$Res_{z=\infty} \frac{z^4}{(t_0 - z)(t_1 - z)(t_2 - z)} = t_0^2 + t_1^2 + t_2^2 + t_0 t_1 + t_0 t_2 + t_1 t_2 \;=$$

- in terms of the elementary symmetric functions:

$$= \sigma_1^2 - \sigma_2 \;= S_2$$

Integral of higher powers

In general

$$\int_{\mathbb{P}^n} c_1^{n+k} \quad = \quad (-1)^n S_k$$

where S_k is the Segre class (a special case of Schur function)

$$S_k = \frac{\begin{vmatrix} t_0^{n+k} & t_0^{n-1} & t_0^{n-2} & \dots & t_0^1 & 1 \\ t_1^{n+k} & t_1^{n-1} & t_1^{n-2} & \dots & t_1^1 & 1 \\ t_2^{n+k} & t_2^{n-1} & t_2^{n-2} & \dots & t_2^1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ t_n^{n+k} & t_n^{n-1} & t_0^{n-2} & \dots & t_n^1 & 1 \end{vmatrix}}{\prod_{i<j}(t_i - t_j)}$$

(Jacobi-Trudy formula, Weyl character formula)

Analogue computation can be performed for Grassmannians.

But still, the calculus of **rational** symmetric functions is not developed enough.

- Equivariant Schubert calculus studied by Knutson, Laksov–Thorup, Gato–Santiago, ...

- Some formulas for flag varieties can be obtained by taking *residue at* ∞, [Berczi–Szenes].

- Assumption: M^T is discrete.

- We apply the Berline-Vergne integration formula for equivariant Chern-Schwartz-MacPherson class of a singular subvariety.

Except from the top gradation

$$\sum_{p \in M^T} \frac{c^T(X)_{|p}}{e_p} = 0.$$

Top equivariant Chern class

- The top component of the local Chern class is easy:

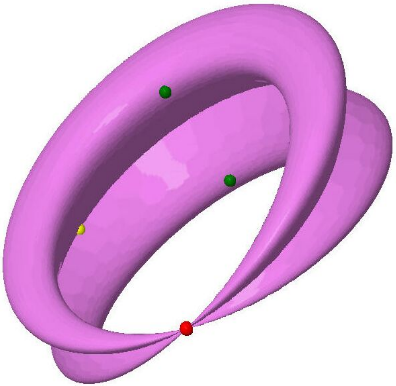
Theorem

If $p \in X^T$ and $w_1, w_2, \ldots, w_n$ are weights of T acting on the tangent space, then

$$c^T(X)_{|p,top} = \prod_{i=1}^{n} w_i \in H_T^{2\dim(M)}(pt)\,,$$

i.e. the top equivariant Chern class is equal to the Euler class at p.

Computation of local equivariant Chern classes



- smooth point
- already computed Chern class
- unknown Chern class

Computation of local equivariant Chern class in $Grass_2(\mathbb{C}^4)$

- Let us compute the local equivariant Chern class of the Schubert variety of codimension 1, i.e.
$V_1 = \{W \,:\, W \cap \langle \varepsilon_1, \varepsilon_2 \rangle \neq 0\}$

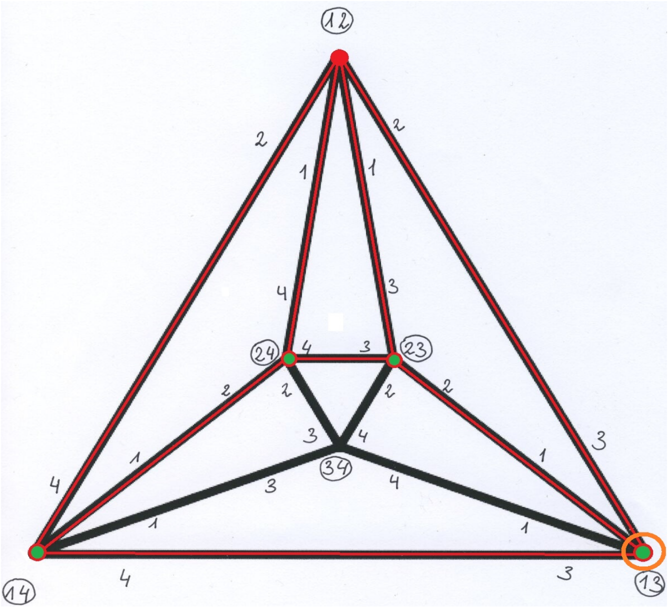
- The neighborhood of the point $p_{1,2}$ in $Grass_2(\mathbb{C}^4)$ is identified with

$$\mathrm{Hom}(span(\varepsilon_1, \varepsilon_2), span(\varepsilon_3, \varepsilon_4))$$

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$$

- The equation of X is $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$

Schubert variety V_1 in $Grass_2(\mathbb{C}^4)$



Computation of local equivariant Chern class in $Grass_2(\mathbb{C}^4)$

- e.g. for $p_{1,3}$, the coordinates are

$$\begin{pmatrix} 1 & a & 0 & b \\ 0 & c & 1 & d \end{pmatrix}$$

and the equation of X is $b = 0$

- For that point the local Chern class is equal to
$(t_4 - t_1)(1 + t_2 - t_1)(1 + t_2 - t_3)(1 + t_4 - t_3)$

- The summand in the integration formula:

$$\frac{(t_4 - t_1)(1 + t_2 - t_1)(1 + t_2 - t_3)(1 + t_4 - t_3)}{(t_4 - t_1)(t_2 - t_1)(t_2 - t_3)(t_4 - t_3)} =$$

$$= \left(1 + \frac{1}{t_2 - t_1}\right)\left(1 + \frac{1}{t_2 - t_3}\right)\left(1 + \frac{1}{t_4 - t_3}\right)$$

Computation of local equivariant Chern class in $Grass_2(\mathbb{C}^4)$

Out[52]= $\left(1+\frac{1}{t1\,u-t2\,u}\right)\left(1+\frac{1}{t1\,u-t3\,u}\right)\left(1+\frac{1}{-t3\,u+t4\,u}\right)$

Out[53]= $\left(1+\frac{1}{t1\,u-t2\,u}\right)\left(1+\frac{1}{t1\,u-t4\,u}\right)\left(1+\frac{1}{t3\,u-t4\,u}\right)$

Out[54]= $\left(1+\frac{1}{-t1\,u+t2\,u}\right)\left(1+\frac{1}{t2\,u-t3\,u}\right)\left(1+\frac{1}{-t3\,u+t4\,u}\right)$

Out[55]= $\left(1+\frac{1}{-t1\,u+t2\,u}\right)\left(1+\frac{1}{t2\,u-t4\,u}\right)\left(1+\frac{1}{t3\,u-t4\,u}\right)$

```
In[61]:= Chern = Expand[Factor[-(a + b + c + d) * (t3 - t1) * (t3 - t2) * (t4 - t1) * (t4 - t2) * u^4]];
In[65]:= Do[Print["deg=", n]; Print[Factor[Coefficient[Chern, u, n]]], {n, 3}]
```

deg=1

$-t1-t2+t3+t4$

deg=2

$(t1+t2-t3-t4)^2$

deg=3

$-(t1+t2-t3-t4)\,(2\,t1\,t2-t1\,t3-t2\,t3-t1\,t4-t2\,t4+2\,t3\,t4)$

Schur expansion

- The coefficients of the expansion in the Schur basis

$$c^T(V_1) = \sum a_{\mathrm{I},\mathrm{J}} S_{\mathrm{I}}(-t_1, -t_2) \cdot S_{\mathrm{J}}(t_3, t_4)$$

	(0)	(1)	(11)	(2)	(21)	(22)
(0)		1	1	1	2	1
(1)	1	1	3	1	1	
(11)	1	3		1		
(2)	1	1	1			
(21)	2	1				
(22)	1					

Computation of local equivariant Chern class in $Grass_3(\mathbb{C}^6)$

- Computations can be continued without problems for $Grass_3(\mathbb{C}^6)$

- The expression which has to be simplified to compute the gradation 1

$$
\begin{aligned}
&\frac{-3 t_1 - 3 t_2 + 2 t_3 + 2 t_4 - 3 t_5 - 3 t_6}{(t_1 - t_3)(t_2 - t_3)(t_3 + t_4)(t_1 + t_5)(t_2 + t_5)(-t_4 + t_5)(t_1 + t_6)(t_2 + t_6)(-t_4 + t_6)} - \frac{-3 t_1 + 2 t_2 - 3 t_3 + 2 t_4 - 3 t_5 - 3 t_6}{(t_1 - t_2)(t_2 - t_3)(t_2 + t_4)(t_1 + t_5)(t_3 + t_5)(-t_4 + t_5)(t_1 + t_6)(t_3 + t_6)(-t_4 + t_6)} +\\
&\frac{2 t_1 - 3 t_2 - 3 t_3 + 2 t_4 - 3 t_5 - 3 t_6}{(t_1 - t_2)(t_1 - t_3)(t_1 + t_4)(t_2 + t_5)(t_3 + t_5)(-t_4 + t_5)(t_2 + t_6)(t_3 + t_6)(-t_4 + t_6)} - \frac{-3 t_1 - 3 t_2 + 2 t_3 - 3 t_4 + 2 t_5 - 3 t_6}{(t_1 - t_3)(t_2 - t_3)(t_1 + t_4)(t_2 + t_4)(t_3 + t_5)(-t_4 + t_5)(t_1 + t_6)(t_2 + t_6)(-t_5 + t_6)} +\\
&\frac{-3 t_1 + 2 t_2 - 3 t_3 - 3 t_4 + 2 t_5 - 3 t_6}{(t_1 - t_2)(t_2 - t_3)(t_1 + t_4)(t_3 + t_4)(t_2 + t_5)(-t_4 + t_5)(t_1 + t_6)(t_3 + t_6)(-t_5 + t_6)} - \frac{2 t_1 - 3 t_2 - 3 t_3 - 3 t_4 + 2 t_5 - 3 t_6}{(t_1 - t_2)(t_1 - t_3)(t_2 + t_4)(t_3 + t_4)(t_1 + t_5)(-t_4 + t_5)(t_2 + t_6)(t_3 + t_6)(-t_5 + t_6)} -\\
&\frac{-3 t_1 + 2 t_2 + 2 t_3 + 2 t_4 + 2 t_5 - 3 t_6}{(t_1 - t_2)(t_1 - t_3)(t_2 + t_4)(t_3 + t_4)(t_2 + t_5)(t_3 + t_5)(t_1 + t_6)(-t_4 + t_6)(-t_5 + t_6)} + \frac{2 t_1 - 3 t_2 + 2 t_3 + 2 t_4 + 2 t_5 - 3 t_6}{(t_1 - t_2)(t_2 - t_3)(t_1 + t_4)(t_3 + t_4)(t_1 + t_5)(t_3 + t_5)(t_2 + t_6)(-t_4 + t_6)(-t_5 + t_6)} -\\
&\frac{2 t_1 + 2 t_2 - 3 t_3 + 2 t_4 + 2 t_5 - 3 t_6}{(t_1 - t_3)(t_2 - t_3)(t_1 + t_4)(t_2 + t_4)(t_1 + t_5)(t_2 + t_5)(t_3 + t_6)(-t_4 + t_6)(-t_5 + t_6)} + \frac{2 t_1 - 3 t_2 - 3 t_3 - 3 t_4 - 3 t_5 + 2 t_6}{(t_1 - t_2)(t_1 - t_3)(t_2 + t_4)(t_3 + t_4)(t_2 + t_5)(t_3 + t_5)(t_1 + t_6)(-t_4 + t_6)(-t_5 + t_6)} -\\
&\frac{-3 t_1 + 2 t_2 - 3 t_3 - 3 t_4 - 3 t_5 + 2 t_6}{(t_1 - t_2)(t_2 - t_3)(t_1 + t_4)(t_3 + t_4)(t_1 + t_5)(t_3 + t_5)(t_2 + t_6)(-t_4 + t_6)(-t_5 + t_6)} + \frac{-3 t_1 - 3 t_2 + 2 t_3 - 3 t_4 - 3 t_5 + 2 t_6}{(t_1 - t_3)(t_2 - t_3)(t_1 + t_4)(t_2 + t_4)(t_1 + t_5)(t_2 + t_5)(t_3 + t_6)(-t_4 + t_6)(-t_5 + t_6)} +\\
&\frac{2 t_1 + 2 t_2 - 3 t_3 + 2 t_4 - 3 t_5 + 2 t_6}{(t_1 - t_3)(t_2 - t_3)(t_1 + t_4)(t_2 + t_4)(t_3 + t_5)(-t_4 + t_5)(t_1 + t_6)(t_2 + t_6)(-t_5 + t_6)} - \frac{2 t_1 - 3 t_2 + 2 t_3 + 2 t_4 - 3 t_5 + 2 t_6}{(t_1 - t_2)(t_2 - t_3)(t_1 + t_4)(t_3 + t_4)(t_2 + t_5)(-t_4 + t_5)(t_1 + t_6)(t_3 + t_6)(-t_5 + t_6)} +\\
&\frac{-3 t_1 + 2 t_2 + 2 t_3 + 2 t_4 - 3 t_5 + 2 t_6}{(t_1 - t_2)(t_1 - t_3)(t_2 + t_4)(t_3 + t_4)(t_1 + t_5)(-t_4 + t_5)(t_2 + t_6)(t_3 + t_6)(-t_5 + t_6)} - \frac{2 t_1 + 2 t_2 - 3 t_3 - 3 t_4 + 2 t_5 + 2 t_6}{(t_1 - t_3)(t_2 - t_3)(t_3 + t_4)(t_1 + t_5)(t_2 + t_5)(-t_4 + t_5)(t_1 + t_6)(t_2 + t_6)(-t_4 + t_6)} +\\
&\frac{2 t_1 - 3 t_2 + 2 t_3 - 3 t_4 + 2 t_5 + 2 t_6}{(t_1 - t_2)(t_2 - t_3)(t_2 + t_4)(t_1 + t_5)(t_3 + t_5)(-t_4 + t_5)(t_1 + t_6)(t_3 + t_6)(-t_4 + t_6)} - \frac{-3 t_1 + 2 t_2 + 2 t_3 - 3 t_4 + 2 t_5 + 2 t_6}{(t_1 - t_2)(t_1 - t_3)(t_1 + t_4)(t_2 + t_5)(t_3 + t_5)(-t_4 + t_5)(t_2 + t_6)(t_3 + t_6)(-t_4 + t_6)}
\end{aligned}
$$

- The result is equal to $[V_1] = -t_1 - t_2 - t_3 + t_4 + t_5 + t_6$

Local Chern class of $V_1 \subset Grass_3(\mathbb{C}^6)$ in Schur basis

$$c^T(V_1) = \sum a_{\mathrm{I},\mathrm{J}} S_{\mathrm{I}}(-t_1, -t_2, -t_3) \cdot S_{\mathrm{J}}(t_4, t_5, t_6)$$

	0	1	11	2	111	21	3	211	31	22	311	221	32	321	222	33	331	322	332	333
0		1	2	2	4	5	1	9	3	4	6	9	3	8	4	1	3	6	3	1
1	1	4	8	5	12	12	2	19	5	8	8	16	4	8	10	1	2	4	1	
11	2	8	12	9	16	16	3	18	6	8	6	10	4	4		1	1			
2	2	5	9	4	11	9	1	13	2	5	3	10	1	2	5			1		
111	4	12	16	11	8	16	4		6	10			4			1				
21	5	12	16	9	16	14	2	15	3	5	3	5	1	1						
3	1	2	3	1	4	2		3		1		2			1					
211	9	19	18	13		15	3		3	5			1							
31	3	5	6	2	6	3		3		1		1								
22	4	8	8	5	10	5	1	5	1		1									
311	6	8	6	3		3				1										
221	9	16	10	10		5	2		1											
32	3	4	4	1	4	1		1												
321	8	8	4	2		1														
222	4	10		5			1													
33	1	1	1		1															
331	3	2	1																	
322	6	4		1																
332	3	1																		
333	1																			

All coefficients are nonnegative.

- Now appears a problem with the size of the expressions since $\dim(Grass_4(\mathbb{C}^8)) = 16$ and $\dim(T) = 8$
- In a polynomial of degree 15 in 8 variables there are

  245 157 monomials.
- The expression is a sums of 79 fractions with factors $t_i - t_j$ in denominators.

- The result written in the Schur basis

$$S_{\mathrm{I}}(-t_1, -t_2, -t_3, -t_4) \cdot S_{\mathrm{J}}(t_5, t_6, t_7, t_8)$$

is the following:

Local equivariant Chern class of $V_1 \subset Grass_4(\mathbb{C}^8)$

Local equivariant Chern class of $V_1 \subset Grass_4(\mathbb{C}^8)$

	0	1	11	2	111	21	3	1111	211	31	22	4	2111	311	221	41	32	3111
0		1	3	3	7	10	3	13	23	11	11	1	44	26	33	4	15	50
1	1	6	16	12	32	39	10	55	83	35	39	3	147	77	108	11	44	137
11	3	16	38	29	70	84	22	105	162	69	75	6	248	135	188	20	78	203
2	3	12	29	18	52	56	12	84	111	40	51	3	186	83	132	10	46	142
111	7	32	70	52	110	139	37	135	225	106	117	10	261	172	238	30	114	180
21	10	39	84	56	139	152	35	194	272	102	122	8	392	187	283	24	103	272
3	3	10	22	12	37	35	6	58	66	19	29	1	108	38	72	3	20	64
1111	13	55	105	84	135	194	58	80	245	145	150	15	80	180	245	40	145	20
211	23	83	162	111	225	272	66	245	400	173	202	15	428	258	371	40	163	266
31	11	35	69	40	106	102	19	145	173	52	74	3	244	92	163	8	47	132
22	11	39	75	51	117	122	29	150	202	74	84	6	272	122	179	16	63	166
4	1	3	6	3	10	8	1	15	15	3	6		24	6	15		3	10
2111	44	147	248	186	261	392	108	80	428	244	272	24	72	266	381	56	217	16
311	26	77	135	83	172	187	38	180	258	92	122	6	266	132	211	14	75	136
221	33	108	188	132	238	283	72	245	371	163	179	15	381	211	282	35	129	211
41	4	11	20	10	30	24	3	40	40	8	16		56	14	35		7	20
32	15	44	78	46	114	103	20	145	163	47	63	3	217	75	129	7	34	101
3111	50	137	203	142	180	272	64	20	266	132	166	10	16	136	211	20	101	8
2211	72	213	316	243	285	444	129	60	399	250	267	27	36	213	297	54	192	-4
411	10	25	40	21	50	44	6	50	60	14	26		60	20	45		11	20
321	47	127	199	125	235	244	53	240	304	108	131	8	308	136	197	16	69	136
222	26	83	127	96	135	176	49	135	184	95	95	10	190	95	103	20	66	101
42	6	15	25	12	35	26	3	45	40	7	15		54	11	30		5	15
33	8	21	34	19	49	38	7	58	59	14	19	1	76	21	40	2	7	28
4111	20	45	60	36	50	64	10		60	20	36			20	45		15	
3211	102	251	325	237	250	385	99	20	314	169	194	15	8	138	200	25	103	
2221	80	219	280	228	185	352	115	20	206	186	194	24	8	97	113	40	132	
421	20	45	65	33	75	61	8	75	75	16	30		75	20	45		10	20
331	28	67	95	57	110	97	21	106	118	35	40	3	118	42	61	5	14	42
322	42	107	145	99	145	161	40	155	161	68	68	6	167	68	68	10	35	70
43	4	9	14	6	19	12	1	24	18	2	6		24	3	12		1	4
4211	45	90	105	63	75	96	15		75	25	45			20	45		15	
3311	64	138	158	114	114	156	42	-4	122	56	59	6		42	63	8	21	
3221	123	280	300	246	165	322	99	20	167	136	136	15		70	70	20	70	
2222	69	150	165	135	30	186	66	-40	20	93	101	15				20	70	
431	15	30	40	18	45	30	3	45	36	5	12		36	6	18		2	6
422	20	40	50	26	50	40	6	50	40	10	15		40	10	15		5	10
332	33	74	89	59	89	80	21	93	80	28	21	3	84	28	21	4	7	28
44	1	2	3	1	4	2		5	3		1		4		2			
4311	36	63	66	36	45	48	6		36	8	18			6	18		3	
4221	60	105	100	66	50	80	15		40	20	30			10	15		10	
3321	100	201	182	156	97	160	56		84	56	42	8		28	21	8	14	
3222	100	195	165	159	10	167	64			70	70	10				10	35	
441	4	7	9	3	10	5		10	6		2		6		3			
432	20	35	40	18	40	24	3	40	24	4	6		24	4	6		1	4

	2211	411	321	222	42	33	4111	3211	2221	421	331	322	43	4211	3311	3221	2222	431	422	332
0	72	10	47	26	6	8	20	102	80	20	28	42	4	45	64	123	69	15	20	33
1	213	25	127	83	15	21	45	251	219	45	67	107	9	90	138	280	150	30	40	74
11	316	40	199	127	25	34	60	325	280	65	95	145	14	105	158	300	165	40	50	89
2	243	21	125	96	12	19	36	237	228	33	57	99	6	63	114	246	135	18	26	59
111	285	50	235	135	35	49	50	250	185	75	110	145	19	75	114	165	30	45	50	89
21	444	44	244	176	26	38	64	385	352	61	97	161	12	96	156	322	186	30	40	80
3	129	6	53	49	3	7	10	99	115	8	21	40	1	15	42	99	66	3	6	21
1111	60	50	240	135	45	58		20	20	75	106	155	24		-4	20	-40	45	50	93
211	399	60	304	184	40	59	60	314	206	75	118	161	18	75	122	167	20	36	40	80
31	250	14	108	95	7	14	20	169	186	16	35	68	2	25	56	136	93	5	10	28
22	267	26	131	95	15	19	36	194	194	30	40	68	6	45	59	136	101	12	15	21
4	27		8	10		1		15	24		3	6			6	15	15			3
2111	36	60	308	190	54	76		8	8	75	118	167	24					36	40	84
311	213	20	136	95	11	21	20	138	97	20	42	68	3	20	42	70		6	10	28
221	297	45	197	103	30	40	45	200	113	45	61	68	12	45	63	70		18	15	21
41	54		16	20		2		25	40		5	10			8	20	20			4
32	192	11	69	66	5	7	15	103	132	10	14	35	1	15	21	70	70	2	5	7
3111	-4	20	136	101	15	28				20	42	70	4					6	10	28
2211	24	45	202	101	45	59				45	63	70	18					18	15	21
411	45		20	20		3		20	20		6	10			6	10				4
321	202	20	104	66	10	14	20	105	70	15	21	35	2	15	21	35		3	5	7
222	101	20	66	8	15	21	20	70		15	21		6	15	21			6		
42	45		10	15		1		15	30		2	5			3	10	15			1
33	59	3	14	21	1		4	21	42	2		7		3		14	21		1	
4111			20	20		4					6	10								4
3211		20	105	70	15	21				15	21	35	3					3	5	7
2221		20	70		30	42				15	21		12					6		
421	45		15	15		2		15	15		3	5			3	5				1
331	63	6	21	21	2		6	21	21	3		7		3		7			1	
322	70	10	35		5	7	10	35		5	7		1	5	7			1		
43	18		2	6				3	12			1				2	6			
4211			15	15		3					3	5								1
3311		6	21	21	3					3		7							1	
3221		10	35		10	14				5	7		2					1		
2222					15	21							6							
431	18		3	6				3	6			1				1				
422	15		5			1		5			1				1					
332	21	4	7		1		4	7		1				1						
44	3			1					2								1			
4311			3	6								1								
4221			5			2					1									
3321		4	7		2					1										
3222					5	7							1							
441	3			1					1											
432	6		1					1												

Positivity

- The local Chern classes are a positive combination of monomials
- But starting from $Grass_4(\mathbb{C}^8)$ they are *not* positive combinations of products of Schur functions.

Conclusion

- This supports the conjecture of Aluffi and Mihalcea that the Chern class (or even equivariant Chern class) of a Schubert variety is effective.

Further directions of work

- Develop a calculus of symmetric rational functions
- Deduce positivity results
- Study global equivariant Chern classes of Schubert varieties and open cells

Thank You

103	20	145	163	47	63	3	217	75	129	7	34	101	192	11	69	66	5
272	64	20	266	132	166	10	16	136	211	20	101	8	-4	20	136	101	15
444	129	60	399	250	267	27	36	213	297	54	192	-4	24	45	202	101	45
44	6	50	60	14	26		60	20	45		11	20	45		20	20	
244	53	240	304	108	131	8	308	136	197	16	69	136	202	20	104	66	10
176	49	135	184	95	95	10	190	95	103	20	66	101	101	20	66	8	15
26	3	45	40	7	15		54	11	30		5	15	45		10	15	
38																	1
64																	
385																	15
352	1																30
61																	
97	21	106	118	35	40	3	118	42	61	5	14	42	63	6	21	21	2
161	40	155	161	68	68	6	167	68	68	10	35	70	70	10	35		5
12	1	24	18	2	6		24	3	12		1	4	18		2	6	
96	15		75	25	45			20	45		15				15	15	
156	42	-4	122	56	59	6		42	63	8	21			6	21	21	3
322	99	20	167	136	136	15		70	70	20	70			10	35		10
186	66	-40	20	93	101	15				20	70						15
30	3	45	36	5	12		36	6	18		2	6	18		3	6	
40	6	50	40	10	15		40	10	15		5	10	15		5		
80	21	93	80	28	21	3	84	28	21	4	7	28	21	4	7		1